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SENSITIVITY ANALYSIS FOR CONVEX SEPARABLE OPTIMIZATION OVER INTEGRAL POLYMATROIDS*

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Abstract. We study the sensitivity of optimal solutions of convex separable optimization problems over an integral polymatroid base polytope with respect to parameters determining both the cost of each element and the polytope. Under convexity and a regularity assumption on the functional dependency of the cost function with respect to the parameters, we show that reoptimization after a change in parameters can be done by elementary local operations. Applying this result, we derive that starting from any optimal solution, there is a new optimal solution to new parameters such that the L_1-norm of the difference of the two solutions is at most two times the L_1-norm of the difference of the parameters. We apply these sensitivity results to a class of noncooperative games with a finite set of players where a strategy of a player is to choose a vector in a player-specific integral polymatroid base polytope defined on a common set of elements. The players' private cost functions are regular, convex-separable, and the cost of each element is a nondecreasing function of the own usage of that element and the overall usage of the other players. Under these assumptions, we establish the existence of a pure Nash equilibrium. The existence is proven by an algorithm computing a pure Nash equilibrium that runs in polynomial time whenever the rank of the polymatroid base polytope is polynomially bounded. Both the existence result and the algorithm generalize and unify previous results appearing in the literature. We finally complement our results by showing that polymatroids are the maximal combinatorial structure enabling these results. For *any* nonpolymatroid region, there is a corresponding optimization problem for which the sensitivity results do not hold. In addition, there is a game where the players' strategies are isomorphic to the nonpolymatroid region and that does not admit a pure Nash equilibrium.

Key words. polymatroid, submodular function, sensitivity, reoptimization, integer optimization, noncooperative games, congestion games, pure Nash equilibrium

AMS subject classifications. 05B35, 90C27, 91A10, 91A46

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1. Introduction.

We consider polymatroid optimization problems, where the objective is to distribute $d \in \mathbb{N}$ discrete units among a set of elements $E = \{1, \dots, m\}$ so as to minimize a convex separable cost function subject to upper bounds on the total amount of units allocated to subsets of elements. These upper bounds are defined via values of an integral polymatroid rank function $f : 2^E \to \mathbb{N}$. Formally, we study the following optimization problem:

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$$
(P(\mathbf{t},d)) \qquad \begin{aligned} \text{minimize } & \sum_{e\in E} C_e(x_e;t_e) \\ \text{subject to: } & \sum_{e\in U} x_e \le f(U) \text{ for all } U \subseteq E \\ & \sum_{e\in E} x_e = d \\ & x_e \in \mathbb{N} \text{ for all } e \in E, \end{aligned}
$$

where the functions $C_e : \mathbb{N} \times \mathbb{N} \to \mathbb{R}_+, e \in E$ are nondecreasing and discrete convex in the first entry and f is a normalized, monotone, and submodular set function. The vector $\mathbf{t}$ and the scalar d are integral parameters. For fixed parameters, this problem is a convex-separable optimization problem over an integral polymatroid base polytope and can be solved in polynomial time by greedy algorithms; see Federgruen and Groenevelt [8], Groenevelt [17], Hochbaum and Shanthikumar [21], and the book by Fujishige [10]. Besides these appealing theoretical properties, the problem has applications in several areas ranging from scheduling problems (cf. Yao [40] and Krysta, Sanders, and Vöcking [25]) and tree packing and matroid optimization (cf. Gabow [12]) to game-theoretic applications (cf. He, Zhang, and Zhang [20]).

1.1. Sensitivity analysis. Suppose we are given an optimal solution $\mathbf{x}^*(\mathbf{t},d)$ with respect to $\mathbf{t}$ and d. The main question addressed in this paper is, how does the structure of an optimal solution change after changes to the parameters $\mathbf{t}$ and d? We motivate this question by a concrete example. Let $G = (V,E)$ be a connected undirected graph with vertex set V and edge set $E \subseteq (V \times V)$. The objective is to compute $k \in \mathbb{N}$ spanning trees of G so that along each spanning tree, a message of unit size can be sent. If $x_e \in \mathbb{N}$ messages are sent along edge e, i.e., e is contained in exactly x_e spanning trees, the resulting (average) delay is defined as

$$
C_e(x_e;u_e) = \begin{cases} \frac{1}{u_e - x_e} & \text{if } x_e < u_e, \\ +\infty & \text{else,} \end{cases}
$$

where $C_e(x_e;u_e)$ is a standard $M/M/1$-delay function frequently used in queueing theory [13, 24]. The parameter $u_e \in \mathbb{N}$ denotes the installed capacity on edge e. The problem to compute k spanning trees to minimize the total delay can be cast as a convex separable integral polymatroid optimization problem by taking f as the kth multiple of the rank function of the graphic matroid on G. In this model, it is natural to ask how optimal solutions change if the edge capacities or k are changed.

1.2. Our results for the sensitivity of polymatroid optimization. The change of an optimal solution for changed parameters clearly depends on the structural dependency of the objective function and the feasible region on the parameters $\mathbf{t}$ and d. To capture this dependency, we introduce the following concept of *regularity*. Informally, we call a function $C(x;t)$ regular if (i) the (left discrete) partial derivative with respect to the first entry is a nondecreasing function in the parameter and (ii) the left discrete partial derivative with respect to the first entry is not larger after a unit increase of the parameter t than after a unit increase of x.

Our main results (Theorems 3.2, 3.4, and 3.5) can be informally summarized as follows: Let $\mathbf{x}^*(\mathbf{t},d)$ be an optimal solution of $P(\mathbf{t},d)$ for regular and convex functions $C_e, e \in E$. Then, for any other integral parameters $\mathbf{t}', d'$, there exists a new optimal solution $\mathbf{x}^*(\mathbf{t}',d')$ *close* to $\mathbf{x}^*(\mathbf{t},d)$ in the following sense:

$$
\|\mathbf{x}^*(\mathbf{t},d) - \mathbf{x}^*(\mathbf{t}',d')\|_1 \le 2\|\mathbf{t} - \mathbf{t}'\|_1 + |d - d'|.
$$

Moreover, given $\mathbf{x}^*(\mathbf{t}, d)$, we can compute $\mathbf{x}^*(\mathbf{t}', d')$ by performing $\|\mathbf{t} - \mathbf{t}'\|_1 + |d - d'|$ elementary exchange steps.

In the context of the tree packing example with $M/M/1$ queueing functions discussed in section 1.1, this sensitivity result implies that if the capacity vector is changed, say, by p units, then there is a new optimal solution for which at most $2p$ edges of the given spanning trees needs to be changed, and these changes can be efficiently computed from the initial optimal solution. On the other hand, when k is increased by p units, then at most $p|V|$ units need to be changed since adding a single additional spanning tree corresponds to adding $|V|$ new units.

In section 6, we show that submodularity of the capacity function defining the base polytope is necessary for the sensitivity results above in the following sense: For any capacity function that is not submodular, both sensitivity results (in terms of a change of $\mathbf{t}$ and d) do not hold anymore.

1.3. Our results for polymatroid games. Our sensitivity results have consequences for the existence of pure Nash equilibria for a new class of noncooperative games that we term *polymatroid games*. In such a game, there is a finite set of players N and a finite set of resources E. Each player $i \in N$ distributes her demand $d_i \in \mathbb{N}$ in integral units among the resources subject to player-specific submodular capacity constraints. This way, the resulting strategy space for each player forms an integral polymatroid base polytope (truncated at the player-specific demand). We further assume that the private cost function is defined as a sum of regular, player-specific, nondecreasing, convex cost functions $C_{i,e}(x_e; t_e)$, where the parameter t_e is interpreted as the load contribution of other players to resource e. This class of games includes as special cases the following variants of congestion games that have been treated separately in the literature:

(i) congestion games with matroidal strategies and player-specific costs (studied by Ackermann, Röglin, and Vöcking [1]) for the case that the submodular capacity function is the rank function of a matroid;

(ii) integer-splittable congestion games with singleton strategies (studied by Tran-Thanh et al. [38]) for the case that the submodular capacity function is equal to a sufficiently large constant;

(iii) integer-splittable congestion games with matroidal strategies (the matroidal counterpart of integer-splittable congestion games studied by Rosenthal [33]) for the case that the submodular capacity function is an integer multiple of the rank function of a matroid.

We show the existence of a pure Nash equilibrium and devise an algorithm for its computation for polymatroid games, thus generalizing and unifying the existence results by Ackermann, Röglin, and Vöcking [1] and Tran-Thanh et al. [38]. Our algorithm maintains preliminary demands, strategy spaces, and strategies of the players that all are set to zero initially. In the course of the algorithm, the demands of the players are increased iteratively by one unit, and a preliminary pure Nash equilibrium (with respect to the current demands) is recomputed by following a sequence of best response moves of the players. While similar algorithms have been proposed before for matroid congestion games (cf. Ackermann, Röglin, and Vöcking [1]) and integer-splittable singleton games (cf. Tran-Thanh et al. [38]), the main difficulty is to show that the sequence of best responses converges even in the more general setting of polymatroid games. This is where the sensitivity results for polymatroid optimization shown in section 3 are applied. Furthermore, we show that the runtime of the algorithm is bounded by $n^2 m \delta^3$, where n is the number of players, m is the number of resources, and δ is an upper bound on the maximum demand. Thus, for polynomially bounded δ, the algorithm is polynomial.

In section 6, we prove that submodularity of the players' capacity constraints is necessary for the existence of a pure Nash equilibrium in a strong sense. For *any* monotonic, normalized, strictly positive, and nonsubmodular set function defining the player-specific base polytope, there is a two-player game without a pure Nash equilibrium. In this sense, our results are best possible, and polymatroids are the maximal combinatorial structure guaranteeing the existence of a pure Nash equilibrium.

1.4. Related work.

Sensitivity in polymatroid optimization. Fujishige et al. [11] studied convex-separable minimization problems over polymatroid base polytopes. They conduct a sensitivity analysis of optimal solutions for the case that the marginal cost function is (componentwise) shifted downwards. As their main result, they show that for any downwards shift of the marginal cost function, any new optimal solution has the property that all optimal cost values of the components decrease monotonically with respect to the original optimal solution. Their motivation is to analyze the Braess paradox, and their sensitivity result implies that there is no Braess paradox in polymatroids. A difference in our work lies in the fact that we conduct a *quantitative* sensitivity analysis; that is, given an optimal solution for an initial parameter, we quantify exactly the difference of a closest new optimal solution for a changed parameter.

He, Zhang, and Zhang [20] considered separable-concave maximization problems over polymatroids in which each cost function component has a second parameter and is concave in the parameter. They assume that the cost functions are discrete convex in both entries. A further difference in our setting is that we consider as feasible domain an *integral* polymatroid base polytope instead of an ordinary polymatroid (defined by a real-valued submodular function). Their main result establishes (without any further regularity assumptions) that the optimal value function as a function of the parameter, or as a function of the support set of the objective function, is submodular. These two results have important consequences for game-theoretical models because, using the submodularity of the optimal value function, they show that the joint replenishment game and the one-warehouse multiple retailer game is submodular and, thus, has desirable properties in terms of existence of core solutions. In contrast to the work of He, Zhang, and Zhang [20], we study in this paper sensitivity properties of underlying optimal solutions and not the optimal value function. Moreover, the structure of an integral polymatroid base polytope differs from an ordinary polymatroid. The integral polymatroid base polytope does not form a lattice when considering the componentwise minimum and maximum as join and meet. Thus, the sensitivity framework of Topkis [36, 37] (to which also He, Zhang, and Zhang [20] refer) is not directly applicable.

