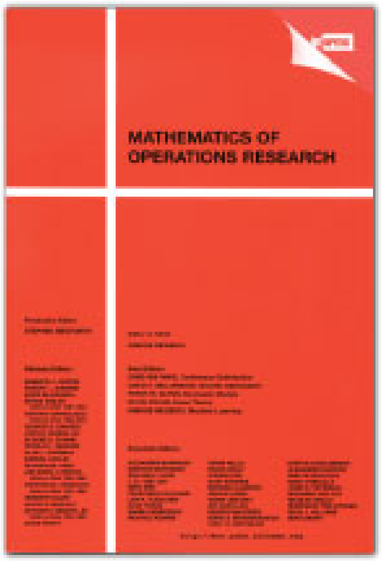

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Congestion Games with Variable Demands

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We initiate the study of congestion games with variable demands in which the players strategically choose both a nonnegative demand and a subset of resources. The players' incentives to use higher demands are stimulated by nondecreasing and concave utility functions. The payoff for a player is defined as the difference between the utility of the demand and the associated cost on the used resources. Although this class of noncooperative games captures many elements of real-world applications, it has not been studied in this generality in the past. Specifically, we study the fundamental problem of the existence of pure Nash equilibria, PNE for short. We call a set of cost functions consistent if every congestion game with variable demands and cost functions from the set possesses a PNE. We show that only affine and homogeneous exponential functions are consistent. En route, we obtain novel characterizations of consistency for congestion games with fixed but resource-dependent demands.

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1. Introduction. Resource allocation problems play a key role in many applications. Whenever a set of demands for scarce resources needs to be satisfied, the goal is to find the most profitable or least costly allocation of the resources to the demands. Typical examples for such situations appear in the area of traffic networks (Beckmann et al. [3], Roughgarden [35], Smith [37], Wardrop [40]) and telecommunication networks (Johari and Tsitsiklis [21], Kelly et al. [22], Srikant [38]). A common characteristic of these examples is that the allocation of the resources is determined by a finite number of independent and selfish players, which are optimizing an individual and private objective function. To understand the behavior of such systems, a common approach is to model them as a noncooperative game, more specifically as a *congestion game* (Rosenthal [33]). In a congestion game, there is a set of resources and a pure strategy of a player is to choose a subset of resources. The cost of a resource depends only on the number of players choosing the resource, and the private cost of a player is the sum of the costs of the chosen resources. Under these assumptions, Rosenthal proved the existence of a pure Nash equilibrium, PNE for short.

In the past, the existence of pure Nash equilibria has been analyzed in many variants of congestion games such as scheduling, routing, facility location, and network design, each variant with unweighted and weighted players (Ackermann et al. [1], Anshelevich et al. [2], Chen and Roughgarden [5], Gairing et al. [10], Ieong et al. [20], Milchtaich [27]). Most of these previous works have a common feature: given a set of resources whose cost increases with increasing congestion, every player allocates a *fixed* demand to an available subset. While obviously important, such models do not take into account a fundamental property of many real-world applications: the intrinsic coupling between the cost (or quality) of the resources and the resulting demands for the resources. A prominent example of this coupling is the flow control problem in telecommunication networks. In this setting, players receive a nonnegative utility from sending data and the perceived costs increase with congestion. In this and other examples, the demands will be reduced if the resources are congested, and increased if the resources are uncongested. Allowing for *variable demands* is, thus, a natural prerequisite for modeling the trade-off between the benefit for demand and the cost or quality of the resources.

There is a large body of work addressing the issue of variable demands, e.g., Cole et al. [6], Low and Lapsley [25], Kelly et al. [22], Shenker [36], and Srikant [38], in the context of telecommunication networks, and Beckmann et al. [3] and Haurie and Marcotte [18] in the context of traffic networks. Most of these works assume that the (variable) demand may be *fractionally* distributed over the available subsets of resources. This assumption together with convexity assumptions on the cost and utility functions implies the existence of a PNE by standard fixed point arguments (Glicksberg [13], Rosen [32]). Allowing a fractional distribution of the demand, however, is obviously not possible in many applications. For instance, the standard TCP/IP protocol suite uses single path routing, because splitting the demand comes with several practical complications, e.g., packets arriving out of order, packet jitter due to different server delays, etc. This issue has been explicitly addressed by Orda et al. [30]. Although the authors investigated the issue of uniqueness of pure Nash equilibria—which they call Nash equilibrium points (NEP)—in congestion games with fixed splittable demands, they raise the following question in the final section (Orda et al. [30, p. 519]):

> Several other open questions of practical value deserve attention. For example, in many networks users are restricted to route their flow along a single path with strict rules of changing them. Under such circumstances an NEP may not exist at all and complicated oscillatory behavior is likely to arise.

We initiate the systematic study of congestion games with variable demands, where the (variable) demand has to be assigned to exactly one subset of resources. We impose the standard economic assumption (cf. Haurie and Marcotte [18], Kelly et al. [22], Shenker [36]) that every player is associated with a nondecreasing and concave utility function that measures her utility for the demand. The payoff for a player is defined as the difference between this utility and the associated congestion cost on the used resources.

There are two fundamental goals from a system design perspective: (i) the system must be *stabilizable*, i.e., there must be a stable state from which no player wants to unilaterally deviate; and (ii) myopic play of the players should guide the system to a stable state. Because the utility functions are private information and not available to the system designer (cf. Kelly et al. [22]), it is natural to study the existence of equilibria and the convergence of selfish behavior with respect to the *cost functions*, which represent the technology associated with the resources, e.g., queuing disciplines at routers, latency functions in transportation networks, etc. To this end, we adopt the notion of *consistency* used in Holzman and Law-Yone [19] and Harks and Klimm [16]. A set $\mathscr{C}$ of cost functions is *consistent* for congestion games with variable demands if every congestion game with variable demands and costs in $\mathscr{C}$ possesses a PNE. We further say that $\mathscr{C}$ is *approximately universally consistent* if every congestion game with variable demands and costs in $\mathscr{C}$ has the approximate finite improvement property (AFIP), i.e., every path of unilateral deviations that increases the payoff of the deviating player by a constant bounded away from 0 is finite.

1.1. Our results. In this paper, we provide a complete characterization of the consistency and approximate universal consistency of cost functions for congestion games with variable demands. We first observe that the existence of a PNE crucially depends on whether the cost functions are interpreted as (monetary) *per-unit costs* or as *latencies*. In the per-unit cost setting, the cost every player incurs for using a certain resource equals the resource cost multiplied with that player's demand. Hence, on every resource every user pays *proportionally* to her demand and such games will be called *proportional games* henceforth. When resource costs represent latencies, it is reasonable to assume that the cost of every player on a certain resource is *not* multiplied with her demand. Thus, each of the users of a particular resource experiences the same cost regardless of her demand and we will refer to these games as *uniform* in the following.

As our main result, we completely characterize the consistency of cost functions for both proportional and uniform congestion games with variable demands. Specifically, we show that a set $\mathscr{C}$ of continuous and nonnegative cost functions is consistent for proportional congestion games with variable demands if and only if exactly one of the following cases holds: (i) $\mathscr{C}$ only contains affine functions of type $c(x) = a_c x + b_c$ with $a_c > 0$, $b_c \geq 0$; *or* (ii) $\mathscr{C}$ only contains homogeneously exponential functions of type $c(x) = a_c e^{\phi x}$ for some $a_c > 0, \phi > 0$, where a_c may depend on c, and ϕ must be equal for all $c \in \mathscr{C}$. Moreover, we prove that $\mathscr{C}$ is approximately universally consistent for proportional congestion games with variable demands if and only if (i) holds. Under the same assumptions on $\mathscr{C}$, we also prove that $\mathscr{C}$ is consistent for uniform games if and only if (ii) holds.

As a byproduct of our analysis we obtain a complete characterization of consistency for congestion games with resource-dependent demands as well. In such a game, every player i is associated with a vector $(d_{i,r})_{r \in R}$ of nonnegative demands and whenever player i chooses resource r, she uses it with the fixed demand $d_{i,r}$. The goal of every player is to minimize the costs of the resources used. As for congestion games with variable demands, it is important to distinguish between proportional games and uniform games. They differ solely in the fact that in the definition of the players' payoff functions, for proportional games the cost of a resource is multiplied with the player's demand, whereas in uniform games it is not.

We prove the following complete characterization for consistency for congestion games with resource-dependent demands: a set $\mathscr{C}$ of continuous cost functions is consistent for proportional congestion games with resource-dependent demands if and only if $\mathscr{C}$ only contains affine functions. We further show that there is no set $\mathscr{C}$ of cost functions that is consistent for uniform congestion games with resource-dependent demands except for the trivial case that $\mathscr{C}$ only contains constant functions. Our results for congestion games with variable and resource-dependent demands are summarized in Table 1.

TABLE 1. Pure Nash equilibria (PNE), the finite improvement property (FIP) and the approximate finite improvement property (AFIP) in congestion games with resource-dependent demands, congestion games with variable demands, and weighted congestion games (with fixed and resource-independent demands). Here, by "exponential," we denote sets $\mathscr{C}$ of cost functions, such that every $c \in \mathscr{C}$ is of type $c(x) = a_c e^{\phi x} + b_c$, where $a_c, b_c \in \mathbb{R}$ may depend on c and ϕ is equal for all $c \in \mathscr{C}$. By "hom. exponential" we denote sets $\mathscr{C}$ with the additional property that $b_c = 0$ for all $c \in \mathscr{C}$. For both congestion games with variable demands and congestion games with resource-dependent demands the nonexistence of a PNE for arbitrary nonaffine and nonexponential cost function follows by a simple corollary from the nonexistence of a PNE in weighted congestion games shown in Harks and Klimm [16]; see Proposition 1 for a formal statement. The existence of a PNE in proportional games with affine cost functions follows directly from the potential function given in Harks et al. [17].

	Resource-dependent demands				Variable demands				Fixed independent demands		
	Proportional		Uniform		Proportional		Uniform				
Costs	PNE	FIP	PNE	FIP	PNE	AFIP	PNE	AFIP	PNE	FIP	
Affine	Yes	Yes	No	No	Yes	Yes	No	No	Yes	Yes	Fotakis et al. [8], Harks et al. [17]
Hom. exponential	No	No	No	No	Yes	No	Yes	No	Yes	Yes	Harks et al. [17], Panagopoulou and Spirakis [31]
Exponential	No	No	No	No	No	No	No	No	Yes	Yes	Harks et al. [17], Panagopoulou and Spirakis [31]
Arbitrary	No	No	No	No	No	No	No	No	No	No	Harks and Klimm [16]

1.2. Techniques and outline of the paper. The main difficulty when analyzing the existence of equilibria in congestion games with variable demands stems from the fact that the players' strategies involve both the discrete choice of a set of resources and the continuous choice of the demand. We approach this issue by relating the existence of equilibria in games with variable demands to the existence of equilibria in congestion games with fixed, but resource-dependent demands.

For a set $\mathscr{C}$ of cost functions containing some c such that c'/c is injective, we show that whenever there is a congestion game with resource-dependent demands and cost functions in $\mathscr{C}$ that does not possess a PNE, then there is also a congestion game with variable demands and cost functions in $\mathscr{C}$ that has no PNE. Thus, in order to understand the existence of equilibria in games with variable demands, in §4 we first investigate games with resource-dependent demands. In contrast to weighted congestion games (in which the demand of a player is independent of the chosen resource) the distinction between monetary per-unit costs and latencies matters for the existence of equilibria. To analyze both types of games at the same time, we consider a more general class of games, which we term g-scaled congestion games with resource-dependent demands, in which the private cost of every player equals the sum of the costs of the chosen resources multiplied with some function g of that player's demand. To precisely characterize the existence of equilibria in these games, we first prove a necessary condition on a single differentiable cost function c to be consistent. This condition requires that two terms involving c and g, their derivatives, and a scalar μ have unequal signs. For $\mu = 0$ this condition simply demands that c is monotonic and we thus call this condition the *generalized monotonicity condition*. Then, we use this condition in order to derive a complete characterization of consistency for g-scaled congestion games.

Given the characterization for resource-dependent demands, in §5 we then formally prove the aforementioned connection between games with variable demands and games with resource-dependent demands. It is interesting to note that for a homogeneously exponential cost function c, the function c'/c is not injective and, thus, our connection between the two classes of games does not apply. However, we can prove that congestion games with variable demands and homogeneously exponential cost functions always have a PNE, which is not true for games with resource-dependent demands.

In §6, we show that all characterizations continue to hold for directed network games. In §7, we conclude the paper with describing open problems.

1.3. Related work. Rosenthal [33] introduced the class of unweighted congestion games and showed that they always possess a PNE. Many interesting generalizations of unweighted congestion games fail to have a PNE in general, e.g., weighted congestion games (Fotakis et al. [8], Goemans et al. [14], Libman and Orda [24]), games with player-specific cost functions (Milchtaich [26, 27], Gairing and Klimm [9]), and games with integer-splittable demands (Rosenthal [34], Tran-Thanh et al. [39]). Bilò [4] and Monderer [29] proved that any finite game can be represented as a congestion game with player-specific costs. Milchtaich [28] additionally showed that any finite game can be represented as a weighted congestion game.