Cook et al. [6] considered general integer programs of the form $\max\{wx \mid Ax \leq b\}$ and conducted a proximity and sensitivity analysis. The proximity analysis is concerned with the difference of optimal solutions for integer linear programs and their continuous relaxations, respectively (the difference is measured by the L_∞- or L_1-norm). The sensitivity analysis investigates the difference of optimal solutions of an integer linear program for changed b. Their main result shows that the L_∞-distance to the nearest optimal solution to the corresponding integer program is at most the number of variables multiplied by the largest subdeterminant of A. Baldick [3] strengthened some of the sensitivity and proximity results of Cook et al. [6] by introducing a finer measure for the constraint matrix A.

Murota and Tamura [31] and later Moriguchi, Shioura, and Tsuchimura [29] derived proximity results for the minimization of M-convex functions in integer variables (see Murota [30] for an introduction to this concept). Moriguchi, Shioura, and Tsuchimura [29] showed for convex-separable objectives that the difference between integral and fractional optimal solutions measured in the L_1-norm is at most $2(n-1)$, where n is the number of variables.

Congestion games. Rosenthal [32] introduced congestion games, a class of strategic games where a finite set of players competes over a finite set of resources. Each player is associated with a set of subsets of resources, e.g., the set of paths connecting a given source and sink in a network. A pure strategy of a player is to choose a subset from this set. The cost of a resource depends only on the number of players choosing the same resource, and each player strives to minimize the sum of the costs of the resources contained in the selected subset. Rosenthal [32] proved the existence of a pure Nash equilibrium by a potential function argument.

Weighted congestion games are a generalization of this model where players are associated with a weight, and the cost of a resource is a function of the aggregated weight of its users. Unlike unweighted games, weighted congestion games may not possess a pure Nash equilibrium in general [5, 9, 16], but a pure Nash equilibrium can be guaranteed under additional restrictions on the set of cost functions [2, 9, 18] or the strategy sets [1].

Similarly, also congestion games with player-specific cost functions need not possess a pure Nash equilibrium [15, 27, 28], but a pure Nash equilibrium exists under additional restrictions on the set of cost functions [14] or the strategy sets [1, 27]. Specifically, Ackermann, Röglin, and Vöcking [1] showed that player-specific congestion games where the strategy set of each player corresponds to the set of bases of a matroid has a pure Nash equilibrium. Our results generalize these existence results towards general polymatroid strategy spaces.

In congestion games with integer-splittable demands, players have an integer weight that they split in integer units over the sets of resources. Rosenthal [33] showed that that these games need not possess a pure Nash equilibrium. Dunkel and Schulz [7] strengthened this result showing that the existence of a pure Nash equilibrium in integer-splittable congestion games is NP-complete to decide. Meyers [26] proved that in games with linear cost functions, a pure Nash equilibrium is always guaranteed to exist. For singleton strategy spaces and nonnegative and convex cost functions, Tran-Thanh et al. [38] showed the existence of pure Nash equilibria. They also showed that pure Nash equilibria need not exist (even for the restricted strategy spaces) if cost functions are semiconvex. Our results generalize the existence result of Tran-Thanh et al. [38] towards general polymatroid strategy spaces.

For the use of congestion games to model decentralized systems involving the selfish allocation of resources, we refer the reader to [4, 34, 39] in the context of selfish route choices in traffic networks and to [22, 23, 35] in the context of flow control in telecommunication networks.

2. Preliminaries. Let $\mathbb{N}$ denote the set of nonnegative integers, and let E be a finite and nonempty set of elements. We write $\mathbb{N}^E$ as shorthand for $\mathbb{N}^{|E|}$. Throughout this paper, vectors $\mathbf{x} = (x_e)_{e \in E} \in \mathbb{N}^E$ will be denoted with boldface. An integral set function $f : 2^E \to \mathbb{N}$ is *submodular* if $f(U) + f(V) \geq f(U \cup V) + f(U \cap V)$ for all $U, V \in 2^E$; f is *monotone* if $f(U) \leq f(V)$ for all $U \subseteq V \subseteq E$; and f is *normalized* if $f(\emptyset) = 0$. We call an integral, submodular, monotone, and normalized function $f : 2^E \to \mathbb{N}$ an *integral polymatroid rank function.* The associated *integral polyhedron* is defined as

$$\mathbb{P}_f := \left\{ \mathbf{x} \in \mathbb{N}^E : x(U) \leq f(U) \text{ for all } U \subseteq E \right\},$$

where for a vector $\mathbf{x} = (x_e)_{e \in E}$ and $U \subseteq E$, we write $x(U)$ as shorthand for $\sum_{e \in U} x_e$. For an element $e \in E$, we write $x(e)$ instead of $x(\{e\})$. Given the integral polyhedron $\mathbb{P}_f$ and an integer $d \in \mathbb{N}$ with $d \leq f(E)$, the *d-truncated integral polymatroid* $\mathbb{P}_f(d)$ is defined as

$$\mathbb{P}_f(d) := \left\{ \mathbf{x} \in \mathbb{N}^E : x(U) \leq f(U) \text{ for all } U \subseteq E,\, x(E) \leq d \right\}.$$

The d-truncated integral polymatroid $\mathbb{P}_f(d)$ is again an integral polyhedron, as the corresponding integral polymatroid rank function $f' : 2^E \to \mathbb{N}$ defined as $f'(U) = \min\{d, f(U)\}$ is submodular. For a d-truncated integral polymatroid $\mathbb{P}_f(d)$, the corresponding *integral polymatroid base polyhedron* is defined as

$$\mathbb{B}_f(d) := \left\{ \mathbf{x} \in \mathbb{N}^E : x(U) \leq f(U) \text{ for all } U \subseteq E, x(E) = d \right\}.$$

The *rank* of $\mathbb{B}_f(d)$ is given by d. We consider the problem of minimizing a parametrized separable discrete convex function over a polymatroid base polytope:

$$(P(\mathbf{t}, d)) \qquad \begin{aligned} &\text{minimize } \sum_{e \in E} C_e(x_e; t_e) \\ &\text{subject to } \mathbf{x} \in \mathbb{B}_f(d), \end{aligned}$$

where $\mathbf{t} = (t_e)_{e \in E} \in \mathbb{N}^E, d \in \mathbb{N}$ are nonnegative integral parameters.

Note that $\mathbf{t}$ influences the cost function, while the parameter d defines the truncation of the integral polymatroid base polytope. Let $\mathbf{x}^*(\mathbf{t}, d) \in \mathbb{N}^E$ be an optimal solution to $P(\mathbf{t}, d)$. We are interested in quantifying the distance between $\mathbf{x}^*(\mathbf{t}, d)$ and a new optimal solution $\mathbf{x}^*(\mathbf{t}', d')$ for the new parameters $\mathbf{t}' \in \mathbb{N}$ and $d' \in \mathbb{N}$. We will measure the distance between solutions using the L_1-norm defined as $\|\mathbf{x}\|_1 := \sum_{e \in E} |x_e|$ for all $\mathbf{x} \in \mathbb{N}^E$. Thus, we are interested in bounding $\|\mathbf{x}^*(\mathbf{t}, d) - \mathbf{x}^*(\mathbf{t}', d')\|_1$ in terms of $\|\mathbf{t} - \mathbf{t}'\|_1$ and $|d - d'|$. Naturally, nontrivial bounds are only possible when imposing additional assumptions on the dependence of the cost function on the parameter.

To state these assumptions formally, we need the following notation of discrete derivatives. For a function $c : \mathbb{N} \to \mathbb{R}_+$ and $x \in \mathbb{N}$, let

$$c^-(x) := c(x) - c(x-1) \qquad \text{and} \qquad c^+(x) := c(x+1) - c(x)$$

denote the left and right discrete derivative, respectively. Note that the left derivative is only defined for $x \geq 1$. The function c is called *discrete convex* if $c^-(x) \leq c^+(x)$ for all $x \in \mathbb{N}$. For a function $C : \mathbb{N} \times \mathbb{N} \to \mathbb{R}_+$, we slightly abuse notation, as we denote by

$$C^-(x; t) := C(x; t) - C(x-1; t) \qquad \text{and} \qquad C^+(x; t) := C(x+1; t) - C(x; t)$$

the left and right derivatives, respectively, with respect to the first argument. We call C discrete convex if $x \mapsto C(x; t)$ is discrete convex for all $t \in \mathbb{N}$.

We introduce the following notion of regularity that bounds the impact of a parameter change on the derivative.

DEFINITION 2.1 (Regularity). *A function* $C : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ *is called* regular *if*

$$C^-(x;t) \leq C^-(x;t+1) \text{ for all } x,t \in \mathbb{N} \tag{2.1}$$

$$C^-(x;t+1) \leq C^-(x+1;t) \text{ for all } x,t \in \mathbb{N}. \tag{2.2}$$

That is, (2.1) requires that the (left) marginal cost function of C is nondecreasing in t and (2.2) bounds the marginal cost of C after adding one unit to parameter t in terms of the marginal cost after adding one unit to x. It can be shown that a regular function C is discrete convex.

Observation 1. A regular function $C : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is discrete convex.

Proof. We calculate

$$C^-(x;t) \leq C^-(x;t+1) \leq C^-(x+1;t) = C^+(x;t),$$

where for the first and second inequality, we used (2.1) and (2.2), respectively. ∎

Throughout this work, we impose the following assumption on $C_e, e \in E$.

Assumption 1. For every $e \in E$, C_e is regular.

If $P(\mathbf{t}, d)$ only involves cost functions satisfying Assumption 1, we speak of a convex and regular optimization problem.

We recapitulate the motivating example given in section 1.1 and state it as a convex and regular optimization problem.

Example 1. Let $G = (V, E)$ be a connected undirected graph. The objective is to compute $k \in \mathbb{N}$ spanning trees of G with minimum cost so that along each spanning tree, a message of unit size can be sent. If $x_e \in \mathbb{N}$ messages are sent along edge e, the resulting cost per edge is defined as

$$C_e(x_e; u_e) = \begin{cases} \frac{1}{u_e - x_e} & \text{if } x_e < u_e, \\ +\infty & \text{else.} \end{cases}$$

By defining $u = \max_{e \in E} u_e$ and $t_e = u - t_e$ for all $e \in E$, we obtain an equivalent problem in which the cost functions are of the form $C_e(x_e; t_e) = 1/(u - t_e - x_e)$. This problem involves regular cost functions because $\partial C_e(x_e; t_e)/\partial x_e = 1/(u - t_e - x_e)^2$ is increasing in t_e; thus, (2.1) is satisfied. Moreover, one easily verifies that also (2.2) is satisfied.

In the above example, there is an additive dependence between the variable x_e and the parameter t_e. We will now discuss the case of multiplicative dependences.

LEMMA 2.2. *Let* $C(x;t) = g(t) \cdot h(x)$ *for functions* $g : \mathbb{N} \to \mathbb{R}$ *and* $h : \mathbb{N} \to \mathbb{R}$ *with* $h^- \not\equiv 0, h^- \geq 0$ *and* $g \not\equiv 0$. *Then* C *is regular if and only if the following properties hold:*

1. g *is nondecreasing.*
2. h *is discrete convex.*
3. $g(t+1) \cdot h^-(x) \leq g(t) \cdot h^-(x+1)$ *for all* $x, t \in \mathbb{N}$.

Proof. ⇒: By Property (2.1) of regularity, we obtain

$$g(t) \cdot h^-(x) \leq g(t+1) \cdot h^-(x) \text{ for all } x, t \in \mathbb{N}.$$

Since $h^- \not\equiv 0, h^- \geq 0$, there is $x_0 \in \mathbb{N}$ with $h^-(x_0) > 0$. Hence, dividing

$$g(t) \cdot h^-(x_0) \leq g(t+1) \cdot h^-(x_0) \text{ for all } t \in \mathbb{N}$$

by $h^-(x_0)$ yields $g(t) \leq g(t+1)$ for all $t \in \mathbb{N}$. By combining Property (2.1) and Property (2.2) (similarly as in Observation 1), we get

$$g(t) \cdot h^-(x) \leq g(t+1) \cdot h^-(x) \leq g(t) \cdot h^-(x+1) \text{ for all } x, t \in \mathbb{N}.$$

Choosing $t_0 \in \mathbb{N}$ with $g(t_0) \neq 0$ and dividing by $g(t_0)$ implies that h is discrete convex. Finally, Property (2.2) just says

$$g(t+1) \cdot h^-(x) \leq g(t) \cdot h^-(x+1) \text{ for all } x, t \in \mathbb{N},$$

which is stated in 3.

$\Leftarrow$: We get $g(t) \cdot h^-(x) \leq g(t+1) \cdot h^-(x)$ for all $t, x \in \mathbb{N}$ by the monotonicity of g; thus, Property (2.1) follows. Property (2.2) follows directly from (3). ∎

We give an example to illustrate the applicability of this characterization.

Example 2. We discuss the regularity of two particular cost functions:

1. The function $C(x;t) = (t+1) \cdot e^x$ is regular.
2. The function $C(x;t) = t \cdot x$ is not regular.

To see the first statement, observe that $g(t) = t+1$ is nondecreasing and $h(x) = e^x$ is discrete convex. Moreover, condition 3 of Lemma 2.2 translates to

$$\begin{aligned} & (t+2) \cdot (e^x - e^{x-1}) \leq (t+1) \cdot (e^{x+1} - e^x) && \text{for all } x, t \in \mathbb{N} \\ \Leftrightarrow\quad & (t+2) \cdot (e^x - e^{x-1}) \leq (t+1) \cdot e \cdot (e^x - e^{x-1}) && \text{for all } x, t \in \mathbb{N} \\ \Leftrightarrow\quad & t+2 \leq (t+1) \cdot e && \text{for all } t \in \mathbb{N}, \end{aligned}$$

which is a true statement. For the second statement, we get with $g(t) = t$ and $h(x) = x$ as a necessary condition

$$t \cdot x - t \cdot (x-1) \underset{by\ (2.1)}{\leq} (t+1) \cdot x - (t+1) \cdot (x-1) \underset{by\ (2.2)}{\leq} t \cdot (x+1) - t \cdot x$$

for all $x, t \in \mathbb{N}$. This implies $t \leq (t+1) \leq t$, a contradiction.

3. Sensitivity results.

For fixed parameters $\mathbf{t} \in \mathbb{N}^E, d \in \mathbb{N}$, we recapitulate the following necessary and sufficient optimality conditions for an optimal solution to problem $P(\mathbf{t}, d)$. Let $\chi_e \in \mathbb{N}^E$ be the indicator vector with all-zero entries except for the eth coordinate which is 1. For $\mathbf{x} \in \mathbb{B}_f(d)$ and $e \in E$, denote by

$$D_e(\mathbf{x}) = \{g \in E \setminus \{e\} \mid \mathbf{x} + \chi_g - \chi_e \in \mathbb{B}_f(d)\}$$

the set of *feasible local exchanges* w.r.t. $\mathbf{x}$ and e and by

$$\Delta_e(\mathbf{x}; \mathbf{t}) = \begin{cases} \min_{g \in D_e(\mathbf{x})} C_g^+(x_g; t_g) & \text{if } D_e(\mathbf{x}) \neq \emptyset, \\ +\infty & \text{else,} \end{cases}$$

the minimum alternative cost when exchanging e. The following theorem gives a necessary and sufficient condition for the optimality of a solution of a polymatroid optimization problem.

THEOREM 3.1 (see Fujishige [10]). *$\mathbf{x}^* \in \mathbb{B}_f(d)$ is an optimal solution for $P(\mathbf{t}, d)$ if and only if $C_e^-(x_e^*; t_e) \leq \Delta_e(\mathbf{x}^*; \mathbf{t})$ for all $e \in E$.*

Using these conditions, we proceed to establish the first sensitivity result, which relates optimal solutions for changed values of $\mathbf{t}$.

THEOREM 3.2. *Let $P(\mathbf{t}, d)$ be a regular convex optimization problem with optimal solution $\mathbf{x}^*(\mathbf{t}, d)$, and let $\mathbf{t}' = \mathbf{t} + \chi_{e^*}$ for some $e^* \in E$. Let*

$$g^* \in \begin{cases} \arg\min_{g \in D_{e^*}(\mathbf{x}^*(\mathbf{t},d))} \{C_g^+(x_g; t_g)\} & \text{if } D_{e^*}(\mathbf{x}^*(\mathbf{t}, d)) \neq \emptyset, \\ \{e^*\} & \text{else.} \end{cases}$$

Then the better of the two solutions, $\mathbf{x}^*(\mathbf{t}, d)$ and $\mathbf{x}^*(\mathbf{t}, d) - \chi_{e^*} + \chi_{g^*}$, is optimal for $P(\mathbf{t}', d)$.

Proof. When $D_{e^*}(\mathbf{x}^*(\mathbf{t}, d)) = \emptyset$, then $\mathbf{x}^*(\mathbf{t}, d)$ trivially satisfies the optimality conditions of Theorem 3.1 for any parameter vector $\mathbf{t}'$, and there is nothing left to show. Thus, let us assume $D_{e^*}(\mathbf{x}) \neq \emptyset$. Let $\mathbf{x} = \mathbf{x}^*(\mathbf{t}, d)$ and $\mathbf{y} = \mathbf{x} - \chi_{e^*} + \chi_{g^*} \neq \mathbf{x}$.

First, consider the case $C_{e^*}^-(x_{e^*}; t_{e^*} + 1) \leq \Delta_{e^*}(\mathbf{x}; \mathbf{t})$. We claim that $\mathbf{x}$ is still an optimal solution to $P(\mathbf{t}', d)$, as it satisfies the optimality conditions

$$C_e^-(x_e; t'_e) \leq \Delta_e(\mathbf{x}; \mathbf{t}') \quad \text{for all } e \in E. \tag{3.1}$$

To see this, note that by Assumption 1 (using (2.1)), we get $\Delta_e(\mathbf{x}; \mathbf{t}') \geq \Delta_e(\mathbf{x}; \mathbf{t})$ for all $e \in E$ as $t'_e \geq t_e$. This directly implies that (3.1) is satisfied for all $e \neq e^*$. To see the inequality also for e^*, observe that

$$C_{e^*}^-(x_{e^*}; t'_{e^*}) = C_{e^*}^-(x_{e^*}; t_{e^*} + 1) \leq \Delta_{e^*}(\mathbf{x}; \mathbf{t}) = \Delta_{e^*}(\mathbf{x}; \mathbf{t}'),$$

where for the last equality we used that $C_g^+(x_g; t_g) = C_g^+(x_g; t'_g)$ for all $g \neq e^*$.

Second, consider the case $C_{e^*}^-(x_{e^*}; t_{e^*} + 1) > \Delta_{e^*}(\mathbf{x}; \mathbf{t})$. We proceed to show that $\mathbf{y}$ is then optimal for $P(\mathbf{t}', d)$ by checking the optimality conditions of Theorem 3.1 for all $e \in E = \{e^*\} \cup \{g^*\} \cup E \setminus \{e^*, g^*\}$.

Case (A): $e = e^*$. We obtain

$$C_{e^*}^-(y_{e^*}; t_{e^*} + 1) = C_{e^*}^-(x_{e^*} - 1; t_{e^*} + 1) \leq C_{e^*}^-(x_{e^*}; t_{e^*}) \leq \Delta_{e^*}(\mathbf{x}; \mathbf{t}),$$

where the first inequality follows by the regularity of C_{e^*} and the second since $\mathbf{x}$ is optimal for $P(\mathbf{t}, d)$. Thus, the optimality conditions for e^* are satisfied if $\Delta_{e^*}(\mathbf{x}; \mathbf{t}) \leq \Delta_{e^*}(\mathbf{y}; \mathbf{t}')$. To prove the latter inequality, we first show that $D_{e^*}(\mathbf{y}) \subseteq D_{e^*}(\mathbf{x})$. Assume by contradiction that there is $g \in D_{e^*}(\mathbf{y}) \setminus D_{e^*}(\mathbf{x})$. This implies $\mathbf{x} - \chi_{e^*} + \chi_g \notin \mathbb{B}_f(d)$. Hence, there must exist $T \subset E$ with $g \in T, e^* \notin T$, and $f(T) = x(T)$. On the other hand, $g \in D_{e^*}(\mathbf{y})$ implies $\mathbf{y}' := \mathbf{y} - \chi_{e^*} + \chi_g = \mathbf{x} - 2\chi_{e^*} + \chi_g + \chi_{g^*} \in \mathbb{B}_f(d)$. Using $g \in T, e^* \notin T, x(T) = f(T)$, this implies $y'(T) \geq x(T) + 1 = f(T) + 1$; hence, $\mathbf{y}' \notin \mathbb{B}_f(d)$, a contradiction.

Finally, let us show $\Delta_{e^*}(\mathbf{x}; \mathbf{t}) \leq \Delta_{e^*}(\mathbf{y}; \mathbf{t}')$. This is trivial if $D_{e^*}(\mathbf{y}) = \emptyset$. Otherwise, we obtain

$$\Delta_{e^*}(\mathbf{x}; \mathbf{t}) = \min_{g \in D_{e^*}(\mathbf{x})} \{C_g^+(x_g; t_g)\} = C_{g^*}^+(x_{g^*}; t_{g^*}) \leq C_{g^*}^+(y_{g^*}; t_{g^*}) = C_{g^*}^+(y_{g^*}; t'_{g^*}),$$

where for the inequality we used discrete convexity. Moreover, we obtain for all $g' \in D_{e^*}(\mathbf{x}) \setminus \{g^*\}$ the inequality

$$\Delta_{e^*}(\mathbf{x}; \mathbf{t}) = \min_{g \in D_{e^*}(\mathbf{x})} \{C_g^+(x_g; t_g)\} \leq C_{g'}^+(x_{g'}; t_{g'}) = C_{g'}^+(y_{g'}; t'_{g'}).$$

Putting things together, we get

$$\Delta_{e^*}(\mathbf{x}; \mathbf{t}) \leq \min_{g \in D_{e^*}(\mathbf{x})} C_g^+(y_g; t'_g) \leq \min_{g \in D_{e^*}(\mathbf{y})} C_g^+(y_g; t'_g) \leq \Delta_{e^*}(\mathbf{y}; \mathbf{t}').$$

Case (B): $e = g^*$. For a contradiction, assume that there is $g \in D_{g^*}(\mathbf{y})$ with

$$C_{g^*}^-(y_{g^*}; t'_{g^*}) > C_g^+(y_g; t'_g). \tag{3.2}$$

Note that $g \neq e^*$ since

$$C^-_{g^*}(y_{g^*}; t'_{g^*}) = C^-_{g^*}(y_{g^*}; t_{g^*}) = C^+_{g^*}(x_{g^*}; t_{g^*}) < C^-_{e^*}(x_{e^*}; t_{e^*}+1) = C^+_{e^*}(y_{e^*}; t_{e^*}+1) = C^+_{e^*}(y_{e^*}; t'_{e^*}),$$

a contradiction to (3.2). Thus, $g \in D_{g^*}(\mathbf{y}) \setminus \{e^*\}$. We obtain

$$\mathbf{y} - \chi_{g^*} + \chi_g = \mathbf{x} - \chi_{e^*} + \chi_g \in \mathbb{B}_f(d),$$

which implies $g \in D_{e^*}(\mathbf{x})$. Since g^* minimizes $C^+_g(\mathbf{x}; \mathbf{t})$ among all $g \in D_{e^*}(\mathbf{x})$, we get $C^-_{g^*}(y_{g^*}; t'_{g^*}) = C^-_{g^*}(y_{g^*}; t_{g^*}) = C^+_{g^*}(x_{g^*}; t_{g^*}) \leq C^+_g(x_g; t_g) = C^+_g(y_g; t'_g)$, contradicting (3.2).