Given these results, several works analyzed under which *additional* assumptions the existence of a PNE can be guaranteed. Ieong et al. [20] showed that *singleton* weighted congestion games with nondecreasing costs possess a PNE. Ackermann et al. [1] proved that weighted congestion games with nondecreasing costs in which the strategy space of every player consists of the bases of a matroid possess a PNE. Fotakis et al. [8] showed that a PNE exists in all weighted congestion games for which the cost of every resource is affine. Panagopoulou and Spirakis [31] proved the same result for the exponential function. In previous work (Harks and Klimm [16]) we gave a complete characterization of the set of cost functions that guarantees the existence of a PNE in all weighted congestion games. Specifically, we showed that only affine functions and the set of exponential functions of type $c(x) = ae^{\phi x} + b$, where $a, b \in \mathbb{R}$ may depend on the function and $\phi \in \mathbb{R}$ is equal for all functions within the set, guarantees the existence of a PNE.

Congestion games with resource-dependent demands have been examined in the context of scheduling games studied by Even-Dar et al. [7]. In such a game, every player controls a job that she wishes to be processed on exactly one machine out of an available set of feasible machines. The induced load of a job depends on which machine it is scheduled. This class of games exactly corresponds to singleton congestion games with resource-dependent demands.

Congestion games with variable demands (and proportional costs) have been introduced first in Harks et al. [17], where it is shown that for affine cost functions, there is an exact potential function and, thus, there always exists a PNE.

An extended abstract with parts of the results appeared in Harks and Klimm [15]. They are also included in the thesis of the second author (Klimm [23, Chapters 4–5]).

2. Preliminaries. In this section, we fix notation and introduce congestion games with variable demands and congestion games with resource-dependent demands. Finally, we state and discuss the regularity assumptions imposed on the cost and utility functions.

Strategic games. A strategic game is a triple $G = (N, S, \pi)$, where $N = \{1, \dots, n\}$ is the nonempty and finite set of players, $S = \times_{i\in N} S_i$ is the nonempty strategy space, and π: $S \to \mathbb{R}^n$ is the combined *payoff function* assigning a payoff vector $\pi(s)$ to every strategy profile $s \in S$. We call G *finite* if S_i is finite for all $i \in N$. For $i \in N$, we write $S_{-i} = \times_{j\neq i} S_j$ and $s = (s_i, s_{-i})$ meaning that $s_i \in S_i$ and $s_{-i} \in S_{-i}$. A strategy profile $s \in S$ is a PNE if $\pi_i(s) \geq \pi_i(t_i, s_{-i})$ for all $i \in N$ and $t_i \in S_i$. For $\rho > 0$, we call a pair of strategy profiles $(s, (t_i, s_{-i})) \in S \times S$ a *ρ-improvement move* of player i if $\pi_i(t_i, s_{-i}) > \pi_i(s) + \rho$. A sequence $\gamma = (s^1, s^2, \dots)$ of strategy profiles is called a *ρ-improvement path* if for every k the tuple (s^k, s^{k+1}) is a ρ-improvement move of some player i. A closed ρ-improvement path $\gamma = (s^1, \dots, s^l, s^1)$ is called a *ρ-improvement cycle* (of length l) for which we drop the duplicate first entry by writing $\gamma = (s^1, \dots, s^l)$. For $\rho = 0$, instead of 0-improvement move, 0-improvement paths, and 0-improvement cycles, we will refer to improvement moves, improvement paths, and improvement cycles, respectively. A game has the FIP if all improvement paths are finite. A slightly weaker notion, which is in particular more suitable for infinite games, is the AFIP. A game has the AFIP if for all $\rho > 0$, all ρ-improvement paths are finite.

Congestion model. The games considered in this paper are all based on a *congestion model* defined as follows. Let R be a nonempty and finite set of resources, and for every $i \in N$ let $\mathscr{A}_i \subset 2^R \setminus \varnothing$ be a nonempty set of nonempty subsets of resources available to player i. Whenever a player i uses the resources in $\alpha_i \in \mathscr{A}_i$, we say that the resources in α_i are allocated to player i; we also call α_i an allocation of player i. Every resource $r \in R$ is endowed with a cost function c_r: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that maps the aggregated demand on r to a cost value. We call the tuple $\mathscr{M} = (N, R, (\mathscr{A}_i)_{i\in N}, (c_r)_{r\in R})$ a congestion model.

Congestion games with variable demands. In a *congestion game with variable demands*, we are given a congestion model $\mathscr{M}$ and, for every player $i \in N$, a utility function U_i: $[\sigma_i, \tau_i] \to \mathbb{R}_{\geq 0}$ where $[\sigma_i, \tau_i] \subseteq \mathbb{R}_{\geq 0}$, $\sigma_i \in \mathbb{R}_{\geq 0}$, $\tau_i \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, $\sigma_i \leq \tau_i$ is the interval of feasible demands of player i. We say that player i has an *unrestricted demand* if $\sigma_i = 0$ and $\tau_i = \infty$. The corresponding *proportional* congestion game with variable demands is the maximization game $G = (N, S, \pi)$ with $S_i = \mathscr{A}_i \times [\sigma_i, \tau_i]$ and $\pi_i(\alpha, d) = U_i(d_i) - \sum_{r\in\alpha_i} d_i c_r(\ell_r(\alpha, d))$ for all $i \in N$, where $\ell_r(\alpha, d) = \sum_{j\in N: r\in\alpha_j} d_j$ is called the *load* or *aggregated demand* of resource r under strategy profile (α, d). In the corresponding *uniform* congestion game with variable demands the private payoff is defined as $\pi_i(\alpha, d) = U_i(d_i) - \sum_{r\in\alpha_i} c_r(\ell_r(\alpha, d))$. To treat uniform games and proportional games simultaneously, it is convenient to introduce games, in which the costs on the resources are scaled by a scaling function g: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $g(x) = \beta x + \eta$, $\beta, \eta \in \{0, 1\}$, $\beta + \eta = 1$, i.e., the private payoff is $\pi_i(\alpha, d) = U_i(d_i) - g(d_i) \sum_{r\in\alpha_i} c_r(\ell_r(\alpha, d))$.

Congestion games with resource-dependent demands. In *congestion games with resource-dependent demands* we are given a congestion model $\mathscr{M}$ and, for every player $i \in N$, a vector $(d_{i,r})_{r\in R}$ of nonnegative resource-dependent demands. The corresponding *proportional* congestion game with resource-dependent demands is the strategic game $G = (N, S, \pi)$ with $S_i = \mathscr{A}_i$ and $\pi_i(s) = -\sum_{r\in s_i} d_{i,r} c_r(\ell_r(s))$ for all $i \in N$, where $\ell_r(s) = \sum_{j\in N: r\in s_j} d_{j,r}$. The corresponding *uniform* congestion game with resource-dependent demands has the same strategies but the payoff function of every player i is defined as $\pi_i(s) = -\sum_{r\in s_i} c_r(\ell_r(s))$ and, for a scaling function g, the g-scaled congestion game with resource-dependent demands has the payoff $\pi_i(s) = \sum_{r\in s_i} -g(d_{i,r})c_r(\ell_r(s))$. For the special case that $d_{i,r} = d_{i,r'}$ for all $i \in N$ and $r, r' \in R$, we call the game a (proportional, uniform, or g-scaled) *weighted congestion game*.

Consistency of cost functions. Let $\mathscr{C}$ be a nonempty set of cost functions and let g be a scaling function. We say that $\mathscr{C}$ is *consistent* for g-scaled congestion games with variable demands if every g-scaled congestion game with variable demands and cost functions in $\mathscr{C}$ possesses a PNE. Furthermore, we call $\mathscr{C}$ *approximately universally consistent* (respectively, universally consistent) for g-scaled congestion games with variable demands if every such game with cost functions in $\mathscr{C}$ has the AFIP (respectively, the FIP). Consistency for g-scaled congestion games with resource-dependent demands and g-scaled weighted congestion games is defined accordingly. For a cost function c, instead of saying that $\{c\}$ is consistent we simply say that c is consistent.

Regularity assumptions. Throughout this paper, we impose the following regularity assumption on the cost and utility functions.

Assumption 1. *For every resource $r \in R$, the cost function c_r: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is locally Lipschitz continuous, nondecreasing, and strictly positive on $\mathbb{R}_{>0}$.*

The assumption that cost functions are locally Lipschitz continuous is rather weak as, e.g., every continuously differentiable function has this property. In contrast to most of the works in the area of congestion games with splittable demands (e.g., Haurie and Marcotte [18], Kelly et al. [22], and Orda et al. [30]), we do not assume semiconvexity of the cost functions.

For congestion games with variable demands, we further impose the following assumption on the utility functions.

Assumption 2. *For every player $i \in N$, the utility function U_i: $[\sigma_i, \tau_i] \to \mathbb{R}_{\geq 0}$ is continuous, nondecreasing, and concave.*

Since every U_i is concave, for every $x \in (\sigma_i, \tau_i)$ the left and right derivatives, denoted by $(\partial^-/\partial x)U_i(x)$, respectively, $(\partial^+/\partial x)U_i(x)$ exist and satisfy $(\partial^-/\partial x)U_i(x) \geq (\partial^+/\partial x)U_i(x) \geq (\partial^-/\partial y)U_i(y) \geq (\partial^+/\partial y)U_i(y)$ for all $\sigma_i < x < y < \tau_i$; see Webster [41, Theorem 5.1.3] for a reference.

3. Necessary conditions on the existence of a pure Nash equilibrium. In previous work (Harks and Klimm [16]) we have shown that a set $\mathscr{C}$ of continuous cost functions is consistent for proportional weighted congestion games if and only if at least one of the following two cases holds: (i) $\mathscr{C}$ only contains affine functions; *or* (ii) $\mathscr{C}$ only contains exponential functions of type $c(x) = a_c e^{\phi x} + b_c$, where $a_c, b_c \in \mathbb{R}$ may depend on c, while ϕ is equal for all $c \in \mathscr{C}$. This characterization is even valid for games with three players. We start with the useful observation that these conditions are also necessary for games with resource-dependent and variable demands, respectively.

Proposition 1. *Let $\mathscr{C} \neq \varnothing$ be a set of continuous functions and let g be a scaling function such that one of the following two cases holds*:

A. *$\mathscr{C}$ is consistent for g-scaled congestion games with resource-dependent demands.*
B. *$\mathscr{C}$ is consistent for g-scaled congestion games with variable demands.*

Then, $\mathscr{C}$ satisfies at least one of the following two conditions:

1. *$\mathscr{C}$ only contains affine functions.*
2. *$\mathscr{C}$ only contains exponential functions of type $c(x) = a_c e^{\phi x} + b_c$, where $a_c, b_c \in \mathbb{R}$ may depend on c, while $\phi \in \mathbb{R}$ is equal for all $c \in \mathscr{C}$.*

Proof. We show that for every set $\mathscr{C}$ of cost functions, that neither satisfies 1 nor 2, both A and B are violated. Let $\mathscr{C}$ be such a set of cost functions. The main result in Harks and Klimm [16, Theorem 5.1] shows the existence of a congestion model $\mathscr{M} = (N, R, (\mathscr{A}_i)_{i\in N}, (c_r)_{r\in R})$ and resource-*independent* demands $d_i^{\mathrm{w}} > 0$, $i \in N$ such that the corresponding proportional weighted congestion game $G^{\mathrm{w}} = (N, S, \pi^w)$ with $S_i = \mathscr{A}_i$ for all $i \in N$ does not have a PNE.

For an arbitrary scaling function g, consider the g-scaled congestion games with resource-dependent demands $G^{\mathrm{rd}} = (N, S, \pi^{\mathrm{rd}})$ with $S_i = \mathscr{A}_i$ for all $i \in N$ and $d_{i,r} = d_i^{\mathrm{w}}$ for all $r \in R$. We observe that $\pi_i^{\mathrm{rd}}(s) = (g(d_i)/d_i)\pi_i^{\mathrm{w}}(s)$ for all $s \in S$ and $i \in N$. As $g(d_i)/d_i > 0$ we derive that G^{rd} does not possess a PNE and, thus, A is violated. Next, let $G^{\mathrm{var}} = (N, S, \pi^{\mathrm{var}})$ be a g-scaled congestion game with variable demands and with $S_i = \mathscr{A}_i \times [\sigma_i, \tau_i]$ for all $i \in N$, where $\sigma_i = \tau_i = d_i^{\mathrm{w}}$. By construction, $\pi_i^{\mathrm{var}}(\alpha, d) = U_i(d_i^{\mathrm{w}}) + (g(d_i)/d_i)\pi_i^{w}(\alpha)$ for all $(\alpha, d) \in S$ and $i \in N$. We conclude that G^{var} does not possess a PNE and B is violated. □

4. Pure Nash equilibria in congestion games with resource-dependent demands. Proposition 1 implies in particular that every consistent cost function is infinitely often differentiable. We proceed to present a strong necessary condition on the consistency of a differentiable cost function c, which we term *generalized monotonicity condition*.

Definition 1 (Generalized Monotonicity Condition (GMC)). Let g be a scaling function. A differentiable cost function c satisfies the *generalized monotonicity condition* (*GMC*) *for* g if for all $x, y \in \mathbb{R}_{>0}$ with $c(x) \neq 0$, $c(y) \neq 0$, and all $\mu \in \mathbb{R}_{\geq 0}$ the following two conditions hold:

G1. If $c(x) > c(x+y) - \mu c'(x+y)$,
then $(1-\mu(g'(y)/g(y)))c(x+y) - \mu c'(x+y) \leq (1-\mu(g'(y)/g(y)))c(y) - \mu c'(y)$.

G2. If $c(x) < c(x+y) - \mu c'(x+y)$,
then $(1-\mu(g'(y)/g(y)))c(x+y) - \mu c'(x+y) \geq (1-\mu(g'(y)/g(y)))c(y) - \mu c'(y)$.

For $\mu = 0$, the GMC is independent of the scaling function g and ensures only that c is monotonic. We proceed to show that any differentiable cost function that is consistent for g-scaled congestion games with resource-dependent demands satisfies the GMC for g.

Lemma 1 (Generalized Monotonicity Lemma). *Let c be a differentiable function and let g be a scaling function. If c is consistent for g-scaled congestion games with resource-dependent demands, then c satisfies the GMC for g.*

Proof. We start to show that c satisfies G1. Suppose not. Then, there are $x, y \in \mathbb{R}_{>0}$ with $c(x) \neq 0$ and $c(y) \neq 0$, and $\mu \in \mathbb{R}_{\geq 0}$ such that the following two inequalities hold:

$$c(x) > c(x+y) - \mu c'(x+y), \tag{1}$$

$$\left(\mu \frac{g'(y)}{g(y)} - 1\right) c(y) + \mu c'(y) > \left(\mu \frac{g'(y)}{g(y)} - 1\right) c(x+y) + \mu c'(x+y). \tag{2}$$

As the terms on the left-hand side and right-hand side of (1) and (2) are continuous in μ, it is without loss of generality to assume that μ is rational and positive, i.e., $\mu = p/q$ for some $p, q \in \mathbb{N}$. Let $\epsilon > 0$ be such that

$$c(x) > c(x+y) - \mu c'(x+y) + \epsilon,$$

$$\left(\mu \frac{g'(y)}{g(y)} - 1\right) c(y) + \mu c'(y) > \left(\mu \frac{g'(y)}{g(y)} - 1\right) c(x+y) + \mu c'(x+y) + \epsilon.$$

Since the functions c and g are differentiable (and thus also continuous), there is $m \in \mathbb{N}$ such that for $\delta = 1/(q \cdot m)$ we have

$$c(y+\delta) \neq 0, \tag{3a}$$

$$\left| \mu \cdot \frac{c(x+y+\delta) - c(x+y)}{\delta} - \mu c'(x+y) \right| \leq \frac{\epsilon}{4}, \tag{3b}$$

$$\left| \mu \cdot \frac{c(y+\delta) - c(y)}{\delta} - \mu c'(y) \right| \leq \frac{\epsilon}{4}, \tag{3c}$$

$$\left| \frac{\mu}{g(y)} \cdot \frac{g(y+\delta) - g(y)}{\delta} \cdot c(x+y+\delta) - \mu \cdot \frac{g'(y)}{g(y)} \cdot c(x+y) \right| \leq \frac{\epsilon}{4}, \tag{3d}$$

$$\left| \frac{\mu}{g(y)} \cdot \frac{g(y+\delta) - g(y)}{\delta} \cdot c(y+\delta) - \mu \cdot \frac{g'(y)}{g(y)} \cdot c(y) \right| \leq \frac{\epsilon}{4}. \tag{3e}$$

Our proof proceeds in two steps. In the first step, we construct a g-scaled congestion game with resource-dependent demands parameterized by $b_1, b_2 \in \mathbb{Z}$, and $a \in \mathbb{N}_{>0}$. Then, in the second step, we specify these parameters such that the corresponding game does not possess a PNE.

For the first step, let the parameters $b_1, b_2 \in \mathbb{Z}$, and $a \in \mathbb{N}$ be fixed. We write $b_1 = b_1^+ - b_1^-$ with $b_1^+, b_1^- \in \{0, |b_1|\}$ and $b_2 = b_2^+ - b_2^-$ with $b_2^+, b_2^- \in \{0, |b_2|\}$. Consider the congestion model $\mathcal{M} = (N, R, (\mathcal{A}_i)_{i \in N}, (c_r)_{r \in R})$ with two players $N = \{1, 2\}$. The set of resources R is partitioned into the mutually disjoint sets R_1, R_2, Q_1^+, Q_1^-, Q_2^+, and Q_2^-. The set R_1 contains $a(p \cdot m + 1)$ resources, R_2 contains $a \cdot p \cdot m$ resources, Q_1^+ contains b_1^+ resources, Q_1^- contains b_1^- resources, Q_2^+ contains b_2^+ resources, and Q_2^- contains b_2^- resources. We set $\mathcal{A}_1 = \{R_1 \cup R_2 \cup Q_1^-, Q_1^+\}$ and $\mathcal{A}_2 = \{R_1 \cup Q_2^-, R_2 \cup Q_2^+\}$. The demand of player 1 equals $d_{1,r} = x$ for all resources

Player 1	Player 2: $R_1 \cup Q_2^-$	Player 2: $R_2 \cup Q_2^+$
$R_1 \cup R_2 \cup Q_1^-$	$-ag(x)((p \cdot m+1)c(x+y)+p \cdot m \cdot c(x)) - b_1^- g(x)c(x)$, $-ag(y)(p \cdot m+1)c(x+y) - b_2^- g(y)c(y)$	$-ag(x)((p \cdot m+1)c(x)+p \cdot m \cdot c(x+y+\delta)) - b_1^- g(x)c(x)$, $-ag(y+\delta)p \cdot m \cdot c(x+y+\delta) - b_2^+ g(y+\delta)c(y+\delta)$
Q_1^+	$-b_1^+ g(x)c(x)$, $-ag(y)(p \cdot m+1)c(y) - b_2^- g(y)c(y)$	$-b_1^+ g(x)c(x)$, $-ag(y+\delta)p \cdot m \cdot c(y+\delta) - b_2^+ g(y+\delta)c(y+\delta)$

FIGURE 1. Bimatrix representation of the g-scaled congestion game with variable demands constructed in the proof of Lemma 1. The rows and columns of the bimatrix correspond to the strategies of players 1 and 2, respectively. Every entry corresponds to a strategy profile of the game and shows the payoff of player 1 on top and the payoff of player 2 below.

$r \in R$ and the demand of player 2 equals $d_{2,r} = y$ for all $r \in R_1 \cup Q_2^-$ and $d_{2,r} = y+\delta$ for all $r \in R_2 \cup Q_2^+$. A bimatrix representation of the so-defined game is shown in Figure 1.

Let $z = y+\delta$, if $b_2 \geq 0$ and $z = y$, if $b_2 < 0$. Consider the cycle

$$\gamma = ((Q_1^+, R_1 \cup Q_2^-), (R_1 \cup R_2 \cup Q_1^-, R_1 \cup Q_2^-), (R_1 \cup R_2 \cup Q_1^-, R_2 \cup Q_2^+), (Q_1^+, R_2 \cup Q_2^+)).$$

Calculating the differences in the payoffs of the deviating players, we obtain

$$\pi_1(R_1 \cup R_2 \cup Q_1^-, R_1 \cup Q_2^-) - \pi_1(Q_1^+, R_1 \cup Q_2^-) = -ag(x)\left((p \cdot m+1)c(x+y) + p \cdot m \cdot c(x) - \frac{b_1}{a}c(x)\right), \tag{4a}$$

$$\begin{aligned}&\pi_2(R_1 \cup R_2 \cup Q_1^-, R_2 \cup Q_2^+) - \pi_2(R_1 \cup R_2, R_1 \cup Q_2^-)\\ &\quad = -ag(y)\left(\frac{g(y+\delta)}{g(y)}(p \cdot m \cdot c(x+y+\delta)) + \frac{b_2}{a} \cdot \frac{g(z)}{g(y)}c(z) - (p \cdot m+1)c(x+y)\right),\end{aligned} \tag{4b}$$

$$\pi_1(Q_1^+, R_2 \cup Q_2^+) - \pi_1(R_1 \cup R_2 \cup Q_1^-, R_2 \cup Q_2^+) = -ag(x)\left(\frac{b_1}{a}c(x) - (p \cdot m+1)c(x) - p \cdot m \cdot c(x+y+\delta)\right), \tag{4c}$$

$$\pi_2(Q_1^+, R_1 \cup Q_2^-) - \pi_2(Q_1, R_2 \cup Q_2^+) = -ag(y)\left((p \cdot m+1)c(y) - \frac{g(y+\delta)}{g(y)}(p \cdot m \cdot c(y+\delta)) - \frac{b_2}{a} \cdot \frac{g(z)}{g(y)}c(z)\right). \tag{4d}$$

We proceed to show that there are $b_1, b_2 \in \mathbb{Z}$, and $a \in \mathbb{N}_{>0}$ such that the expressions in (4a)–(4d) are strictly positive, implying that γ is an improvement cycle and that the corresponding game does not possess a PNE. Using that a, $g(x)$, and $g(y)$ are strictly positive and substituting $\beta_1 = b_1/a$ and $\beta_2 = b_2/a$, this is equivalent to finding $\beta_1, \beta_2 \in \mathbb{Q}$ such that

$$p \cdot m \cdot c(x) + (p \cdot m+1)c(x+y) < \beta_1 c(x) < p \cdot m \cdot c(x+y+\delta) + (p \cdot m+1)c(x), \tag{5}$$

$$\left(\mu\frac{g'(y)}{g(y)} - 1\right)c(x+y) + \mu c'(x+y) + \frac{\epsilon}{2} < -\beta_2 c(z)\frac{g(z)}{g(y)} < \left(\mu\frac{g'(y)}{g(y)} - 1\right)c(y) + \mu c'(y) - \frac{\epsilon}{2}, \tag{6}$$

where the inequalities in (5) and (6) are associated with the payoff differences of the first and second player in γ, respectively.

First, we show that there is $\beta_1 \in \mathbb{Q}$ such that (5) holds. To this end, note that $c(x) \neq 0$ and $c(x) > c(x+y) - \mu c'(x+y) + \epsilon$ together imply the existence of $\beta_1 \in \mathbb{Q}$ with

$$c(x) > \beta_1 c(x) - p \cdot m \cdot c(x+y+\delta) - p \cdot m \cdot c(x) > c(x+y) - \mu c'(x+y) + \epsilon.$$

Adding $p \cdot m \cdot c(x+y+\delta) + p \cdot m \cdot c(x)$ to all terms and using $p \cdot m = \mu/\delta$ we obtain

$$\begin{aligned}&p \cdot m \cdot c(x+y+\delta) + (p \cdot m+1)c(x)\\ &\quad > \beta_1 c(x) > \mu\frac{c(x+y+\delta) - c(x+y)}{\delta} + \mu\frac{c(x+y)}{\delta} + p \cdot m \cdot c(x) + c(x+y) - \mu c'(x+y) + \epsilon.\end{aligned}$$

Rearranging terms, using (3b) and again $p \cdot m = \mu/\delta$, we obtain

$$p \cdot m \cdot c(x+y+\delta) + (p \cdot m+1)c(x) > \beta_1 c(x) > p \cdot m \cdot c(x) + (p \cdot m+1)c(x+y),$$

as claimed.

We proceed in a similar fashion to show that there is $\beta_2 \in \mathbb{Q}$ such that (6) holds. First, we use $(\mu(g'(y)/g(y))-1)\cdot c(x+y)+\mu c'(x+y)+\epsilon/2<(\mu(g'(y)/g(y))-1)c(y)+\mu c'(y)-\epsilon/2$ and (3a) to derive the existence of $\beta_2 \in \mathbb{Q}$ with

$$\left(\mu\frac{g'(y)}{g(y)}-1\right)c(x+y)+\mu c'(x+y)+\frac{\epsilon}{2}<-\beta_2 c(z)\frac{g(z)}{g(y)}<\left(\mu\frac{g'(y)}{g(y)}-1\right)c(y)+\mu c'(y)-\frac{\epsilon}{2}.$$

Using (3b) and (3c), we incur at most an error of $\epsilon/4$ when replacing $\mu c'(x+y)$ by $(\mu/\delta)(c(x+y+\delta)-c(x+y))$, and $\mu c'(y)$ by $(\mu/\delta)(c(y+\delta)-c(y))$. Using $p\cdot m=\mu/\delta$ this gives rise to

$$\begin{aligned}&\mu\frac{g'(y)}{g(y)}c(x+y)+p\cdot m\cdot c(x+y+\delta)-(p\cdot m+1)c(x+y)+\frac{\epsilon}{4}\\&\quad<-\beta_2 c(z)\frac{g(z)}{g(y)}<\mu\frac{g'(y)}{g(y)}c(y)+p\cdot m\cdot c(y+\delta)-(p\cdot m+1)c(y)-\frac{\epsilon}{4}.\end{aligned}$$

Now, using (3d) and (3e), we obtain

$$\begin{aligned}&\left(p\cdot m\frac{g(y+\delta)}{g(y)}-p\cdot m\right)c(x+y+\delta)+p\cdot m\cdot c(x+y+\delta)-(p\cdot m+1)c(x+y)\\&\quad<-\beta_2 c(z)\frac{g(z)}{g(y)}<\left(p\cdot m\frac{g(y+\delta)}{g(y)}-p\cdot m\right)c(y+\delta)+p\cdot m\cdot c(y+\delta)-(p\cdot m+1)c(y).\end{aligned}$$

Rearranging terms, we finally obtain

$$\frac{g(y+\delta)}{g(y)}p\cdot m\cdot c(x+y+\delta)-(p\cdot m+1)c(x+y)<-\beta_2 c(z)\frac{g(z)}{g(y)}<\frac{g(y+\delta)}{g(y)}p\cdot m\cdot c(y+\delta)-(p\cdot m+1)c(y),$$

as desired. By construction, γ is an improvement cycle and because every strategy combination is contained in γ, we conclude that the constructed game does not possess a PNE, which contradicts the consistency of c. To see that c also satisfies G2 we proceed as above, but traverse the cycle γ in the opposite direction. □

The games constructed to prove the generalized monotonicity lemma (Lemma 1) have a simple structure: every game has only two players with two strategies each. The first player has a single demand $x\in\mathbb{R}_{>0}$ that she places on all resources. For some $y,\delta\in\mathbb{R}_{>0}$, the second player's demand equals y for all resources contained in her first strategy and $y+\delta$ for all other resources. With these observations, Lemma 1 can be strengthened in the following way.