Case (C): $e \in E \setminus \{e^*, g^*\}$. Assume by contradiction that there is $g \in D_e(\mathbf{y})$ with

$$C^-_e(y_e; t'_e) > C^+_g(y_g; t'_g). \tag{3.3}$$

We first treat the case $g = e^*$, where (3.3) becomes

$$C^-_e(y_e; t_e) > C^+_{e^*}(y_{e^*}; t_{e^*}+1). \tag{3.4}$$

With $e^* \in D_e(\mathbf{y})$, we get $\mathbf{y} - \chi_e + \chi_{e^*} = \mathbf{x} - \chi_e + \chi_{g^*} \in \mathbb{B}_f(d)$; hence, $g^* \in D_e(\mathbf{x})$. We get

(using (3.4))	$C^+_{e^*}(y_{e^*}; t_{e^*}+1) < C^-_e(y_e; t_e)$
(using $e \in E \setminus \{e^*, g^*\}$)	$= C^-_e(x_e; t_e)$
(since $\mathbf{x}$ was optimal for $\mathbf{t}$)	$\leq C^+_{g^*}(x_{g^*}; t_{g^*})$
(since $\mathbf{x}$ was not optimal for $\mathbf{t}'$)	$< C^-_{e^*}(x_{e^*}; t_{e^*}+1)$
(since $\mathbf{y} = \mathbf{x} - \chi_{e^*} + \chi_{g^*}$)	$= C^+_{e^*}(y_{e^*}; t_{e^*}+1)$,

a contradiction.

From now on, we may assume $g \in E \setminus \{e^*\}$. We claim that the following two properties are satisfied:

(a) $\mathbf{x} - \chi_e + \chi_{g^*} \in \mathbb{B}_f(d)$;
(b) $\mathbf{x} - \chi_{e^*} + \chi_g \in \mathbb{B}_f(d)$.

This claim has also been used in the proof of Theorem 4.11 in Fujishige [10]. In order to keep our analysis self-contained, we provide an alternative proof of fact (a) and (b) below.

Before proving these properties, however, we show that they give the desired contradiction to (3.3). First note that (a) implies $g^* \in D_e(\mathbf{x})$, which together with the optimality of $\mathbf{x}$ for $\mathbf{t}$ implies $C^-_e(y_e; t'_e) = C^-_e(x_e; t_e) \leq C^+_{g^*}(x_{g^*}; t_{g^*})$. Similarly, (b) implies $g \in D_{e^*}(\mathbf{x})$, which by the choice of g^* implies $C^+_{g^*}(x_{g^*}; t_{g^*}) \leq C^+_g(x_g; t_g)$. Finally, we have $C^+_g(x_g; t_g) \leq C^+_g(y_g; t_g) = C^+_g(y_g; t'_g)$ by discrete convexity and the fact that $y_g \geq x_g$ for all $g \in E \setminus \{e^*\}$. Combining all three inequalities, we obtain $C^-_e(y_e; t'_e) \leq C^+_g(y_g; t'_g)$, a contradiction to (3.3).

It remains to prove properties (a) and (b). Denote

$$\mathbf{y}' := \mathbf{y} - \chi_e + \chi_g = \mathbf{x} + \chi_{g^*} - \chi_{e^*} - \chi_e + \chi_g \in \mathbb{B}_f(d).$$

We will first show that $\mathbf{x} - \chi_e + \chi_g \notin \mathbb{B}_f(d)$. Assume by contradiction that $\mathbf{x} - \chi_e + \chi_g \in \mathbb{B}_f(d)$. We then obtain

$$C^-_e(x_e; t_e) = C^-_e(y_e; t'_e) > C^+_g(y_g; t'_g) = C^+_g(y_g; t_g) \geq C^+_g(x_g; t_g), \tag{3.5}$$

where for the first identity we used $e \in E \setminus \{e^*, g^*\}$, the first inequality used (3.3), the second identity used $g \neq e^*$, and the last inequality used $g \neq e^*$, discrete convexity and $y_g \geq x_g$ for all $g \in E \setminus \{e^*\}$. This implies that $\mathbf{x}$ was not optimal for $P(\mathbf{t}, d)$, a contradiction.

We conclude that $\mathbf{x} - \chi_e + \chi_g \notin \mathbb{B}_f(d)$, but $\mathbf{y}' = \mathbf{x} - \chi_e + \chi_g - \chi_{g^*} + \chi_{e^*} \in \mathbb{B}_f(d)$. Thus, there exists some set $S \subset E$ with $x(S) = f(S)$ and

$$\{g, e^*\} \subseteq S \quad \text{and} \quad \{e, g^*\} \cap S = \emptyset. \tag{3.6}$$

It follows that $x(S) = y'(S) = f(S)$.

We proceed to show (a). For the sake of a contradiction, suppose that $\mathbf{x} - \chi_e + \chi_{g^*} \notin \mathbb{B}_f(d)$. Using $g \in D_e(\mathbf{y})$ and, thus, $x_e = y_e \geq 1$, this implies the existence of a set $T \subset E$ with $x(T) = f(T)$, $g^* \in T$, and $e \notin T$. Since $\mathbf{y} = \mathbf{x} - \chi_{e^*} + \chi_{g^*} \in \mathbb{B}_f(d)$, it follows that $e^* \in T$. Moreover, since $\mathbf{y}' = \mathbf{y} - \chi_e + \chi_g \in \mathbb{B}_f(d)$ and $e \notin T$, we have $g \notin T$. Hence, $y'(T) = x(T) = f(T)$. By submodularity of f, the identities $x(S) = y'(S) = f(S)$ and $x(T) = y'(T) = f(T)$ imply $x(S \cap T) = y'(S \cap T) = f(S \cap T)$ and $x(S \cup T) = y'(S \cup T) = f(S \cup T)$. Together with $e^* \in S \cap T$ and $\{e, g^*, g\} \cap (S \cap T) = \emptyset$, we arrive at the desired contradiction

$$f(S \cap T) = x(S \cap T) = y'(S \cap T) - 1 = f(S \cap T) - 1. \tag{3.7}$$

Thus, $x - \chi_e + \chi_{g^*} \in \mathbb{B}_f(d)$, as claimed.

In a similar way, we show (b). For the sake of a contradiction, suppose $\mathbf{x} - \chi_{e^*} + \chi_g \notin \mathbb{B}_f(d)$. Since $D_{e^*}(\mathbf{x}) \neq \emptyset$ and, thus, $x_{e^*} \geq 1$, this implies the existence of a set $U \subset E$ with $x(U) = f(U)$, $g \in U$, and $e^* \notin U$. Since $\mathbf{y}' = x - \chi_{e^*} + \chi_{g^*} - \chi_e + \chi_g \in \mathbb{B}_f(d)$, we further have $e \in U$ and $g^* \notin U$, implying $y'(U) = f(U)$ and $g \in S \cap U$. Just as for (a), $x(S) = y'(S) = f(S)$ and $x(U) = y'(U) = f(U)$ leads to the desired contradiction

$$f(U \cap S) = x(U \cap S) = y'(U \cap S) - 1 = f(U \cap S) - 1.$$

We have treated all cases, and the theorem follows. ∎

In a similar manner, it can be shown that at most a single local improvement step suffices to obtain a new optimal solution for any parameter shift of type $\mathbf{t}' = \mathbf{t} - \chi_{e^*}$. We conclude with the following corollary.

COROLLARY 3.3. *Let $P(\mathbf{t}, d)$ be a regular convex optimization problem. Then, for every optimal solution $\mathbf{x}^*(\mathbf{t}, d)$ to $P(\mathbf{t}, d)$ and every $\mathbf{t}'$ with $\|\mathbf{t} - \mathbf{t}'\|_1 = 1$, there is an optimal solution $\mathbf{x}^*(\mathbf{t}', d)$ for $P(\mathbf{t}', d)$ satisfying*

$$\|\mathbf{x}^*(\mathbf{t}, d) - \mathbf{x}^*(\mathbf{t}', d)\|_1 \leq 2. \tag{3.8}$$

We now turn to the impact of a change of the parameter d on optimal solutions of $P(\mathbf{t}, d)$.

THEOREM 3.4. *Let $D \subset \mathbb{N}$ denote the set of possible integral values of d. Let $P(\mathbf{t}, d)$ be a family of regular optimization problems with $\mathbb{B}_f(d) \neq \emptyset$ for all $d \in D$.*

Then, for every $d, d' \in D$ with $|d - d'| = 1$ and every optimal solution $\mathbf{x}^*(\mathbf{t}, d)$ to $P(\mathbf{t}, d)$, there is an optimal solution $\mathbf{x}^*(\mathbf{t}, d')$ for $P(\mathbf{t}, d')$ with

$$\|\mathbf{x}^*(\mathbf{t}, d) - \mathbf{x}^*(\mathbf{t}, d')\|_1 \leq |d - d'| = 1. \tag{3.9}$$

Proof. We here only prove the case $d' = d+1$; the case $d' = d-1$ follows similarly. Define the set of resources with *slack* with respect to $\mathbf{x} := \mathbf{x}^*(\mathbf{t}, d)$ as

$$\mathcal{S}(\mathbf{x}) := \{e \in E : \mathbf{x} + \chi_e \in \mathbb{B}_f(d+1)\}.$$

Since by assumption $d + 1 \in D$ and thus $\mathbb{B}_f(d+1) \neq \emptyset$, we obtain $\mathcal{S}(\mathbf{x}) \neq \emptyset$ (cf. Fujishige [10, Theorem 2.3, p. 35]).

We claim that the solution $\mathbf{y} := \mathbf{x} + \chi_{g^*}$ is optimal for problem $P(\mathbf{t}, d')$, where

$$g^* \in \arg\min_{g \in \mathcal{S}(\mathbf{x})} C_g^+(x_g; t_g).$$

To prove the claim, we show that the optimality conditions of Theorem 3.1 are satisfied, i.e., $C_e^-(y_e; t_e) \leq \Delta_e(\mathbf{y}; \mathbf{t})$ for all $e \in E$. This is trivial if $D_e(\mathbf{y}) = \emptyset$. Otherwise, we distinguish two cases. If $g^* \in \arg\min_{g \in D_e(\mathbf{y})}\{C_g^+(y_g; t_g)\}$, we obtain

$$\Delta_e(\mathbf{y}; \mathbf{t}) = C_{g^*}^+(y_{g^*}; t_{g^*}) \geq C_{g^*}^+(x_{g^*}; t_{g^*}) \geq C_e^-(x_e; t_e) = C_e^-(y_e; t_e),$$

where for the first inequality we used discrete convexity and for the second inequality we used the optimality of $\mathbf{x}$ for $\mathbb{B}_f(d)$. If $g^* \notin \arg\min_{g \in D_e(\mathbf{y})}\{C_g^+(y_g; t_g)\}$, we obtain

$$\begin{aligned}\Delta_e(\mathbf{y}; \mathbf{t}) &= \min_{g \in D_e(\mathbf{y})}\{C_g^+(y_g; t_g)\} \geq \min_{g \in D_e(\mathbf{x})}\{C_g^+(y_g; t_g)\} = \min_{g \in D_e(\mathbf{x})}\{C_g^+(x_g; t_g)\} \\ &\geq C_e^-(x_e; t_e) = C_e^-(y_e; t_e),\end{aligned}$$

where for the first inequality we used $D_e(\mathbf{x}) \supseteq D_e(\mathbf{y})$ for all $e \in E \setminus \{g^*\}$ and for the second inequality we used the optimality of $\mathbf{x}$ for $\mathbb{B}_f(d)$. ∎

By inductively applying Theorems 3.2 and 3.4, we arrive at the following result.