Corollary 1. *Let g be a scaling function and let c be a differentiable function not satisfying the GMC for g. Then, there are $x,y,\epsilon\in\mathbb{R}_{>0}$ such that for every $\delta\in(0,\epsilon)$, there is a congestion model $\mathcal{M}=(N,R,(\mathcal{A}_i)_{i\in N},(c_r)_{r\in R})$ with the following properties*:

1. $N=\{1,2\}$.
2. *Every player i has two disjoint allocations, i.e., $\mathcal{A}_i=\{\alpha_{i,1},\alpha_{i,2}\}$ with $\alpha_{i,1},\alpha_{i,2}\in 2^R$.*
3. *All cost functions are equal to c, i.e., $c_r=c$ for all $r\in R$.*
4. *The corresponding g-scaled congestion game with the resource-dependent demands $d_{1,r}=x$ for all $r\in R$, $d_{2,r}=y$ for all $r\in\alpha_{2,1}$, and $d_{2,r}=y+\delta$ for all $r\in\alpha_{2,2}$ does not possess a PNE.*

In other words, for every cost function c not satisfying the GMC for g, there is a *threshold* value $\epsilon>0$ such for all $\delta\in(0,\epsilon)$ one can construct a simple game (two players with two feasible allocations each, all costs equal to c) without a PNE. Moreover, the players' demands in this game are *almost resource independent*, that is, only the demand of the second player on the resources contained in her second allocation is increased by δ, which can be made arbitrarily small. This insight will be important in §5 where we characterize consistency for congestion games with variable demands for congestion games with variable demands.

4.1. Consistency of affine functions. As noted in Proposition 1, every set of continuous cost functions that is consistent for g-scaled congestion games with resource-dependent demands contains either only affine or only certain exponential functions. In this section, we investigate the question whether affine functions are indeed consistent. Whereas in weighted congestion games, the distinction between proportional and uniform games is irrelevant for the existence of a PNE, it matters for games with resource-dependent demands. We will show that the set of affine functions is consistent for proportional games with resource-dependent demands, but is not consistent for uniform games with resource-dependent demands.

THEOREM 1. *Let $\mathscr{C}$ be a set of affine functions that contains a nonconstant function. Then, $\mathscr{C}$ is consistent and universally consistent for proportional congestion games with resource-dependent demands, but not consistent for uniform congestion games with resource-dependent demands.*

PROOF. For proportional games, the statement follows from the potential function that has been given in Harks et al. [17]. Thus, it is sufficient to show that $\mathscr{C}$ is not consistent for uniform games. For uniform games, we have $g'(x) = 0$ for all $x \geq 0$. As c is nonconstant, we find $x, y \in \mathbb{R}_{>0}$ such that $c(x) \neq 0$, $c(y) \neq 0$. We set $\mu = y + 1$ and observe that

$$c(x) - c(x+y) + \mu c'(x+y) = -c'(x+y)y + \mu c'(x+y) = c'(x+y),$$
$$\left(\mu \frac{g'(y)}{g(y)} - 1\right) c(x+y) + \mu c'(x+y) - \left(\mu \frac{g'(y)}{g(y)} - 1\right) c(y) - \mu c'(y) = c(y) - c(x+y) = -xc'(x+y),$$

which have opposite signs. Thus, either G1 or G2 is violated. □

4.2. Consistency of exponential functions. We proceed to show that there is no scaling function such that exponential functions are consistent for g-scaled congestion games with resource-dependent demands.

THEOREM 2. *Let $\phi > 0$ and let c be a nonconstant exponential function of type $c(x) = ae^{\phi x} + b$, $a \neq 0$, $b \in \mathbb{R}$. Then, c satisfies the GMC for a scaling function g if and only if g satisfies the differential equation $g'(x) = (\phi/(e^{\phi x} - 1))g(x)$.*

PROOF. Let c be of the demanded form. For a contradiction, let us assume that c satisfies the GMC for g, but g is not as claimed. The GMC implies that for all $x, y, \mu > 0$ with $c(x) \neq 0$ and $c(y) \neq 0$ the expressions

$$c(x) - c(x+y) + \mu c'(x+y) = ae^{\phi x}(1 + e^{\phi y}(\mu\phi - 1)) \tag{7}$$
$$\left(\mu \frac{g'(y)}{g(y)} - 1\right) c(x+y) + \mu c'(x+y) - \left(\mu \frac{g'(y)}{g(y)} - 1\right) c(y) - \mu c'(y) = ae^{\phi y}(e^{\phi x} - 1)\left(\mu \frac{g'(y)}{g(y)} + \mu\phi - 1\right) \tag{8}$$

have equal signs.

We first make the useful observation that the monotonicity of exponential functions implies that c has at most one root. Together with the continuity of the expressions in (7) and (8), this implies that if there are $x, y \in \mathbb{R}_{>0}$ such that (7) and (8) are nonzero and have different signs, then we can choose x and y such that additionally $c(x) \neq 0$ and $c(y) \neq 0$. Furthermore, we observe that altering the sign of a also alters the sign of the right-hand sides both of (7) and (8). From there, it is without loss of generality to assume that $a > 0$.

If there is $y > 0$ such that $g'(y)/g(y) + \phi \leq 0$, we choose $\mu > 1/\phi$ and $x > 0$ arbitrarily. Then, (7) is positive whereas (8) is negative, and we reach a contradiction. We conclude that $g'(y)/g(y) + \phi > 0$ for all $y > 0$. This implies, that (8) is positive if and only if $\mu > g(y)/(g'(y) + \phi g(y))$ whereas (7) is positive if and only if $\mu > (e^{\phi y} - 1)/\phi e^{\phi y}$. For every

$$\mu \in \left(\min\left\{\frac{g(y)}{g'(y) + \phi g(y)}, \frac{e^{\phi y} - 1}{\phi e^{\phi y}}\right\}, \max\left\{\frac{g(y)}{g'(y) + \phi g(y)}, \frac{e^{\phi y} - 1}{\phi e^{\phi y}}\right\}\right),$$

the expressions are nonzero and have different signs. We conclude that $g(y)/(g'(y) + \phi g(y)) = (e^{\phi y} - 1)/\phi e^{\phi y}$ for all $y > 0$. Rearranging the terms, we derive that g satisfies the claimed differential equation. □

It is easy to verify that there is no affine function that satisfies the differential equation of Theorem 2. In fact, one can show that all solutions to this differential equation are of the form $g(x) = \beta(e^{-\phi x} - 1)$ with $\beta \in \mathbb{R}$.

4.3. A characterization of consistency for games with resource-dependent demands. We are now ready to state the main result of this section.

THEOREM 3. *Let $\mathscr{C} \neq \varnothing$ be a set of continuous functions and let g be a scaling function. Then, the following are equivalent*:

(1) *$\mathscr{C}$ is consistent for g-scaled congestion games with resource-dependent demands.*
(2) *$\mathscr{C}$ is universally consistent for g-scaled congestion games with resource-dependent demands.*
(3) *At least one of the following two cases holds*:
  (a) *$\mathscr{C}$ contains only constant functions.*
  (b) *$g(x) \equiv x$, and $\mathscr{C}$ contains only affine functions.*

Proof. (2) $\Rightarrow$ (1) follows because every game with the FIP has a PNE.

(1) $\Rightarrow$ (3): Referring to Proposition 1, consistency of $\mathscr{C}$ implies that one of the following two cases holds: (i) $\mathscr{C}$ contains only affine functions; (ii) $\mathscr{C}$ contains only exponential functions of type $c(x) = a_c e^{\phi x} + b_c$, where $a_c, b_c \in \mathbb{R}$ may depend on c while $\phi \in \mathbb{R}$ is equal for all $c \in \mathscr{C}$. Let us first consider the case that $\mathscr{C}$ contains only affine functions. If all functions in $\mathscr{C}$ are constant, then (3)(a) is satisfied. If $\mathscr{C}$ contains a nonconstant function, Theorem 1 establishes that g is linear and (3)(b) is satisfied. For the case that $\mathscr{C}$ contains a nonconstant exponential function, the generalized monotonicity lemma (Lemma 1) and Theorem 2 imply that $\mathscr{C}$ is not consistent.

(3) $\Rightarrow$ (2): First, we assume (3)(a). Let $G = (N, S, \pi)$ be a g-scaled congestion game with resource-dependent demands where all cost functions are constant. Then, the payoff of every player i does not depend on the strategies of all other players. We derive that the function $P\colon S \to \mathbb{R}$ defined as $P(s) = \sum_{i \in N} \pi_i(s)$ for all $s \in S$ is an exact potential function of G. Finite potential games have the FIP. The implication (3)(b). $\Rightarrow$ (2) is shown in Theorem 1. □

As Proposition 1 is even valid for three-player games, and the proof of the general monotonicity lemma 1 requires only two players, the characterization of consistency stated in Theorem 3 is even valid for three-player games.

5. Pure Nash equilibria in congestion games with variable demands. The characterization of consistency for games with resource-dependent demands obtained in §4 will be the main building block when analyzing the consistency for games with variable demands. The following lemma states necessary conditions for a PNE in games with differentiable cost functions and will be useful in the remainder of this section.

Lemma 2. *Let G be a g-scaled congestion game with variable demands and differentiable cost functions. If (α, d) is a PNE of G, then for all $i \in N$ the following two conditions hold*:

(1) *If $d_i < \tau_i$, then $(\partial^+/\partial d_i) U_i(d_i) \le (\partial/\partial d_i)(g(d_i) \sum_{r \in \alpha_i} c_r(\ell_r(\alpha, d)))$.*

(2) *If $d_i > \sigma_i$, then $(\partial^-/\partial d_i) U_i(d_i) \ge (\partial/\partial d_i)(g(d_i) \sum_{r \in \alpha_i} c_r(\ell_r(\alpha, d)))$.*

Sketch of proof. In a PNE, no player may gain from unilaterally altering her demand. Thus, the above conditions on the local optimality of the demands are necessary for a PNE. □

5.1. Homogeneously exponential cost functions. In this section, we show that for any scaling function g, homogeneously exponential functions are consistent for g-scaled congestion games with variable demands. For proving this, we introduce a novel concept termed *essential improvement moves*. A subset of improvement moves is called *essential* if every player that has an improvement move from a strategy profile s also has an essential improvement move from s. Formally, let $G = (N, S, \pi)$ be a maximization game and let $I = \{(s, (s_i', s_{-i})) \in S \times S\colon \pi_i(s) < \pi_i(s_i', s_{-i})\}$ denote the set of improvement moves of G. A subset $I' \subseteq I$ of improvement moves is called *essential* if $\{s'\colon (s, s') \in I'\} = \varnothing$ implies $\{s'\colon (s, s') \in I\} = \varnothing$ for all $s \in S$. Such subsets exist since the set of improvement moves I itself is essential.

We proceed to show that for congestion games with variable demands in which all cost functions are homogeneously exponential, there is an essential subset of improvement moves that is a strict subset of the set of improvement moves. In fact, we will show that for every strategy profile $s = (\alpha, d)$ for which there is a player i who improves when switching from $s_i = (\alpha_i, d_i)$ to $s_i' = (\alpha_i', d_i')$, this player i can also improve by either adapting her demand or changing her allocation. That is, one of the two strategies of the form $s_i'' = (\alpha_i, d_i')$ or $s_i''' = (\alpha_i', d_i)$ also yields an improvement for player i.

Lemma 3. *Let G be a g-scaled congestion game with variable demands such that all cost functions are of type $c(x) = a_c e^{\phi x}$, where a_c may depend on c and ϕ is equal for all $c \in \mathscr{C}$. Let I be the set of improvement moves of G. Then, $I' = \{((\alpha, d), (\alpha', d')) \in I\colon \alpha = \alpha'\} \cup \{((\alpha, d), (\alpha', d')) \in I\colon d = d'\}$ is an essential subset of improvement moves.*

Proof. For a contradiction, let us assume that $((\alpha, d), (\alpha_i', \alpha_{-i}, d_i', d_{-i}))$ is an improvement move of player i but $((\alpha, d), (\alpha, d_i', d_{-i}))$ and $((\alpha, d), (\alpha_i', \alpha_{-i}, d))$ are not. We use $\ell_r(\alpha_{-i}, d_{-i})$ to denote the aggregated demands of all players $j \in N \setminus \{i\}$ when playing $s_j = (\alpha_j, d_j)$. We obtain

$$\begin{aligned}\pi_i(\alpha, d_i', d_{-i}) - \pi_i(\alpha, d)\\ = U_i(d_i') - U_i(d_i) - g(d_i') e^{\phi d_i'} \sum_{r \in \alpha_i} a_r e^{\phi \ell_r(\alpha_{-i}, d_{-i})} + g(d_i) e^{\phi d_i} \sum_{r \in \alpha_i} a_r e^{\phi \ell_r(\alpha_{-i}, d_{-i})} \le 0,\end{aligned} \tag{9a}$$

$$\pi_i(\alpha_i', \alpha_{-i}, d) - \pi_i(\alpha, d) = -g(d_i)e^{\phi d_i} \sum_{r\in\alpha_i'} a_r e^{\phi \ell_r(\alpha_{-i}, d_{-i})} + g(d_i)e^{\phi d_i} \sum_{r\in\alpha_i} a_r e^{\phi \ell_r(\alpha_{-i}, d_{-i})} \le 0, \tag{9b}$$

$$\pi_i(\alpha_i', \alpha_{-i}, d_i', d_{-i}) - \pi_i(\alpha, d) = U_i(d_i') - U_i(d_i) - g(d_i')e^{\phi d_i'} \sum_{r\in\alpha_i'} a_r e^{\phi \ell_r(\alpha_{-i}, d_{-i})} + g(d_i)e^{\phi d_i} \sum_{r\in\alpha_i} a_r e^{\phi \ell_r(\alpha_{-i}, d_{-i})} > 0. \tag{9c}$$

The last inequality expresses the fact that $((\alpha, d), (\alpha_i', \alpha_{-i}, d_i', d_{-i}))$ is an improvement move for G. Subtracting (9a) from (9c), we obtain

$$-g(d_i')e^{\phi d_i'}\left(\sum_{r\in\alpha_i'} a_r e^{\phi \ell_r(\alpha_{-i}, d_{-i})} - \sum_{r\in\alpha_i} a_r e^{\phi \ell_r(\alpha_{-i}, d_{-i})}\right) > 0,$$

a contradiction to (9b). □

Next, we use the above lemma to prove that every congestion game with variable demands and homogeneously exponential costs possesses a PNE.

Theorem 4. *Let g be a scaling function and $\mathscr{C} \neq \emptyset$ be a set of functions of type $c(x) = a_c e^{\phi x}$, where $a_c \in \mathbb{R}_{>0}$ may depend on c and $\phi \in \mathbb{R}_{>0}$ is equal for all $c \in \mathscr{C}$. Then, $\mathscr{C}$ is consistent for g-scaled congestion games with variable demands.*

Proof. Let $\phi \in \mathbb{R}_{>0}$ and let $\mathscr{M} = (N, R, (\mathscr{A}_i)_{i\in N}, (c_r)_{r\in R})$ be a congestion model with the property that for all $r \in R$, there is $a_r \in \mathbb{R}_{>0}$ such that $c_r(x) = a_r e^{\phi x}$ for all $x \in \mathbb{R}_{\ge 0}$. Let g be a scaling function, $(U_i)_{i\in N}$ a set of utility functions, and G a corresponding g-scaled congestion game with variable demands. We first show that for every $i \in N$, there is $\omega_i < \infty$ such that for every PNE $s = (\alpha, d)$ of G we have $d_i \le \omega_i$ for all $i \in N$. For players with $\tau_i < \infty$, we trivially obtain $\omega_i = \tau_i$. So, let i be a player with $\tau_i = \infty$ and let $a = \min_{r\in R} a_r$. (The minimum exists as R is finite.) The marginal cost of every player i when playing a demand $x > 0$ can be bounded from below by $g'(x)ae^{\phi x} + g(x)a\phi e^{\phi x} \ge g(x)a\phi e^{\phi x}$ because each allocation contains at least one resource. Using that g is nondecreasing, we derive that $g(x)a\phi e^{\phi x}$ diverges to ∞ as x goes to ∞. This implies that for every player i there is $\omega_i > \sigma_i$ such that $g(x)a\phi e^{\phi x} > (\partial^+/\partial d_i)U_i(0)$ for all $x > \omega_i$. Using Lemma 2 together with the fact that utility functions are concave, we obtain that $d_i \le \omega_i$ for all $i \in N$ and every PNE $s = (\alpha, d)$ of G.

Consider the function Φ: $S \to \mathbb{R}$ defined as $\Phi(\alpha, d) = \sum_{i\in N} \int_0^{d_i} ((\partial^+ U_i(x)/\partial x)/(g(x) + g'(x)/\phi))\,dx - \sum_{r\in R} c_r(\ell(\alpha, d))$. Let $\bar{S} = \{(\alpha, d) \in S\colon\ d_i \in [\sigma_i, \omega_i] \text{ for all } i \in N\}$. As $\bar{S}$ is compact and Φ is continuous, Φ attains its maximum and we may choose $(\alpha^*, d^*) \in \arg\max_{(\alpha, d)\in\bar{S}}\colon \Phi(\alpha, d)$. We proceed to show that (α^*, d^*) is a PNE. In light of Lemma 3, it suffices to show that there is no improvement move from (α^*, d^*) in which exclusively either the demand or the allocation of a single player is adapted.

We first show that there is no improvement move from (α^*, d^*) in which a single player only changes her demand. This is trivial for players with $\sigma_i = \tau_i$. For all other players, the optimality conditions of (α^*, d^*) give rise to $\partial\Phi(\alpha^*, d^*)/\partial d_i^* \ge 0$ for all $i \in N$ with $d_i^* > \sigma_i$ and $\partial\Phi(\alpha^*, d^*)/\partial d_i^* \le 0$ for all $i \in N$ with $d_i^* < \tau_i$. For $i \in N$, we thus obtain

$$\frac{\partial^+}{\partial d_i^*} U_i(d_i^*) \ge \left(g(d_i^*) + \frac{1}{\phi} g'(d_i^*)\right) \sum_{r\in\alpha_i^*} \phi a_r e^{\phi \ell_r(\alpha^*, d^*)} = \frac{\partial}{\partial d_i^*}\left(g(d_i^*) \sum_{r\in\alpha_i^*} a_r e^{\phi \ell_r(\alpha^*, d^*)}\right), \quad \text{if } d_i^* > \sigma_i,$$

$$\frac{\partial^+}{\partial d_i^*} U_i(d_i^*) \le \left(g(d_i^*) + \frac{1}{\phi} g'(d_i^*)\right) \sum_{r\in\alpha_i^*} \phi a_r e^{\phi \ell_r(\alpha^*, d^*)} = \frac{\partial}{\partial d_i^*}\left(g(d_i^*) \sum_{r\in\alpha_i^*} a_r e^{\phi \ell_r(\alpha^*, d^*)}\right), \quad \text{if } d_i^* < \tau_i.$$

As the utility functions are concave we further obtain $\partial^- U_i(d_i^*)/\partial d_i^* \ge \partial^+ U_i(d_i^*)/\partial d_i^*$. Using that the private payoff function of every player is concave in her demand, this implies that the demand d_i^* is optimal for player i when the allocation profile α^* is played. Thus, there is no improvement move in which player i solely changes her demand.

Next, we show that there is no improvement move from (α^*, d^*) in which a single player only changes her allocation. For a contradiction, suppose there is a player i that deviates profitably from strategy (α_i^*, d_i^*) to strategy $(\alpha_i', d_i^*) \in S_i$. If $g(d_i^*) = 0$, then player i does not improve switching from (α_i^*, d_i^*) to (α_i', d_i^*). Thus, we may assume that $g(d_i^*) > 0$. We obtain

$$\Phi(\alpha_i', \alpha_{-i}^*, d^*) - \Phi(\alpha^*, d^*) = \left(\frac{1}{e^{\phi d_i^*}} - 1\right)\left(\sum_{r\in\alpha_i'} a_r e^{\phi \ell_r(\alpha_i', \alpha_{-i}^*, d^*)}\right) + \left(1 - \frac{1}{e^{\phi d_i^*}}\right)\left(\sum_{r\in\alpha_i^*} a_r e^{\phi \ell_r(\alpha^*, d^*)}\right)$$

$$
\begin{aligned}
&= \frac{1}{g(d_i^*)}\left(1-\frac{1}{e^{\phi d_i^*}}\right)\left(-g(d_i^*)\sum_{r\in\alpha_i'} a_r e^{\phi \ell_r(\alpha_i', \alpha_{-i}^*, d^*)} + g(d_i^*)\sum_{r\in\alpha_i^*} a_r e^{\phi\ell_r(\alpha^*, d^*)}\right)\\
&= \frac{1}{g(d_i^*)}\left(1-\frac{1}{e^{\phi d_i^*}}\right)\left(\pi_i(\alpha_i', \alpha_{-i}^*, d^*) - \pi_i(\alpha^*, d^*)\right)\\
&> 0.
\end{aligned}
$$

This is a contradiction to the fact that (α^*, d^*) maximizes Φ. We derive that (α^*, d^*) is a PNE. □

5.2. Necessary conditions for the existence of a pure Nash equilibrium. Although Proposition 1 shows that every set of consistent cost functions only consists of affine functions or only of exponential functions, the positive result of Theorem 4, however, holds only for *homogeneously* exponential functions. This leaves it open whether there are additional sets of consistent cost functions. In this section, we will close this gap by showing that inhomogeneously exponential functions are not consistent neither for proportional games nor for uniform games and that affine functions are consistent only for proportional games.

To prove these results, we will show that if a cost function c is consistent for g-scaled congestion games with variable demands *and* has the property that c'/c is injective on $\mathbb{R}_{>0}$, then c satisfies the GMC for g. The additional condition that c'/c must be injective precisely explains why homogeneously exponential cost functions are consistent for (uniform or proportional) congestion games with *variable* demands although they are not consistent for (uniform or proportional) congestion games with *resource-dependent* demands.

Before we give the formal proof of the result, we first give some intuition. The proof of the generalized monotonicity lemma (Lemma 1) relies on the construction of a prototypical game with a simple structure. As noted in Corollary 1, the GMC is still necessary for games with two players that have two feasible allocations each, that is, $\mathscr{A}_1^{\mathrm{rd}} = \{\alpha_{1,1}^{\mathrm{rd}}, \alpha_{2,1}^{\mathrm{rd}}\}$, $\mathscr{A}_2^{\mathrm{rd}} = \{\alpha_{2,1}^{\mathrm{rd}}, \alpha_{2,2}^{\mathrm{rd}}\}$ with $\alpha_{i,k}^{\mathrm{rd}} \subseteq R$, $i,k\in\{1,2\}$. In addition, there are $x,y,z\in\mathbb{R}_{>0}$ with $y<z$ such that the demand of player 1 equals $d_{1,r}=x$ for all $r\in R$ and the demand of player 2 equals $d_{2,r}=y$, if $r\in\alpha_{2,1}^{\mathrm{rd}}$, and $d_{2,r}=z$, otherwise.

We will design the utility function $U_i\colon [\sigma_i,\tau_i]\to\mathbb{R}_{\geq 0}$ of every player i such that in equilibrium she uses the (fixed) demand of the game with resource-dependent demands G^{rd}. That is, player 1 always plays $d_1=x$ and player 2 plays $d_2=y$ when using $\alpha_{2,1}^{\mathrm{rd}}$ and $d_2=z$ when using $\alpha_{2,2}^{\mathrm{rd}}$. If such a construction is possible, then the necessary conditions for consistency in the case of resource-dependent demands translate to the case of variable demands. For player 1, we may simply set $\sigma_1=\tau_1=x$ and $U_1(\sigma_1)=0$, that is, we allow player 1 only to use the demand x. For player 2 the situation is more subtle since we want her to use two distinct demand values depending on which resources she uses.

We show that player 2 can be forced to use the right equilibrium demands, if c'/c is injective on $\mathbb{R}_{>0}$. The main idea of the construction is to add additional resources to each of the allocations of player 2. The key point is to also introduce an additional third player to the game who has only a single feasible demand and whose only feasible allocation contains all of the additional resources added to *one* of the allocations of player 2. This way the demand of player 3 increases the aggregated demand on some of the additional resources by a certain *offset*. The condition that c'/c is injective ensures that adding an offset to the functional argument has a different impact on the derivative of the function than scalar multiplication. By carefully choosing the number of supplementary resources added to each of the allocations of player 2 and the feasible demand of player 3, we can show that the marginal costs of player 2 can be manipulated as desired. Note that for a homogeneously exponential function of type $\tilde{c}(x)=a_{\tilde{c}}e^{\phi x}$ it actually holds that $\tilde{c}'/\tilde{c}$ is constant, thus, adding an offset q to the argument has the same effect as multiplying the function by the constant $e^{\phi q}$.