THEOREM 3.5. *Let $D \subset \mathbb{N}$ denote the set of possible integral values of d, and let $P(\mathbf{t}, d), d \in D$ be a set of regular optimization problem with $\mathbb{B}_f(d) \neq \emptyset$ for all $d \in D$. Then, for every optimal solution $\mathbf{x}^*(\mathbf{t}, d)$ of $P(\mathbf{t}, d)$, $d \in D$, every $d' \in D$, and every $\mathbf{t}'$, there is an optimal solution $\mathbf{x}^*(\mathbf{t}', d')$ of $P(\mathbf{t}', d')$ with*

$$\|\mathbf{x}^*(\mathbf{t}, d) - \mathbf{x}^*(\mathbf{t}', d')\|_1 \leq 2\|\mathbf{t} - \mathbf{t}'\|_1 + |d - d'|. \tag{3.10}$$

Note that for the proof, we can safely use induction on d also for those d's with $d_1, d_2 \in D, d_1 \leq d \leq d_2, d \notin D$, as $\mathbb{B}_f(d_i) \neq \emptyset, i = 1, 2$ implies $\mathbb{B}_f(d) \neq \emptyset$. Moreover, regularity of $P(\mathbf{t}, d)$ does not depend on d.

The proofs of Theorems 3.2 and 3.4 also show that after a parameter change, a new optimal solution can be recovered by elementary exchange steps, that is, by iteratively shifting one unit from one element to another or by adding one unit to an element. We can summarize this discussion as follows.

COROLLARY 3.6. *Let $D \subset \mathbb{N}$ denote the set of possible integral values of d. Let $P(\mathbf{t}, d), d \in D$ be a set of regular optimization problems with $\mathbb{B}_f(d) \neq \emptyset$ for all $d \in D$. Then, for every optimal solution $\mathbf{x}^*(\mathbf{t}, d)$ of $P(\mathbf{t}, d)$, $d \in D$, every $d' \in D$, and every $\mathbf{t}'$, there is an optimal solution $\mathbf{x}^*(\mathbf{t}', d')$ for $P(\mathbf{t}', d')$ that can be computed from $\mathbf{x}^*(\mathbf{t}, d)$ by performing at most $\|\mathbf{t} - \mathbf{t}'\|_1 + |d - d'|$ elementary exchange steps.*

In the following section, we apply the above sensitivity results to a quite general class of noncooperative games, thereby establishing an existence and computability result of pure Nash equilibria.

4. Noncooperative games on polymatroids. We consider the following class of games. There is a finite set $N = \{1, \dots, n\}$ of players and a finite set $E = \{1, \dots, m\}$ of elements. As it is standard in the congestion game literature, in this section we refer to the elements $e \in E$ as *resources*. Each player i is associated with a demand $d_i \in \mathbb{N}$ and an integral polymatroid rank function $f_i : 2^E \to \mathbb{N}$ that together define a d_i-truncated integral polymatroid $\mathbb{P}_{f_i}(d_i)$ with base polytope $\mathbb{B}_{f_i}(d_i)$. A strategy of player $i \in N$ is to choose a vector $\mathbf{x}_i = (x_{i,e})_{e \in E} \in \mathbb{B}_{f_i}(d_i)$; i.e., player i chooses an integral resource consumption $x_{i,e} \in \mathbb{N}$ for each resource e such that the demand d_i is exactly distributed among the resources and for each $U \subseteq E$ not more than $f_i(U)$ units of demand are distributed to the resources contained in U. Using the notation $\mathbf{x}_i = (x_{i,e})_{e \in E}$, the set X_i of feasible strategies of player i is defined as

$$X_i = \mathbb{B}_{f_i}(d_i) = \left\{ \mathbf{x}_i \in \mathbb{N}^E : x_i(U) \le f_i(U) \text{ for all } U \subseteq E, x_i(E) = d_i \right\},$$

where for a set $U \subseteq E$, we write $x_i(U)$ as shorthand for $\sum_{e \in U} x_{i,e}$. The Cartesian product $X = X_1 \times X_2 \times \cdots \times X_n$ of the players' sets of feasible strategies is the joint strategy space. An element $\mathbf{x} = (\mathbf{x}_i)_{i \in N} \in X$ is a strategy profile. For a resource e and a strategy profile $\mathbf{x} \in X$, we write $x_e = \sum_{i \in N} x_{i,e}$ and $x_{-i,e} = \sum_{j \in N \setminus \{i\}} x_{j,e}$. The private cost of player i under strategy profile $\mathbf{x} \in X$ is defined as

$$\pi_i(\mathbf{x}) = \sum_{e \in E} C_{i,e}(x_{i,e}; x_{-i,e}).$$

We assume that every $C_{i,e}$ fulfills the conditions of Assumption 1. In the remainder of the paper, we will compactly represent the strategic game by the tuple $G = (N, X, (d_i)_{i \in N}, (C_{i,e})_{i \in N, e \in E})$. We use standard game theory notation. For a player $i \in N$ and a strategy profile $\mathbf{x} \in X$, we write $\mathbf{x}$ as $(\mathbf{x}_i, \mathbf{x}_{-i})$. A *best response* of player i to $\mathbf{x}_{-i}$ is a strategy $\mathbf{x}_i \in X_i$ with $\pi_i(\mathbf{x}_i, \mathbf{x}_{-i}) \le \pi_i(\mathbf{y}_i, \mathbf{x}_{-i})$ for all $\mathbf{y}_i \in X_i$. A pure Nash equilibrium is a strategy profile $\mathbf{x} \in X$ such that for each player i, the strategy $\mathbf{x}_i$ is a best response to $\mathbf{x}_{-i}$.

4.1. Notable special cases. We proceed to illustrate that we obtain the well-known classes of integer-splittable singleton congestion games and matroid congestion games as special cases of congestion games on integer polymatroids.

4.1.1. Singleton integer-splittable congestion games. Tran-Thanh et al. [38] consider singleton integer-splittable congestion games where each player i is associated with an integral demand $d_i \in \mathbb{N}$ that needs to be distributed integrally over a player-specific subset $E_i \subseteq E$ of resources. Every resource has a nondecreasing and convex cost function $c_e : \mathbb{N} \to \mathbb{R}_+$, and the private cost of a player i is equal to

$$\pi_i(\mathbf{x}) = \sum_{e \in E} c_e(x_{i,e} + x_{-i,e}) x_{i,e}.$$

We proceed to show that this class of games is contained in the class of polymatroid games as a special case.

PROPOSITION 4.1. *Singleton integer-splittable congestion games are polymatroid games.*

Proof. For $i \in N$ and $U \subseteq E$, we let

$$f_i(U) = \begin{cases} d_i & \text{if } U \cap E_i \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

For $i \in N$ and $e \in E$, we let $C_{i,e}(x_{i,e}; x_{-i,e}) = c_e(x_{i,e} + x_{-i,e})x_{i,e}$. First, we show that the functions f_i are normalized, monotone, and submodular. For submodularity, it suffices to show that $f(U \cup \{v\}) - f(U) \geq f(V \cup \{v\}) - f(V)$ for all $U \subseteq V$ and $v \notin U$. The inequality can only be violated if $f(V \cup \{v\}) - f(V) = d_i$, which implies $v \in E_i$ and $V \cap E_i = \emptyset$. This, however, implies $U \cap E_i = \emptyset$ and, thus, $f(U \cup \{v\}) - f(U) = d_i$.

We proceed to show that the cost functions are regular provided that c_e is nondecreasing and convex. Discrete convexity is easy to verify. For regularity, we compute for arbitrary $i \in N$ and $e \in E$

$$\begin{aligned} C^-_{i,e}(x; t+1) &= c_e(x+t+1)x - c_e(x+t)(x-1) \\ &\leq c_e(x+t+1)(x+1) - c_e(x+t)x \\ &= C^-_{i,e}(x+1; t), \end{aligned}$$

where the inequality follows since c is nondecreasing. ∎

4.1.2. Matroid congestion games with player-specific costs. Ackermann, Röglin, and Vöcking [1] studied matroid congestion games with player-specific costs, where each player i is associated with a *matroid* $M_i = (E_i, \mathcal{I}_i,)$ defined on some player-specific subset $E_i \subseteq E$. The strategy space for every $i \in N$ is equal to the set $\mathcal{B}_i$ of bases of M_i. Given a strategy profile $(B_1, \ldots, B_n)$ with $B_j \in \mathcal{B}_j$ for all players j, the private cost of player i is defined as

$$\pi_i(B_1, \ldots, B_n) = \sum_{e \in B_i} c_{i,e}(|\{j \in N : e \in B_j\}|),$$

where the functions $c_{i,e} : \mathbb{N} \to \mathbb{R}_+$ with $i \in N$ and $e \in E$ are nondecreasing. We proceed to show that this class of games is contained in the class of polymatroid games as a special case.

PROPOSITION 4.2. *Matroid congestion games are polymatroid games.*

Proof. For a player i, we associate with each basis $B_i \in \mathcal{B}_i$ its characteristic vector $\mathbf{x}_i(B_i) = (x_{i,e}(B_i))_{e \in E}$ defined as

$$x_{i,e}(B_i) = \begin{cases} 1 & \text{if } e \in B_i \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that for each matroid $M_i = (E_i, \mathcal{I}_i)$, there is a function $\mathrm{rk} : E_i \to \mathbb{N}$, called the rank function of the matroid, such that

$$\{\mathbf{x}_i(B_i) : B_i \in \mathcal{B}_i\} = \{\mathbf{x}_i \in \mathbb{N}^{E_i} : x_i(E_i) = \mathrm{rk}(E_i) \text{ and } x_i(U) \leq \mathrm{rk}(U) \text{ for all } U \subseteq E_i\}.$$

Moreover, the rank function is normalized, monotone, and submodular. We let $f_i(U) = \mathrm{rk}(U \cap E_i)$ and let $d_i = \mathrm{rk}(E_i)$. Then f_i is submodular since for all $U, V \in 2^E$, we have

$$\begin{aligned} f_i(U) + f_i(V) &= \mathrm{rk}(U \cap E_i) + \mathrm{rk}(V \cap E_i) \\ &\geq \mathrm{rk}((U \cap E_i) \cup (V \cap E_i)) + \mathrm{rk}(U \cap V \cap E_i) \\ &= f_i(U \cup V) + f_i(U \cap V), \end{aligned}$$

where the inequality uses the submodularity of the rank function.

For $i \in N$ and $e \in E$, let

$$C_{i,e}(x_{i,e}; x_{-i,e}) = \begin{cases} c_{i,e}(x_{i,e} + x_{-i,e}) & \text{if } x_{i,e} = 1, \\ 0 & \text{if } x_{i,e} = 0, \end{cases}$$

and note that the rank function is subcardinal, i.e., $\mathrm{rk}(U) \leq |U|$ for all $U \subseteq E$, so that $\mathrm{rk}(\{r\}) \leq 1$ for all $e \in E$, and thus $C_{i,e}$ is well defined. As a consequence, we need to require regularity and discrete convexity only for $x \in \{0, 1\}$. As for regularity, we obtain

$$C_{i,e}(0; t + 1) = 0 \leq c_{i,e}(1 + t) = C_{i,e}(1; t)$$

for all $t \in \mathbb{N}$ by the nonnegativity of $c_{i,e}$. As for discrete convexity, we do not need to require discrete convexity of the function $x \mapsto C(x; t)$, as x only takes two different values. Moreover, since $C_{i,e}(0; t) = 0$ for all t, we only have to check that $t \mapsto C_{i,e}(1; t)$ is discrete convex. To this end, we calculate

$$\begin{aligned} C^-_{i,e}(1; t) &= c_{i,e}(1 + t_e) - c_{i,e}(1 + t_e - 1) \\ &\leq c_{i,e}(1 + t_e + 1) - c_{i,e}(1 + t_e), \end{aligned}$$

where we used that $c_{i,e}$ is nondecreasing. □

5. Equilibrium existence. In this section, we give an algorithm that computes a pure Nash equilibrium for polymatroid games. Our algorithm relies on the two sensitivity results stated in Theorems 3.2 and 3.4.