We first need the following technical lemma.

Lemma 4. *Let $c\colon \mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ be a strictly increasing and differentiable function and let g be a scaling function. If c'/c is injective on $\mathbb{R}_{>0}$, then for all $y\in\mathbb{R}_{>0}$ and $\delta\in\mathbb{R}_{>0}$ there are $\theta,\mu\in\mathbb{R}_{>0}$ and $z\in(y,y+\delta)$ such that one of the following two cases holds*:

$$(1)\ \frac{\partial}{\partial y}(g(y)c(y+\theta)) > \frac{\mu g(z)c(z)-g(y)c(y+\theta)}{z-y} > \frac{\partial}{\partial z}(\mu g(z)c(z));$$

$$(2)\ \frac{\partial}{\partial y}(\mu g(y)c(y)) > \frac{g(z)c(z+\theta)-\mu g(y)c(y)}{z-y} > \frac{\partial}{\partial z}(g(z)c(z+\theta)).$$

Proof. Let c satisfy the demanded properties and let $y > 0$ be given. As c'/c is injective there is $\tilde{\theta} \in \mathbb{R}_{>0}$ with $c'(y+\tilde{\theta})/c(y+\tilde{\theta}) \neq c'(y)/c(y)$. We distinguish two cases.

First case. $c'(y+\tilde{\theta})/c(y+\tilde{\theta}) > c'(y)/c(y)$. Multiplying with $g(y)c(y+\tilde{\theta})$ and setting $\tilde{\mu} = c(y+\tilde{\theta})/c(y)$ we obtain $g(y)c'(y+\tilde{\theta}) > \tilde{\mu}g(y)c'(y)$. Adding $g'(y)c(y+\tilde{\theta})$ to both sides gives

$$g'(y)c(y+\tilde{\theta}) + g(y)c'(y+\tilde{\theta}) > \tilde{\mu}g'(y)c(y) + \tilde{\mu}g(y)c'(y). \tag{10}$$

As the expression on the left-hand side of (10) is continuous in $\tilde{\theta}$, there is $\theta < \tilde{\theta}$ such that

$$g'(y)c(y+\theta) + g(y)c'(y+\theta) > \tilde{\mu}g'(y)c(y) + \tilde{\mu}g(y)c'(y). \tag{11}$$

Using the fact that c is strictly increasing, we obtain $0 = \tilde{\mu}g(y)c(y) - g(y)c(y+\tilde{\theta}) < \tilde{\mu}g(y)c(y) - g(y)c(y+\theta)$ and hence

$$a_0 := \tilde{\mu}g(y)c(y) - g(y)c(y+\theta) > 0. \tag{12}$$

Because the right-hand side of (11) is continuous in y, there is a sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in (y, y+\delta)$ for all $n \in \mathbb{N}$ that converges to y and satisfies the inequality

$$g'(y)c(y+\theta) + g(y)c'(y+\theta) > \tilde{\mu}g'(z_n)c(z_n) + \tilde{\mu}g(z_n)c'(z_n) \tag{13}$$

for all $n \in \mathbb{N}$. Further, using the definition of a_0 in (12) and the fact that both g and c are nondecreasing, we obtain for all $n \in \mathbb{N}$ the inequality

$$\tilde{\mu}g(z_n)c(z_n) - g(y)c(y+\theta) > \tilde{\mu}g(y)c(y) - g(y)c(y+\theta) = a_0,$$

which implies

$$\frac{\tilde{\mu}g(z_n)c(z_n) - g(y)c(y+\theta)}{z_n - y} > \frac{a_0}{z_n - y} \xrightarrow{n\to\infty} \infty.$$

Using that by (13), $-\tilde{\mu}g'(z_n)c(z_n) - \tilde{\mu}g(z_n)c'(z_n)$ is bounded from below by a constant not depending on n, we derive the existence of $m \in \mathbb{N}$ such that

$$\frac{\tilde{\mu}g(z_m)c(z_m) - g(y)c(y+\theta)}{z_m - y} - \tilde{\mu}g'(z_m)c(z_m) - \tilde{\mu}g(z_m)c'(z_m) > 0. \tag{14}$$

Furthermore, (13) trivially implies

$$g'(y)c(y+\theta) + g(y)c'(y+\theta) - \tilde{\mu}g'(z_m)c(z_m) - \tilde{\mu}g(z_m)c'(z_m) =: a_1 > 0. \tag{15}$$

To finish the proof of the first case, observe that the left-hand side of (15) increases as $\tilde{\mu}$ decreases. The left-hand side of (14) is continuous as a function of $\tilde{\mu}$ and negative for $\tilde{\mu} = (g(y)/g(z_m)) \cdot (c(y+\theta)/c(z_m))$. Thus, there is $\mu \in \mathbb{R}_{>0}$ with $(g(y)/g(z_m)) \cdot (c(y+\theta)/c(z_m)) \leq \mu \leq c(y+\tilde{\theta})/c(y) = \tilde{\mu}$ such that

$$0 < \frac{\mu g(z_m)c(z_m) - g(y)c(y+\theta)}{z_m - y} - \mu g'(z_m)c(z_m) - \mu g(z_m)c'(z_m) < a_1.$$

Adding $\mu g'(z_m)c(z_m) + \mu g(z_m)c'(z_m)$, we obtain

$$\mu g'(z_m)c(z_m) + \mu g(z_m)c'(z_m) < \frac{\mu g(z_m)c(z_m) - g(y)c(y+\theta)}{z_m - y} < a_1 + \mu g'(z_m)c(z_m) + \mu g(z_m)c'(z_m).$$

Finally, we observe that

$$\begin{aligned} a_1 + \mu g'(z_m)c(z_m) + \mu g(z_m)c'(z_m) &= g'(y)c(y+\theta) + g(y)c'(y+\theta) - (\tilde{\mu}-\mu)g'(z_m)c(z_m) - (\tilde{\mu}-\mu)g(z_m)c'(z_m) \\ &\leq g'(y)c(y+\theta) + g(y)c'(y+\theta), \end{aligned}$$

where we used that $\mu \leq \tilde{\mu}$. This finishes the proof of the first case.

Second case. $c'(y+\tilde{\theta})/c(y+\tilde{\theta}) < c'(y)/c(y)$. This case essentially works as the first case but with reversed inequality signs. First, we find $y, \tilde{\theta} \in \mathbb{R}_{>0}$ such that for $\tilde{\mu} = c(y+\tilde{\theta})/c(y)$ the inequality (10) holds with reversed sign, i.e.,

$$g'(y)c(y+\tilde{\theta}) + g(y)c'(y+\tilde{\theta}) < \tilde{\mu}g'(y)c(y) + \tilde{\mu}g(y)c'(y). \tag{16}$$

With the same continuity arguments as in the first case, there is $\theta > \tilde{\theta}$ such that

$$g'(y)c(y+\theta) + g(y)c'(y+\theta) < \tilde{\mu}g'(y)c(y) + \tilde{\mu}g(y)c'(y). \tag{17}$$

Using that c is strictly increasing, we obtain $a_2 := g(y)c(y+\theta) - \tilde{\mu}g(y)c(y) > g(y)c(y+\tilde{\theta}) - \tilde{\mu}g(y)c(y) = 0$. We proceed analogous to the first case but exchange the roles of z_n and y, i.e., we derive from (17) the existence of sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in (y, y+\delta)$ for all $n \in \mathbb{N}$ that converges to y and satisfies the inequality

$$g'(z_n)c(z_n+\theta) + g(z_n)c'(z_n+\theta) < \tilde{\mu}g'(y)c(y) + \tilde{\mu}g(y)c'(y) \tag{18}$$

for all $n \in \mathbb{N}$. Further, we use that both g and c are nondecreasing to obtain for all $n \in N$ the inequality

$$g(z_n)c(z_n+\theta) - \tilde{\mu}g(y)c(y) > g(y)c(y+\theta) - \tilde{\mu}g(y)c(y) = a_2,$$

which implies

$$\frac{g(z_n)c(z_n+\theta) - \tilde{\mu}g(y)c(y)}{z_n - y} > \frac{a_2}{z_n - y} \xrightarrow{n\to\infty} \infty.$$

Using that, by (18), $-g'(z_n)c(z_n+\theta) - g(z_n)c'(z_n+\theta)$ is bounded from below by a constant not depending on n, we derive as in the first case the existence of $m \in \mathbb{N}$ such that

$$\frac{g(z_m)c(z_m+\theta) - \tilde{\mu}g(y)c(y)}{z_m - y} - g'(z_m)c(z_m+\theta) - g(z_m)c'(z_m+\theta) > 0. \tag{19}$$

Furthermore, (18) trivially implies

$$\tilde{\mu}g'(y)c(y) + \tilde{\mu}g(y)c'(y) - g'(z_m)c(z_m+\theta) - g(z_m)c'(z_m+\theta) =: a_3 > 0. \tag{20}$$

We proceed with the same arguments as in the first case. The left-hand side of (20) increases as $\tilde{\mu}$ increases and the left-hand side of (19) is continuous as a function of $\tilde{\mu}$ and negative for $\tilde{\mu} = (g(z_m)/g(y)) \cdot (c(z_m+\theta)/c(y))$. Thus, there is $\mu \in \mathbb{R}_{>0}$ with $\tilde{\mu} = (c(y+\tilde{\theta})/c(y)) \le \mu \le (g(z_m)/g(y)) \cdot (c(z_m+\theta)/c(y))$ such that

$$0 < \frac{g(z_m)c(z_m+\theta) - \mu g(y)c(y)}{z_m - y} - g'(z_m)c(z_m+\theta) - g(z_m)c'(z_m+\theta) < a_3.$$

Adding $-g'(z_m)c(z_m+\theta) - g(z_m)c'(z_m+\theta)$, we obtain

$$g'(z_m)c(z_m+\theta) + g(z_m)c'(z_m+\theta) < \frac{g(z_m)c(z_m+\theta) - \mu g(y)c(y)}{z_m - y} < a_3 + g'(z_m)c(z_m+\theta) + g(z_m)c'(z_m+\theta).$$

Finally,

$$\begin{aligned} a_3 + g'(z_m)c(z_m+\theta) + g(z_m)c'(z_m+\theta) &= \tilde{\mu}g'(y)c(y) + \tilde{\mu}g(y)c'(y) - g'(z_m)c(z_m+\theta) - g(z_m)c'(z_m+\theta) \\ &\le \mu g'(y)c(y) + \mu g(y)c'(y), \end{aligned}$$

which finishes the proof of the second case. □

The next lemma establishes a necessary condition on the consistency of cost functions for g-scaled congestion games with variable demands.

Lemma 5. *Let g be a scaling function and let c be a differentiable and convex cost function that is strictly positive on $\mathbb{R}_{>0}$ and has the property that c'/c is injective on $\mathbb{R}_{>0}$. If c is consistent for g-scaled congestion games with variable demands, then c satisfies the GMC for g.*

Proof. Let c be a function with the demanded properties that is consistent for g-scaled congestion games with variable demands and let us assume that c does not satisfy the GMC for g. Applying Corollary 1, we derive the existence of $x, y \in \mathbb{R}_{>0}$ and $\epsilon > 0$ such that for every $\delta \in (0, \epsilon)$, there is a congestion model $\mathcal{M}_\delta^{\mathrm{rd}} = (N^{\mathrm{rd}}, R_\delta^{\mathrm{rd}}, (\mathcal{A}_{\delta,i}^{\mathrm{rd}})_{i\in N^{\mathrm{rd}}}, (c_r^{\mathrm{rd}})_{r\in R_\delta^{\mathrm{rd}}})$ with two players that have access to two disjoint allocations each and all resources have cost function c (i.e., $N^{\mathrm{rd}} = \{1, 2\}$, and $\mathcal{A}_i^{\mathrm{rd}} = \{\alpha_{i,1}^{\mathrm{rd}}, \alpha_{i,2}^{\mathrm{rd}}\}$ for some $\alpha_{i,1}^{\mathrm{rd}}, \alpha_{i,2}^{\mathrm{rd}} \subseteq R^{\mathrm{rd}}$ with $\alpha_{i,1}^{\mathrm{rd}} \cap \alpha_{i,2}^{\mathrm{rd}} = \varnothing$, $i \in \{1, 2\}$, $c_r^{\mathrm{rd}} = c$ for all $r \in R$). Further there is a corresponding g-scaled congestion game with resource-dependent demands G_δ^{rd} that does not have a PNE and for which the players' resource-dependent demands equal $d_{1,r} = x$ for all $r \in R^{\mathrm{rd}}$, $d_{2,r} = y$, if $r \in \alpha_{2,1}^{\mathrm{rd}}$, and $d_{2,r} = y + \delta$, otherwise.

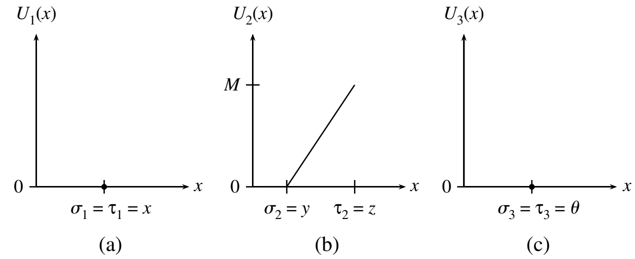


FIGURE 2. The players' utility functions U_i in the three-player g-scaled congestion game with variable demands G^k constructed in the proof of Lemma 5. For the second player's utility the parameter M is chosen such that the slope of the function equals $(k/(z-y))\cdot(pg(z)c(z)-qg(y)c(y+\theta))$, that is, $M = kpg(z)c(z)-kqg(y)c(y+\theta)$.

Let $x, y, \epsilon \in \mathbb{R}_{>0}$ be fixed accordingly. Referring to Lemma 4, there are $z\in(y, y+\epsilon)$ and $\mu, \theta \in \mathbb{R}_{>0}$ such that one of the following two cases holds:

$$\frac{\partial}{\partial x}(g(y)c(y+\theta)) > \frac{\mu g(z)c(z)-g(y)c(y+\theta)}{z-y} > \frac{\partial}{\partial z}(\mu g(z)c(z)), \tag{21}$$

$$\frac{\partial}{\partial y}(\mu g(y)c(y)) > \frac{g(z)c(z+\theta)-\mu g(y)c(y)}{z-y} > \frac{\partial}{\partial z}(g(z)c(z+\theta)). \tag{22}$$

We fix such z, μ, and θ and set $\delta = z-y$. In the following, we omit the subscript δ and denote by $\mathcal{M}^{\mathrm{rd}} = (N^{\mathrm{rd}}, R^{\mathrm{rd}}, (\mathcal{A}_i^{\mathrm{rd}})_{i\in N^{\mathrm{rd}}}, (c_r^{\mathrm{rd}})_{r\in R^{\mathrm{rd}}})$ the congestion model and by G^{rd} the corresponding g-scaled congestion game with demands x, y, and z not possessing a PNE.

We proceed to show the proof for the case (21), the other case follows along the same lines. As all expressions occurring in (21) are continuous in μ, it is without loss of generality to assume that μ is rational, i.e., $\mu = p/q$ for some $p, q\in\mathbb{N}$. Multiplying (21) with q, we obtain

$$\frac{\partial}{\partial y}(qg(y)c(y+\theta)) > \frac{pg(z)c(z)-qg(y)c(y+\theta)}{z-y} > \frac{\partial}{\partial z}(pg(z)c(z)). \tag{23}$$

For $k\in\mathbb{N}$ we define a new congestion model $\mathcal{M}^k = (N, R^k, (\mathcal{A}_i^k)_{i\in N}, (c_r)_{r\in R})$. The set of players $N = N^{\mathrm{rd}}\cup\{3\}$ contains an additional third player, the set of resources R^k contains additional $k(p+q)$ resources partitioned into two subsets R_1^k, R_2^k of cardinality $|R_1^k| = kq$ and $|R_2^k| = kp$, respectively. We obtain $R^k = R^{\mathrm{rd}}\cup R_1^k\cup R_2^k$. Every resource $r\in R^k$ is endowed with the cost function c. Player 1 has the same set of feasible allocations as in G^{rd}, that is, $\mathcal{A}_1^k = \{\alpha_{1,1}^{\mathrm{rd}}, \alpha_{1,2}^{\mathrm{rd}}\}$. For player 2, we add the kq new resources contained in R_1^k to the first, and the kp new resources contained in R_2^k to the second allocation, that is, $\mathcal{A}_2^k = \{\alpha_{2,1}^{\mathrm{rd}}\cup R_1^k, \alpha_{2,2}^{\mathrm{rd}}\cup R_2^k\}$. Player 3 has a single feasible allocation where she uses the kq new resources contained in R_1^k.

The players' sets of feasible demands are given by $\sigma_1 = \tau_1 = x$, $\sigma_2 = y$, $\tau_2 = z$, and $\sigma_3 = \tau_3 = \theta$. The utility functions of players 1 and 3 are arbitrary. We may simply define them as $U_1^k(\sigma_1) = U_3^k(\sigma_3) = 0$. The utility function $U_2^k\colon [y,z]\to\mathbb{R}_{\geq 0}$ of player 2 is the linear function with slope $(k/(z-y))\cdot(pg(z)c(z)-qg(y)c(y+\theta))$ through the point $(y, 0)$. The three utility functions are shown in Figure 2.

We claim that there is $m\in\mathbb{N}$ such that $d_2^m = y$ for every PNE (α^m, d^m) of G^m with $\alpha_2^m = \alpha_{2,1}^{\mathrm{rd}}\cup R_1^m$ and $d_2^m = z$ for every PNE (α^m, d^m) of G^m with $\alpha_2^m = \alpha_{2,2}^{\mathrm{rd}}\cup R_2^m$. For a contradiction, suppose for every $k\in\mathbb{N}$, there is a PNE (α^k, d^k) such that one of the following cases holds: (i) $\alpha_2^k = \alpha_{2,1}^{\mathrm{rd}}\cup R_1^k$ and $d_2^k\in(y,z]$; (ii) $\alpha_2^k = \alpha_{2,2}^{\mathrm{rd}}\cup R_2^k$ and $d_2^k\in[y,z)$. Considering subsequences, it is without loss of generality to assume that either (i) holds for all $k\in\mathbb{N}$ or (ii) holds for all $k\in\mathbb{N}$.

Let us first assume, that (i) holds for all $k\in\mathbb{N}$. We calculate

$$\frac{\partial \pi_2(\alpha^k, d^k)}{\partial d_2^k} = k\left(\frac{pg(z)c(z)-qg(y)c(y+\theta)}{z-y} - q\frac{\partial}{\partial d_2^k}g(d_2^k)c(d_2^k+\theta)\right) - \frac{\partial}{\partial d_2^k}\sum_{r\in\alpha_{2,1}^{\mathrm{rd}}} c_r(\ell_r(\alpha^k, d^k)).$$

Using that $d_2^k > y$ and that c and g are convex and nondecreasing, we obtain

$$\frac{\partial \pi_2(\alpha^k, d^k)}{\partial d_2^k} \leq k\left(\frac{pg(z)c(z)-qg(y)c(y+\theta)}{z-y} - q\frac{\partial}{\partial y}g(y)c(y+\theta) - \frac{1}{k}\cdot\frac{\partial}{\partial d_2^k}\sum_{r\in\alpha_{2,1}^{\mathrm{rd}}} c_r(\ell_r(\alpha^k, d^k))\right).$$

From the left inequality of (23) we derive $\lim_{k\to\infty} \partial\pi_2(\alpha^k, d^k)/\partial d_2^k < 0$. This implies the existence of $m \in \mathbb{N}$ with $\partial\pi_2(\alpha^m, d^m)/\partial d_2^m < 0$. Using Lemma 2, this contradicts the assumption that (α^m, d^m) is a PNE of G^m.

If, on the other hand, (ii) holds for all $k \in \mathbb{N}$, we calculate

$$\begin{aligned}\frac{\partial\pi_2(\alpha^k, d^k)}{\partial d_2^k} &= k\left(\frac{pg(z)c(z) - qg(y)c(y+\theta)}{z-y} - p\frac{\partial}{\partial d_2^k} g(d_2^k)c(d_2^k) - \frac{1}{k}\cdot\frac{\partial}{\partial d_2^k}\sum_{r\in\alpha_{2,2}^{\mathrm{rd}}} c_r(\ell_r(\alpha^k, d^k))\right)\\ &\geq k\left(\frac{pg(z)c(z) - qg(y)c(y+\theta)}{z-y} - p\frac{\partial}{\partial z} g(z)c(z) - \frac{1}{k}\cdot\frac{\partial}{\partial d_2^k}\sum_{r\in\alpha_{2,2}^{\mathrm{rd}}} c_r(\ell_r(\alpha^k, d^k))\right),\end{aligned}$$

where we use the convexity of c and g and the fact that $d_2^k < z$. Using the right inequality of (23), we derive $\lim_{k\to\infty} \partial\pi_2(\alpha^k, d^k)/\partial d_2^k > 0$. In light of Lemma 2, this is a contradiction to the assumption that (α^k, d^k) is a PNE for all $k \in \mathbb{N}$. We conclude that there is $m \in \mathbb{N}$ such that $d_2^m = y$ for every PNE (α^m, d^m) of G^m with $\alpha_2^m = \alpha_{2,1}^{\mathrm{rd}} \cup R_1^m$ and $d_2^m = z$ for every PNE (α^m, d^m) of G^m with $\alpha_2^m = \alpha_{2,2}^{\mathrm{rd}} \cup R_2^m$.

To finish the proof, we show that G^m does not possess a PNE. For a contradiction, let (α^m, d^m) be a PNE of G^m. Here we show the contradiction only for the case that every player plays her first allocation, that is, $\alpha_1^m = \alpha_{1,1}^{\mathrm{rd}}$ and $\alpha_2^m = \alpha_{2,1}^{\mathrm{rd}} \cup R_1^m$. The other three cases can be treated with the same arguments. Consider the strategy profile $(\alpha_{1,1}^{\mathrm{rd}}, \alpha_{2,1}^{\mathrm{rd}})$ of G^{rd}. Because G^{rd} does not possess a PNE, at least one of the players 1 or 2 improves switching to her second allocation. We distinguish two cases.

First case. $\pi_1^{\mathrm{rd}}(\alpha_{1,2}^{\mathrm{rd}}, \alpha_{2,1}^{\mathrm{rd}}) > \pi_1^{\mathrm{rd}}(\alpha_{1,1}^{\mathrm{rd}}, \alpha_{2,1}^{\mathrm{rd}})$. Consider the strategy profile $(\alpha_{1,2}^{\mathrm{rd}}, x) \in S_1^m$. We calculate

$$\begin{aligned}&\pi_1^m((\alpha_{1,2}^{\mathrm{rd}}, x), (\alpha_{2,1}^{\mathrm{rd}} \cup R_1^m, y), (R_1^m, \theta)) - \pi_1^m((\alpha_{1,1}^{\mathrm{rd}}, x), (\alpha_{2,1}^{\mathrm{rd}} \cup R_1^m, y), (R_1^m, \theta))\\ &\quad= -g(x)\sum_{r\in\alpha_{1,2}^{\mathrm{rd}}} c_r(\ell_r((\alpha_{1,2}^{\mathrm{rd}}, x), (\alpha_{2,1}^{\mathrm{rd}} \cup R_1^m, y), (R_1^m, \theta))) + g(x)\sum_{r\in\alpha_{1,1}^{\mathrm{rd}}} c_r(\ell_r((\alpha_{1,1}^{\mathrm{rd}}, x), (\alpha_{2,1}^{\mathrm{rd}} \cup R_1^m, y), (R_1^m, \theta)))\\ &\quad= \pi_1^{\mathrm{rd}}(\alpha_{1,2}^{\mathrm{rd}}, \alpha_{2,1}^{\mathrm{rd}}) - \pi_1^{\mathrm{rd}}(\alpha_{1,1}^{\mathrm{rd}}, \alpha_{2,1}^{\mathrm{rd}}) > 0.\end{aligned}$$

Second case. $\pi_2^{\mathrm{rd}}(\alpha_{1,1}^{\mathrm{rd}}, \alpha_{2,2}^{\mathrm{rd}}) > \pi_2^{\mathrm{rd}}(\alpha_{1,1}^{\mathrm{rd}}, \alpha_{2,1}^{\mathrm{rd}})$. Consider the strategy profile $(\alpha_{2,2}^{\mathrm{rd}} \cup R_2^m, z)$ where player 2 chooses allocation $\alpha_{2,2}^{\mathrm{rd}} \cup R_2^m$ and her demand equals z. We obtain

$$\begin{aligned}&\pi_2^m((\alpha_{1,1}^{\mathrm{rd}}, x), (\alpha_{2,2}^{\mathrm{rd}} \cup R_2^m, z), (R_1^m, \theta)) - \pi_2^m((\alpha_{1,1}^{\mathrm{rd}}, x), (\alpha_{2,1}^{\mathrm{rd}} \cup R_1^m, y), (R_1^m, \theta))\\ &\quad= U_2(z) - g(z)\sum_{r\in\alpha_{2,2}^{\mathrm{rd}}} c_r(\ell_r((\alpha_{1,1}^{\mathrm{rd}}, x), (\alpha_{2,2}^{\mathrm{rd}} \cup R_2^m, z), (R_1^m, \theta))) - kpg(z)c(z)\\ &\qquad -U_2(y) + g(y)\sum_{r\in\alpha_{2,1}^{\mathrm{rd}}} c_r(\ell_r((\alpha_{1,1}^{\mathrm{rd}}, x), (\alpha_{2,1}^{\mathrm{rd}} \cup R_1^m, y), (R_1^m, \theta))) + kqg(y)c(y+\theta)\\ &\quad= -g(z)\sum_{r\in\alpha_{2,2}^{\mathrm{rd}}} c_r(\ell_r((\alpha_{1,1}^{\mathrm{rd}}, x), (\alpha_{2,2}^{\mathrm{rd}} \cup R_2^m, z), (R_1^m, \theta))) + g(y)\sum_{r\in\alpha_{2,1}^{\mathrm{rd}}} c_r(\ell_r((\alpha_{1,1}^{\mathrm{rd}}, x), (\alpha_{2,1}^{\mathrm{rd}} \cup R_1^m, y), (R_1^m, \theta)))\\ &\quad= \pi_2^{\mathrm{rd}}(\alpha_{1,1}^{\mathrm{rd}}, \alpha_{2,2}^{\mathrm{rd}}) - \pi_2^{\mathrm{rd}}(\alpha_{1,1}^{\mathrm{rd}}, \alpha_{2,1}^{\mathrm{rd}}) > 0.\end{aligned}$$

This is a contradiction to the assumption that (α^m, d^m) is a PNE of G^m. $\square$

5.3. A characterization of consistency. We are ready to give the main results of this paper—a complete characterization of consistency and approximate universal consistency for congestion games with variable demands. We start with the characterization of the approximate universal consistency.