5.1. The algorithm. Both sensitivity results are used as the main building blocks for the Algorithm in Figure 1 that computes a pure Nash equilibrium for congestion games on integral polymatroids. The algorithm maintains *preliminary demands, strategy spaces, and strategies* of the players denoted by $\bar{d}_i \leq d_i$, $\bar{X}_i = X_i(\bar{d}_i)$, and $\mathbf{x}_i \in \bar{X}_i$, respectively. Initially, $\bar{d}_i$ is set to zero for all $i \in N$, and the strategy profile, where the strategy of each player equals the zero vector, is a pure Nash equilibrium for this game, in which the demand of each player is zero.

Then in each round, for some player i, the demand is increased from $\bar{d}_i$ to $\bar{d}_i + 1$, and a best response $\mathbf{y}_i \in X(\bar{d}_i + 1)$ with $\|\mathbf{x}_i - \mathbf{y}_i\|_1 = 1$ is computed; see line 5 in the algorithm. By Theorem 3.4, such a best response always exists. In effect, the load on exactly one resource e increases, and only those players j with $x_{j,e} > 0$ on this resource can potentially decrease their private cost by a deviation. By Theorem 3.2, a best response of such players consists w.l.o.g. of moving a single unit from this resource to another resource; see line 8 in the algorithm. As a consequence, during the while-loop (lines 7–9), only one additional unit (compared to the previous iteration) is moved, preserving the invariant that only players using a resource to which this additional unit is assigned may have an incentive to profitably deviate. Thus, if the while-loop is left, the current strategy profile $\mathbf{x}$ is a pure Nash equilibrium for the reduced game $\bar{G} = (N, \bar{X}, \bar{d}, (C_{i,e})_{i \in N, e \in E})$.

Now we are ready to prove the main existence result.

THEOREM 5.1. *Polymatroid games possess a pure Nash equilibrium.*

Proof. We prove by induction on the total demand $d = \sum_{i \in N} d_i$ of the input game $G = (N, X, (d_i)_{i \in N}, (C_{i,e})_{i \in N, e \in E})$ that the algorithm computes a pure Nash equilibrium of G.

Input: $G = (N, X, (d_i)_{i\in N}, (C_{i,e})_{i\in N, e\in E})$
Output: pure Nash equilibrium $\mathbf{x}$

1 $\bar{d}_i \leftarrow 0, \bar{X}_i \leftarrow X_i(0)$ and $\mathbf{x}_i \leftarrow \mathbf{0}$ for all $i \in N$;
2 **for** $k = 1, \dots, \sum_{i\in N} d_i$ **do**
3   Choose $i \in N$ with $\bar{d}_i < d_i$;
4     $\bar{d}_i \leftarrow \bar{d}_i + 1$; $\bar{X}_i \leftarrow X_i(\bar{d}_i)$;
5   Choose a best response $\mathbf{y}_i \in \bar{X}_i$ with $\|\mathbf{y}_i - \mathbf{x}_i\|_1 = 1$;
6     $\mathbf{x}_i \leftarrow \mathbf{y}_i$;
7   **while** $\exists i \in N$ *who can improve in* $\bar{G} = (N, \bar{X}, \bar{d}, (C_{i,e})_{i\in N, e\in E})$ **do**
8   Compute a best response $\mathbf{y}_i \in \bar{X}_i$ with $\|\mathbf{y}_i - \mathbf{x}_i\|_1 = 2$;
9         $\mathbf{x}_i \leftarrow \mathbf{y}_i$;

10 Return $\mathbf{x}$;

FIG. 1. *Computation of a pure Nash equilibrium.*

For $d = 0$, this is trivial. Suppose that the algorithm works correctly for games with total demand $d - 1$, for some $d \geq 1$, and consider a game G with total demand d. Let us assume that in line 3, the algorithm always chooses a player with minimum index. Consider the game $G' = (N, X, (d'_i)_{i\in N}, (C_{i,e})_{i\in N, e\in E})$ that differs from G only in the fact that the demand of the last player n is reduced by one, i.e., $d'_i = d_i$ for all $i < n$ and $d'_n = d_n - 1$. Then, when running the algorithm with G' as input, the $d - 1$ iterations (of the for-loop) are equal to the first $d - 1$ iterations when running the algorithm with G as input. Thus, with G as input, we may assume that after the first $d - 1$ iterations, the preliminary strategy profile that we denote by $\mathbf{x}'$ is a pure Nash equilibrium of G'.

We analyze the final iteration $k = d$ of the algorithm in which the demand of player n is increased by 1 (see line 4). In line 5, a best reply $\mathbf{y}_n$ with $\|\mathbf{x}_n - \mathbf{y}_n\|_1 = 1$ is computed which exists by Lemma 3.4. Then, as long as there is a player i that can improve unilaterally, in line 8, a best response $\mathbf{y}_i$ with $\|\mathbf{y}_i - \mathbf{x}_i\|_1 = 2$ is computed which exists by Lemma 3.2.

It remains to show that the while-loop in lines 7–9 terminates. To prove this, we give each unit of demand of each player $i \in N$ an identity denoted by $i_j, j = 1, \dots, d_i$. For a strategy profile $\mathbf{x}$, we define $r(i_j, \mathbf{x}) \in E$ to be the resource to which unit i_j is assigned in strategy profile $\mathbf{x}$. Let $\mathbf{x}^l$ be the strategy profile after line 8 in the algorithm has been executed the lth time, where we use the convention that $\mathbf{x}^0$ denotes the preliminary strategy profile when entering the while-loop. As we chose in line 5 a best reply $\mathbf{y}_n$ of player n with $\|\mathbf{x}_n - \mathbf{y}_n\|_1 = 1$, there is a unique resource e_0 such that $x^0_{e_0} = x'_{e_0} + 1$ and $x^0_e = x'_e$ for all $e \in E \setminus \{e_0\}$. Furthermore, because we choose in line 8 a best response $\mathbf{y}_i$ with $\|\mathbf{y}_i - \mathbf{x}_i\|_1 = 2$, a simple inductive claim shows that after each iteration l of the while-loop, there is a unique resource $e_l \in E$ such that $x^l_{e_l} = x'_{e_l} + 1$ and $x^l_e = x'_e$ for all $e \in E \setminus \{e_l\}$.

For any $\mathbf{x}^l$ during the course of the algorithm, we define the *marginal cost* of unit i_j under strategy profile $\mathbf{x}^l$ as

$$\Delta_{i_j}(\mathbf{x}^l) = \begin{cases} C^-_{i,e}(x^l_{i,e}; x^l_{-i,e}) & \text{if } e = e(i_j, \mathbf{x}^l) = e_l, \\ C^-_{i,e}(x^l_{i,e}; x^l_{-i,e} + 1) & \text{if } e = e(i_j, \mathbf{x}^l) \neq e_l. \end{cases} \tag{5.1}$$

Intuitively, if $e(i_j, \mathbf{x}^l) = e_l$, then the value $\Delta_{i_j}(\mathbf{x}^l)$ measures the *cost saving* on resource $e(i_j, \mathbf{x}^l)$ if i_j (or any other unit of player i on resource $e(i_j, \mathbf{x}^l)$) is removed from $e(i_j, \mathbf{x}^l)$. If $e(i_j, \mathbf{x}^l) \neq e_l$, then the value $\Delta_{i_j}(\mathbf{x}^l)$ measures the cost saving if i_j is removed from $e(i_j, \mathbf{x}^l)$ after the total allocation has been increased by one unit by some other player. For a strategy profile $\mathbf{x}$, we define $\Delta(\mathbf{x}) = (\Delta_{i_j}(\mathbf{x}))_{i=1,\dots,n,j=1,\dots,d_i}$ to be the vector of marginal costs and let $\bar{\Delta}(\mathbf{x})$ be the vector of marginal costs sorted in nonincreasing order. We claim that $\bar{\Delta}(\mathbf{x})$ decreases lexicographically during the while-loop. To see this, consider an iteration l in which some unit i_j of player i is moved from resource e_{l-1} to resource e_l.

For proving $\bar{\Delta}(\mathbf{x}^l) <_{\text{lex}} \bar{\Delta}(\mathbf{x}^{l-1})$, we first observe that we only have to care for Δ-values that correspond to units i_j of the deviating player i because for all players $h \neq i$, we obtain $\Delta_{h_j}(\mathbf{x}^{l-1}) = \Delta_{h_j}(\mathbf{x}^l)$ for all $j = 1, \dots, d_h$. This follows immediately if h_j is assigned to neither e_{l-1} nor e_l. If h_j is assigned to e_{l-1} or e_l, then we switch the case in (5.1), and the claimed equality still holds. It remains to consider the Δ-values corresponding to the units of the deviating player i. Recall that the deviation of player i consists of moving unit i_j from resource e_{l-1} to resource e_l. We obtain

$$\begin{aligned}\Delta_{i_j}(\mathbf{x}^{l-1}) &= C^-_{i,e_{l-1}}(x^{l-1}_{i,e_{l-1}}; x^{l-1}_{-i,e_{l-1}})\\ &> C^+_{i,e_l}(x^{l-1}_{i,e_l}; x^{l-1}_{-i,e_l})\\ &= C^-_{i,e_l}(x^{l-1}_{i,e_l}+1; x^{l-1}_{-i,e_l})\\ &= C^-_{i,e_l}(x^{l}_{i,e_l}; x^{l}_{-i,e_l})\\ &= \Delta_{i_j}(\mathbf{x}^l),\end{aligned}$$

where the strict inequality follows since player i strictly improves. For every unit i_m of player i that is assigned to resource e_l as well, i.e, $e(i_m, \mathbf{x}^l) = e(i_j, \mathbf{x}^l) = e_l$, we have $\Delta_{i_j}(\mathbf{x}^l) = \Delta_{i_m}(\mathbf{x}^l)$ since the Δ-value is the same for all units of a single player assigned to the same resource. The Δ-values of such units i_m might have increased, but only to the Δ-value of unit i_j.

Next, consider the Δ-values of a unit i_m assigned to resource e_{l-1}, i.e., $e(i_m, \mathbf{x}^l) = e(i_j, \mathbf{x}^{l-1}) = e_{l-1}$. We obtain

$$\begin{aligned}\Delta_{i_m}(\mathbf{x}^l) &= C^-_{i,e_{l-1}}(x^{l}_{i,e_{l-1}}; x^{l}_{-i,e_{l-1}}+1)\\ &= C^-_{i,e_{l-1}}(x^{l-1}_{i,e_{l-1}}-1; x^{l-1}_{-i,e_{l-1}}+1)\\ &\leq C^-_{i,e_{l-1}}(x^{l-1}_{i,e_{l-1}}; x^{l-1}_{-i,e_{l-1}})\\ &= \Delta_{i_m}(\mathbf{x}^{l-1}),\end{aligned}$$

where for the inequality we used that $C_{i,e_{l-1}}$ is regular. Altogether, the Δ-values of all units of all players $h \neq i$ have not changed, for player i the Δ-values of remaining units assigned to resource e_{l-1} decreased, and the Δ-values assigned to resource e_l increased exactly to $\Delta_{i_j}(\mathbf{x}^l)$, which is strictly smaller than $\Delta_{i_j}(\mathbf{x}^{l-1})$. Thus, $\bar{\Delta}(\mathbf{x}^l) <_{\text{lex}} \bar{\Delta}(\mathbf{x}^{l-1})$ follows. ∎

The following corollary states an upper bound on the number of iterations of the algorithm in terms of $\delta = \max_{i \in N} d_i$.

THEOREM 5.2. *The number of best responses computed by Algorithm* 1 *is bounded from above by* $n^2 m \delta^3$.

Proof. The algorithm successively adds units of demands to the system starting with zero units and terminating with $\sum_{i \in N} d_i \leq n\delta$ units. To prove the theorem, we show that for a fixed number of k units in the system, the number of best replies computed by the algorithm during the while loop in lines 7–9 is bounded by $nm\delta^2$.

Fix an arbitrary unit of demand i_j of an arbitrary player i, and consider its marginal cost value Δ_{i_j} during the execution of the while loop in lines 7–9. By definition of the Δ_{i_j}, the value Δ_{i_j} only changes either when another unit i_l of player i enters or leaves the resource that i_j is assigned to or i_j is assigned to another resource. Since the number of resources i_j can be assigned to is bounded by m and the number of other units of player i assigned to the same resource as i_j is bounded by δ, there are at most $m\delta$ different values that Δ_{i_j} can take. Moreover, for every best reply where i_j is moved, the value Δ_{i_j} strictly decreases so that we can conclude that unit i_j is moved at most $m\delta$ times during the while loop in lines 7–9. Note that the strict monotonicity of the Δ_{i_j}-values follows directly if we assume (for the analysis) that units for any player i are moved in last-in-first-out order. The same reasoning applies to any of the $k \leq n\delta$ units; thus, the total number of best replies during the execution of the while loop is bounded by $nm\delta^2$. This implies the claimed result. ∎

6. Nonpolymatroid regions.

The proofs of the results obtained in sections 3 and 4 relied on the fact that the function f is submodular, and thus the feasible region of the optimization problem and the strategy spaces of the players, respectively, are polymatroids. One may wonder whether polymatroids are the maximal combinatorial structure for which these results hold. In this section, we will give an affirmative answer to this question. In fact, we will work towards showing that for *every* normalized, monotonic, and nonsubmodular function f, there is a convex and regular optimization problem with feasible set

$$\mathbb{B}_f(d) = \left\{\mathbf{x} \in \mathbb{N}^E \ : x(U) \leq f(U) \text{ for all } U \subseteq E,\, x(E) = d\right\} \tag{6.1}$$

with $d \in \mathbb{N}$ such that the sensitivity results of Theorems 3.2 and 3.4 do not hold. Moreover, there is a game with convex and regular cost functions where the players' strategies are isomorphic to (6.1) that does not have a pure Nash equilibrium. This implies that also for the existence result of Theorem 5.1, the polymatroid structure is maximal.

For ease of exposition, we assume that f is strictly positive in the sense that $f(U) > 0$ for all $U \in 2^E \setminus \emptyset$. This assumption can be made without loss of generality since a nonempty set of resources U with $f(U) = 0$ is not used in any strategy anyway and, thus, can effectively be removed from E.

Formally, let

$$\begin{aligned}\mathcal{X}(d) &= \{\mathbb{B}_f(d) : f \text{ is strictly positive, normalized and monotonic}\},\\ \mathcal{X}^*(d) &= \{\mathbb{B}_f(d) : f \text{ is strictly positive, normalized, monotonic, and submodular}\}\end{aligned}$$

denote the feasible regions that can be described by arbitrary and submodular functions f, respectively.

First, note that $\mathcal{X}(1) = \mathcal{X}^*(1)$. To see this, note that $f(\{e\}) \geq 1$ as f is strictly positive, so f does not encode any constraints on where the single unit of demand can be put. Thus, the set of strategies can be described equivalently by a function f' with $f'(U) = 1$ for all $U \in 2^E \setminus \{\emptyset\}$. It is straightforward to verify that f' is submodular.

More interestingly, already for $d = 2$, the feasible regions contained in $\mathcal{X}(2)$ and $\mathcal{X}^*(2)$ differ. We start with the following observations regarding the feasible regions in $\mathcal{X}(2) \setminus \mathcal{X}^*(2)$.

LEMMA 6.1. *For any* $X \in \mathcal{X}(2) \setminus \mathcal{X}^*(2)$, *there are* $f : 2^E \to \mathbb{N}$ *and* $S, T \in 2^E$ *such that*

1. $X = \mathbb{B}_f(2) = \{\mathbf{x} \in \mathbb{N}^E \; : x(U) \leq f(U) \text{ for all } U \subseteq E,\, x(E) = 2\}$;
2. *for any constraint* $x(U) \leq f(U)$, *there is* $\mathbf{x} \in \mathbb{B}_f(2)$ *with* $x(U) = f(U)$;
3. $f(S) = f(T) = f(S \cap T) = 1$ *and* $f(S \cup T) = 2$.

Proof. Since $X \in \mathcal{X}(2)$, there is a strictly positive, normalized, and monotonic function f with $X = \mathbb{B}_f(d)$. Note that f is not submodular since $X \in \mathcal{X}^*(2)$ otherwise.

To prove the second observation, suppose that there is $U' \subseteq R$ such that $x(U') < f(U')$ for all $\mathbf{x} \in \mathbb{B}_f(2)$. Consider the new function $f' : 2^E \to \mathbb{N}$ defined as

$$f'(U) = \begin{cases} f(U) & \text{if } U \neq U', \\ f(U) - 1 & \text{if } U = U'. \end{cases}$$

By construction, $\mathbb{B}_f(2) = \mathbb{B}_{f'}(2)$. Applying the above argumentation on f', we derive that f' is not submodular as well. Decreasing the right-hand side of nontight constraints iteratively in this manner, we finally obtain a nonsubmodular function $f : 2^E \to \mathbb{N}$ with $X = \mathbb{B}_f(2)$ such that for any constraint of type $x(U) \leq f(U)$, there is $\mathbf{x} \in \mathbb{B}_f(2)$ with $x(U) = f(U)$.

To prove the third observation, first note that $x(E) = 2$ together with the second statement of the lemma implies that $f(U) \leq 2$ for all $U \subseteq E$. Since f is not submodular, there are sets $S, T \in 2^E$ such that

$$f(S) + f(T) < f(S \cap T) + f(S \cup T). \tag{6.2}$$

It is straightforward to verify that this inequality can only be satisfied when S and T are nonempty. We claim that $S \cap T$ is nonempty as well. For the sake of a contradiction, let us assume that (6.2) is satisfied by $S, T \in 2^E \setminus \emptyset$ with $S \cap T = \emptyset$. Using the second statement of the lemma, there is a vector $\mathbf{x} \in \mathbb{B}_f(2)$ such that the constraint $x(S \cup T) \leq f(S \cup T)$ is tight, i.e., $x(S \cup T) = f(S \cup T)$. Since S and T are disjoint, we obtain $x(S \cup T) = x(S) + x(T) \leq f(S) + f(T) < f(S \cup T)$, a contradiction. We have established that S, T, and $S \cap T$ are nonempty. Using the strict positivity of f, this implies that $f(S) \geq 1$, $f(T) \geq 1$, and $f(S \cap T) \geq 1$. Together with $f(S \cup T) \leq 2$ and the monotonicity of f, this implies that

$$f(S) = f(T) = f(S \cap T) = 1 \qquad \text{and} \qquad f(S \cup T) = 2, \tag{6.3}$$

as claimed. ∎

We proceed to give two additional lemmas that give further structural results regarding the sets $S, T \in 2^E \setminus \emptyset$ for which the submodularity constraints are violated. First, we show that each strategy whose support is contained in $S \cup T$ does not use a resource in $S \cap T$.

LEMMA 6.2. *Let* $f : 2^E \to \mathbb{N}$ *and* $S, T \in 2^E$ *be as guaranteed by Lemma* 6.1. *Then, for any* $\mathbf{x} \in \mathbb{B}_f(2)$ *with* $\operatorname{supp}(\mathbf{x}) \subseteq S \cup T$, *we have* $\operatorname{supp}(\mathbf{x}) \cap (S \cap T) = \emptyset$.

Proof. Suppose there is $\mathbf{x} \in \mathbb{B}_f(d)$ and a resource $e \in E$ with $\operatorname{supp}(\mathbf{x}) \subseteq S \cup T$ and $e \in \operatorname{supp}(\mathbf{x}) \cap S \cap T$. Because $\mathbf{x}$ is integral and $f(S \cap T) = 1$, this implies that $x(e) = 1$. Since $x(R) = 2 = x(S \cup T)$ and $x(S \cap T) \leq 1$, there is another element $e' \neq e$ with $e' \in S \Delta T$ and $x(e') = 1$. It is without loss of generality to assume that $e' \in S \setminus T$. This, however, implies that $x(S) \geq x(e) + x(e') = 2$, a contradiction to $x(S) \leq 1$. ∎

Combining the two previous lemmas, we derive the existence of four distinct *critical elements* which are used by two vectors with disjoint supports.

LEMMA 6.3. *Let* $f : 2^E \to \mathbb{N}$ *be as guaranteed by Lemma* 6.1. *Then there are four distinct elements* $e_1, e_2, e_3, e_4 \in E$ *and two vectors* $\mathbf{x}, \mathbf{y} \in \mathbb{B}_f(2)$ *with the following properties:*

1. $x(e_1) = x(e_2) = 1$ *for some* $e_1 \in E \setminus (S \cup T)$ *and* $e_2 \in S \cap T$.
2. $y(e_3) = y(e_4) = 1$ *for some* $e_3 \in S \setminus T$ *and* $e_4 \in T \setminus S$.
3. *For all other strategies* $\mathbf{z} \in \mathbb{B}_f(2) \setminus \{\mathbf{x}, \mathbf{y}\}$ *with* $\operatorname{supp}(\mathbf{z}) \subseteq \{e_1, e_2, e_3, e_4\}$, *one of the following three cases holds:*
   (a) $\operatorname{supp}(\mathbf{z}) = \{e_1, e_3\}$.
   (b) $\operatorname{supp}(\mathbf{z}) = \{e_1, e_4\}$.
   (c) $\operatorname{supp}(\mathbf{z}) = \{e_1\}$.

Proof. By Lemma 6.1, there is a strategy $\mathbf{x}$ for which the constraint $x(S \cap T) \leq 1$ is tight. This implies the existence of a element $e_2 \in S \cap T$ with $x(e_2) = 1$. By Lemma 6.2, $\operatorname{supp}(\mathbf{x}) \not\subseteq S \cup T$, implying the existence of a element $e_1 \in E \setminus (S \cup T)$ with $x(e_1) = 1$.

By Lemma 6.1, there is also another strategy $\mathbf{y}$ for which the constraint $x(S \cup T) \leq 2$ is tight. First, note that $y(r) \leq 1$ for all $r \in S \cup T$, as otherwise the constraint $y(S) \leq f(S) = 1$ or $y(T) \leq f(T) = 1$ would be violated. This implies the existence of two distinct elements $e_3, e_4 \in S \cup T$ such that $y(e_3) = y(e_4) = 1$. Further, note that $e_3, e_4 \notin S \cap T$, as otherwise Lemma 6.2 is violated. Using $y(S) \leq 1$ and $y(T) \leq 1$, we derive that $e_3 \in S \setminus T$ and $e_4 \in T \setminus S$.

To see the last part of the claim, note that any strategy $\mathbf{z}$ with $e_2 \in \operatorname{supp}(\mathbf{z}) \subseteq \{e_1, e_2, e_3, e_4\}$ must have $\operatorname{supp}(\mathbf{z}) = \{e_1, e_2\}$, as otherwise the constraint $z(S) \leq 1$ or $z(T) \leq 1$ is violated. Thus, $\mathbf{z} = \mathbf{x}$ for any such $\mathbf{z}$. Similarly, note that any strategy $\mathbf{z}$ with a singleton support $\operatorname{supp}(\mathbf{z}) = \{e\}$ with $e \in \{e_1, e_2, e_3, e_4\}$ must have $\operatorname{supp}(\mathbf{z}) = \{e_1\}$, as otherwise the constraint $z(S) \leq 1$ or $z(T) \leq 1$ is violated. These two observations leave only room for the strategies as in part 3 of the statement of the lemma. ∎

6.1. Violation of sensitivity results (Corollary 3.3) and Theorem 3.4.

We first show that for any feasible region $X \in \mathcal{X}(2) \setminus X^*(2)$ that is not described by a submodular capacity constraint, the sensitivity results of Corollary 3.3 do not hold.

THEOREM 6.4. *For any* $X \in \mathcal{X}(2) \setminus X^*(2)$, *there is an optimization problem of the form*

$$\begin{aligned} &\textit{minimize} \sum_{e \in E} C_e(x_e; t_e) \\ &\textit{subject to } \mathbf{x} \in X \end{aligned}$$

with regular functions $C_e, e \in E$, *and parameters* $\mathbf{t}, \mathbf{t}' \in N^E$ *such that* $\|\mathbf{t} - \mathbf{t}'\|_1 = 1$ *but* $\|\mathbf{x}^*(\mathbf{t}, 2) - \mathbf{x}^*(\mathbf{t}', 2)\|_1 > 2$.

Proof. By Lemma 6.3, there are four critical elements e_1, e_2, e_3, e_4 such that X can be decomposed in the following way:

$$X = \{\mathbf{x}, \mathbf{y}\} \cup X^{\text{crit}} \cup X^{\text{out}},$$

where $\operatorname{supp}(\mathbf{x}) = \{e_1, e_2\}$, $\operatorname{supp}(\mathbf{y}) = \{e_3, e_4\}$. The set

$$X^{\text{crit}} = \{\mathbf{z} \in X : \operatorname{supp}(z) \in \{\{e_1, e_3\}, \{e_1, e_4\}, \{e_1\}\}\}$$

contains an arbitrary subset of vectors whose support is a subset of the four critical resources but that are not contained in $\{\mathbf{x}, \mathbf{y}\}$. By Lemma 6.3, the only supports that can occur for vectors in X^{crit} are $\{e_1, e_3\}$,$\{e_1, e_4\}$, and $\{e_1\}$. Finally, the set X^{out} contains all vectors whose support contains a noncritical element.

Let

$$\begin{aligned} C_e(x_e; t_e) &= (x_e + t_e)^2 && \text{for } e \in \{e_1, e_2\},\\ C_e(x_e; t_e) &= (x_e + t_e)^2 + 1/2 && \text{for } e \in \{e_3, e_4\},\\ C_e(x_e; t_e) &= 20 && \text{for all } e \in E \setminus \{e_1, e_2, e_3, e_4\}, \end{aligned}$$

and consider the parameter vectors $\mathbf{t} = \mathbf{0}$ and $\mathbf{t}' = \chi_{e_2}$. It is easy to see that $\mathbf{x}^*(\mathbf{t}, 2) = \chi_{e_1} + \chi_{e_2}$ is the unique optimal solution for parameter vector $\mathbf{t}$. On the other hand, for $\mathbf{t}'$ the unique optimal solution is $x^*(\mathbf{t}', 2) = \chi_3 + \chi_4$. We note that $\|\mathbf{x}^*(\mathbf{t}, 2) - \mathbf{x}^*(\mathbf{t}', 2)\|_1 = 4$ even though $\|\mathbf{t} - \mathbf{t}'\|_1 = 1$ proving the claimed result. ∎

With the same construction, it is also not hard to verify that also Theorem 3.4 does not continue to hold for any feasible region that is not a polymatroid.

THEOREM 6.5. *For any $X \in \mathcal{X}(2) \setminus X^*(2)$, there is an optimization problem of the form*

$$\begin{aligned} &\textit{minimize} \sum_{e \in E} C_e(x_e; t_e)\\ &\textit{subject to } \mathbf{x} \in X \end{aligned}$$

with regular functions $C_e, e \in E$ and $d, d' \in N^E$, and $\mathbf{t} \in N^E$ such that $|d - d'| = 1$ but $\|\mathbf{x}^*(\mathbf{t}, d) - \mathbf{x}^*(\mathbf{t}, d')\|_1 > 1$.

Proof. With the same construction as in the proof of Theorem 6.4, we compute that the unique optimal solution for $\mathbf{t} = \chi_{e_2}$ and $d = 1$ is $\mathbf{x}^*(\chi_{e_2}, 1) = \chi_{e_1}$. However, as argued in the proof of Theorem 6.4, $\mathbf{x}^*(\chi_{e_2}, 2) = \chi_{e_3} + \chi_{e_4}$. We obtain $\|\mathbf{x}^*(\chi_{e_1}, 2) - \mathbf{x}^*(\chi_{e_2}, 2)\|_1 = 3$, proving the claimed result. ∎

6.2. Violation of the existence of equilibria (Theorem 5.1). We proceed to show that also the existence result for pure Nash equilibria of Theorem 5.1 does not continue to hold. In fact, for any nonpolymatroid structure $X \in \mathcal{X}(2) \setminus \mathcal{X}^*(2)$, there is a two-player game where both players' strategies are isomorphic to X that does not have a pure Nash equilibrium.

THEOREM 6.6. *For any $X \in \mathcal{X}(2) \setminus \mathcal{X}^*(2)$, there is a two-player game with regular player-specific costs and strategy sets isomorphic to $\mathbb{B}_f(2)$ that does not have a pure Nash equilibrium.*

Proof. Let f be as guaranteed by Lemma 6.1. By Lemma 6.3, for every player $i \in \{1, 2\}$, there are four critical resources $e_i^1, e_i^2, e_i^3, e_i^4$ such that the strategy set X_i of player i can be decomposed as

$$X_i = \{\mathbf{x}_i, \mathbf{y}_i\} \cup X_i^{\mathrm{opt}} \cup X_i^{\mathrm{out}},$$

where $\mathrm{supp}(\mathbf{x}_i) = \{e_i^1, e_i^2\}$ and $\mathrm{supp}(\mathbf{y}_i) = \{e_i^3, e_i^4\}$. The set

$$X_i^{\mathrm{crit}} = \bigl\{\mathbf{z}_i \in X_i : \mathrm{supp}(\mathbf{z}_i) \in \bigl\{\{e_i^1, e_i^3\}, \{e_i^1, e_i^4\}, \{e_i^1\}\bigr\}\bigr\}$$

strategy	supp	$\mathbf{y}_2$ $\{a,h\}$	$\mathbf{x}_2$ $\{b,g\}$	X_2^{crit} $\{a,g\}$	X_2^{crit} $\{h,g\}$	X_2^{crit} $\{g\}$	X_2^{out}
$\mathbf{y}_1$	$\{a,b\}$	$1{+}1, 1{+}0$	$0{+}1, 0{+}2$	$1{+}1, 1{+}2$	$0{+}1, 0{+}2$	$0{+}1, 2{+}2$	$\cdot\ , \geq 20$
$\mathbf{x}_2$	$\{h,g\}$	$0{+}0, 1{+}2$	$0{+}3, 0{+}2$	$0{+}3, 1{+}2$	$0{+}3, 2{+}2$	$0{+}6, 2{+}2$	$\cdot\ , \geq 20$
X_1^{crit}	$\{a,g\}$	$1{+}0, 1{+}0$	$0{+}3, 0{+}2$	$1{+}3, 1{+}2$	$0{+}3, 0{+}2$	$0{+}6, 2{+}2$	$\cdot\ , \geq 20$
X_1^{crit}	$\{b,g\}$	$1{+}0, 1{+}0$	$1{+}3, 0{+}2$	$1{+}3, 1{+}2$	$1{+}3, 0{+}2$	$1{+}6, 2{+}2$	$\cdot\ , \geq 20$
X_1^{crit}	$\{g\}$	$3{+}3, 1{+}0$	$6{+}6, 0{+}2$	$6{+}6, 1{+}2$	$6{+}6, 0{+}2$	$9{+}9, 2{+}2$	$\cdot\ , \geq 20$
X_1^{out}		$\geq 20, \ \cdot$	$\geq 20, \ \cdot$	$\geq 20, \ \cdot$	$\geq 20, \ \cdot$	$\geq 20, \ \cdot$	$\geq 20, \geq 20$

FIG. 2. *Game without a pure Nash equilibrium as constructed in the proof of Theorem* 6.6.

contains a possibly empty subset of strategies whose support is contained in the set of critical resources $\{e_i^1, e_i^2, e_i^3, e_i^4\}$, and the set

$$X_i^{\text{out}} = \left\{\mathbf{z}_i \in X_i : \text{supp}(\mathbf{z}_i) \setminus \{e_i^1, e_i^2, e_i^3, e_i^4\} \neq \emptyset\right\}$$

contains a possibly empty subset of strategies that contains a noncritical resource $e \in E \setminus \{e_i^1, e_i^2, e_i^3, e_i^4\}$. Next, we describe how the set of strategies of both players are interweaved. For our purposes, it is sufficient to specify the critical resources of the two players. To this end, let a, b, h, g be four resources such that

$$\begin{aligned} e_1^1 &= g, & e_2^1 &= g,\\ e_1^2 &= e, & e_2^2 &= h,\\ e_1^3 &= a, & e_2^3 &= a,\\ e_1^4 &= b, & e_2^4 &= b. \end{aligned}$$

Consider the following player-specific cost functions:

$$\begin{aligned} c_{1,a}(x) &= \max\{0, x-1\}, & c_{2,a}(x) &= 1,\\ c_{1,b}(x) &= 1, & c_{2,b}(x) &= 0,\\ c_{1,h}(x) &= 0, & c_{2,h}(x) &= \max\{0, 2x-2\},\\ c_{1,g}(x) &= \max\{0, 3x-3\}, & c_{2,g}(x) &= 2. \end{aligned}$$

For any noncritical resource $e \in E \setminus \{a, b, e, g\}$, we define

$$c_{1,e}(x) = 20, \qquad c_{2,e}(x) = 20.$$

For a player $i \in \{1,2\}$ and $e \in E$, we let $C_{i,e}(x_e; x_{-i,e}) := c_{i,e}(x_{i,e}+x_{-i,e})x_{i,e}$. The resulting private costs of the players are shown in Figure 2. Note that by Lemma 6.3, for every player i we are only guaranteed the existence of the two strategies $\mathbf{x}_i$ and $\mathbf{y}_i$ shown in the upper left part of the bimatrix. As shown in Lemma 6.3, any other strategy $\mathbf{z}_i$ contains either a noncritical resource and is thus contained in the subset of strategies X_i^{out} or contains only critical resources and is contained in the subset of strategies X_i^{crit}. The bimatrix in Figure 2 has the property that there is no pure Nash equilibrium no matter which subset of the strategies in X_i^{crit} or whether a strategy in X_i^{out} is actually present. We may thus conclude that no matter how the sets of strategies described by f specifically look like, no pure Nash equilibrium exists. ∎

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