THEOREM 5. *Let $\mathscr{C} \neq \varnothing$ be a set of continuous functions and let g be a scaling function. Then, $\mathscr{C}$ is approximately universally consistent for g-scaled congestion games with variable demands if and only if the following two conditions are satisfied*:

(1) $g(x) \equiv x$.

(2) *$\mathscr{C}$ only contains affine functions of type $c(x) = a_c x + b_c$ where $a_c \in \mathbb{R}_{>0}$ and $b_c \in \mathbb{R}_{\geq 0}$.*

If (1) *and* (2) *are satisfied, then $\mathscr{C}$ is even consistent for g-scaled congestion games with variable demands.*

PROOF. We first show that conditions (1) and (2) imply consistency and approximate universal consistency of $\mathscr{C}$. Let $\mathscr{M} = (N, R, (\mathscr{A}_i)_{i\in N}, (c_r)_{r\in R})$ be a congestion model such that for every resource $r \in R$, there are $a_r \in \mathbb{R}_{>0}$ and $b_r \in \mathbb{R}_{\geq 0}$ with $c_r(x) = a_r x + b_r$ for all $x \in \mathbb{R}_{>0}$. For a set $(U_i)_{i\in N}$ of utility functions let G be a corresponding g-scaled congestion game with variable demands. Analogously as in the proof of Theorem 4, for every player i there is $\omega_i \in \mathbb{R}_{>0}$ such that every demand $d_i > \omega_i$ is strictly dominated by the demand $d_i' = \omega_i$, that is, $\pi_i(\alpha_i, d_i, \alpha_{-i}, d_{-i}) < \pi_i(\alpha_i, d_i', \alpha_{-i}, d_{-i})$ for all $\alpha_i \in \mathcal{A}_i$ and $(\alpha_{-i}, d_{-i}) \in S_{-i}$.

In previous work (Harks et al. [17]) we noted that the function $P\colon S \to \mathbb{R}$ defined as $P(\alpha, d) = \sum_{i\in N}(U_i(d_i) - d_i \sum_{r\in\alpha_i} a_r(\sum_{j\in\{1,\dots,i\}: r\in s_j} d_j) + b_r)$ is an exact potential function for proportional congestion games with variable demands. Let $\bar{S} = \{(\alpha, d) \in S\colon d_i \in [\sigma_i, \omega_i] \text{ for all } i \in N\}$. As $\bar{S}$ is compact and P is continuous, we may choose $(\alpha^*, d^*) \in \arg\max_{(\alpha, d)\in\bar{S}} P(\alpha, d)$. Using that strategies with demands larger than ω_i are strictly dominated we derive that (α^*, d^*) also maximizes P over S. This implies that (α^*, d^*) is a PNE.

To see that $\mathscr{C}$ is also approximately universally consistent, let $\rho > 0$ and $(\alpha^0, d^0) \in S$ be arbitrary. As $P(\alpha^0, d^0)$ is finite, P is bounded from above and every ρ-improvement move increases the value of P by at least ρ, we conclude that every ρ-improvement path is finite.

We proceed to prove that if $\mathscr{C}$ is approximately universally consistent for g-scaled congestion games with variable demands, then (1) and (2) hold. If $\mathscr{C}$ contains a constant function we can construct a one-player game where player 1 can always improve her payoff by an arbitrary constant raising her demand. If $\mathscr{C}$ contains a nonaffine function $\tilde{c}$ *or* g is not linear, Theorem 3 establishes the existence of a g-scaled congestion game with resource-dependent demands G^{rd} that does not possess a PNE and has the additional property that $c_r = \tilde{c}$ for all $r \in R$. In light of Corollary 1 the game G^{rd} can be chosen such that for every player all resources contained in one allocation are accessed with the same resource-dependent demand, that is for all $i \in N$ and $\alpha_i \in \mathcal{A}_i$, there is $d_i(\alpha_i)$ such that $d_{i,r} = d_i(\alpha_i)$ for all $r \in \alpha_i$. As G^{rd} has no PNE there is an improvement cycle $\gamma^{\mathrm{rd}} = (\alpha^0, \dots, \alpha^k)$ in G^{rd} and as γ^{rd} is finite there is $\epsilon > 0$ such that γ^{rd} is an ϵ-improvement cycle.

We consider the g-scaled congestion game with variable demands G with the same set of players, resources, and feasible allocations as G^{rd} where for all players i we have $[\sigma_i, \tau_i] = \mathbb{R}_{\ge 0}$ and $U_i(x) = 0$ for all $x \ge 0$. By construction, the cycle

$$\gamma = ((\alpha_1^0, d_1(\alpha_1^0), \dots, \alpha_n^0, d_1(\alpha_n^0)), \dots, (\alpha_1^k, d_1(\alpha_1^k), \dots, \alpha_n^k, d_1(\alpha_n^k)), (\alpha_1^0, d_1(\alpha_1^0), \dots, \alpha_n^0, d_1(\alpha_n^0)))$$

in which every player chooses the demand specified in G^{rd} is an ϵ-improvement cycle of G. We derive that $\mathscr{C}$ is not approximately universally consistent. □

The following theorem provides a complete characterization of the consistency of cost functions in g-scaled congestion games with variable demands.

Theorem 6. *Let $\mathscr{C} \neq \varnothing$ be a set of continuous functions and g be a scaling function. Then, the following are equivalent*:

(1) *$\mathscr{C}$ is consistent for g-scaled congestion games with variable demands.*

(2) *Exactly one of the following two holds*:

(a) *$\mathscr{C}$ only contains homogeneously exponential functions of type $c(x) = a_c e^{\phi x}$, where $a_c \in \mathbb{R}_{>0}$ may depend on c, and $\phi \in \mathbb{R}_{>0}$ must be equal for all $c \in \mathscr{C}$.*

(b) *$g(x) \equiv x$, and $\mathscr{C}$ only contains affine functions of type $c(x) = a_c x + b_c$, where $a_c \in \mathbb{R}_{>0}$ and $b_c \in \mathbb{R}_{\ge 0}$.*

Proof. (1) ⇒ (2): For a contradiction, suppose there is a scaling function g and a set of cost functions $\mathscr{C}$ that is consistent for g-scaled congestion games with variable demands, but neither (2)(a) nor (2)(b) are satisfied. Referring to Proposition 1, $\mathscr{C}$ consists only of affine functions or only of exponential functions.

Let us first assume that $\mathscr{C}$ only contains affine functions. Because (2)(a) is not satisfied, at least one affine function $c \in \mathscr{C}$ is nonconstant. Furthermore, g is not linear since (2)(b) is violated. As shown in the proof of Theorem 1, this implies that c does not satisfy the GMC for g. Furthermore, because c is linear but not constant, we derive that c'/c is injective on $\mathbb{R}_{>0}$. Applying Lemma 5 we derive the existence of a g-scaled congestion game with variable demands not possessing a PNE. This is a contradiction to the consistency of $\mathscr{C}$.

For the second case, let us assume that $\mathscr{C}$ only contains functions of type $c(x) = a_c e^{\phi x} + b_c$. If $\mathscr{C}$ contains a function c such that $c'(x) < K$ for some constant K and all $x \ge 0$, it is easy to construct a one-player game with unrestricted demands in which the single player can always improve her payoff by raising her demand. This observation implies that $\phi > 0$. The nonnegativity of the cost functions further implies that $a_c > 0$ and $b_c \ge -a_c$. Because (2)(a) is violated, $\mathscr{C}$ contains at least one inhomogeneously exponential function c, i.e., a function with $b_c \neq 0$. Theorem 2 implies that c does not satisfy the GMC for g. Using that c is inhomogeneously exponential, we further see that $c'(x)/c(x) = \phi/(1 + (b_c/a_c)e^{-\phi x})$ for all $x \in \mathbb{R}_{>0}$. Thus, c'/c is injective on $\mathbb{R}_{>0}$. Applying Lemma 5, the existence of a g-scaled congestion game with variable demands not possessing a PNE follows.

(2)(a) ⇒ (1) and (2)(b) ⇒ (1) are shown in Theorems 4 and 5, respectively. □

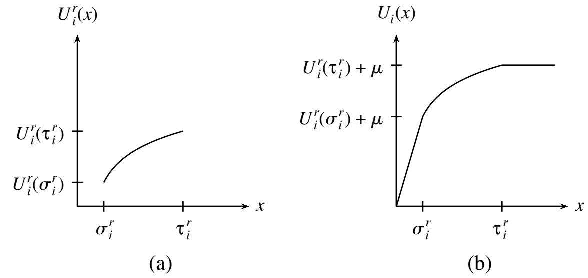


Figure 3. Lifting of the players' utility functions used in the proof of Theorem 7. (a) the utility function U_i^r of player i in the game with restricted demands; (b) the lifted utility function U_i of player i in the game with unrestricted demands.

5.4. A characterization for unrestricted demands. In the previous sections, we characterized the sets of cost functions that are consistent for g-scaled congestion games with variable demands. We assumed that for every player i, the set of feasible demands is restricted to an interval $[\sigma_i, \tau_i] \subseteq \mathbb{R}_{\geq 0}$, with $\sigma_i \in \mathbb{R}_{\geq 0}$, $\tau_i \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, and $\sigma_i \leq \tau_i$. In particular, we allowed the somewhat degenerate case $\sigma_i = \tau_i$. This section is devoted to the case of unrestricted demands, i.e., the case $[\sigma_i, \tau_i] = \mathbb{R}_{\geq 0}$ for all $i \in N$. As the main result, we show that our characterizations of consistency for the case of restricted demands continuous to hold even for the case of unrestricted demands. We additionally strengthen the above result by requiring that utility functions must be smooth.

Theorem 7. *Let G^r be a g-scaled congestion game with variable demands such that all utility functions $U_i^r\colon [\sigma_i^r, \tau_i^r] \to \mathbb{R}_{\geq 0}$ satisfy $\tau_i^r < \infty$. If G^r does not possess a PNE, then there is a corresponding g-scaled congestion game with unrestricted variable demands that does not possess a PNE as well.*

Proof. Let $T = \sum_{i \in N} \tau_i^{\mathrm{r}}$. For every $r \in R$, the cost function $c_r\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is locally Lipschitz continuous on $\mathbb{R}_{\geq 0}$ and hence globally Lipschitz continuous with Lipschitz constant L_r on the compact $[0, T]$. Let $L = \sum_{r \in R} L_r$ and $\sigma_{\max} = \max_{i \in N} \sigma_i^{\mathrm{r}}$. As g is continuously differentiable, it is globally Lipschitz continuous on $[0, \sigma_{\max}]$ and we denote the Lipschitz constant by M. We define $C_{\max} = \sum_{r \in R} \max_{x \in [0,T]} c_r(x)$. For some

$$\mu > \max\left\{L \cdot \sigma_{\max} \cdot g(\sigma_{\max}) + \sigma_{\max} \cdot M \cdot C_{\max}, \max_{i \in N}\left\{\frac{\partial^+ U_i^{\mathrm{r}}(\sigma_i^{\mathrm{r}})}{\partial \sigma_i^{\mathrm{r}}} \cdot \sigma_i^{\mathrm{r}}\right\}\right\},$$

we define the utility function of player i as the function $U_i\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, where

$$U_i(x) = \begin{cases} \dfrac{U_i^{\mathrm{r}}(\sigma_i^{\mathrm{r}}) + \mu}{\sigma_i^{\mathrm{r}}} \cdot x, & \text{if } x \in [0, \sigma_i^{\mathrm{r}}) \\ U_i^{\mathrm{r}}(x) + \mu, & \text{if } x \in [\sigma_i^{\mathrm{r}}, \tau_i^{\mathrm{r}}] \\ U_i^{\mathrm{r}}(\tau_i^{\mathrm{r}}) + \mu, & \text{otherwise.} \end{cases}$$

On $[\sigma_i^{\mathrm{r}}, \tau_i^{\mathrm{r}}]$, the new utility function equals the old utility function raised by μ, on $[\tau_i^{\mathrm{r}}, \infty)$ it is constant and on $[0, \sigma_i^{\mathrm{r}}]$ it equals the linear function through the origin and the point $(\sigma_i^{\mathrm{r}}, U_i(\sigma_i^{\mathrm{r}}) + \mu)$. Note that U_i is concave as U_i^{r} is concave and $\mu > (\partial^+ U_i^{\mathrm{r}}(\sigma_i^{\mathrm{r}})/\partial \sigma_i^{\mathrm{r}}) \cdot \sigma_i^{\mathrm{r}}$; see also Figure 3 for an illustration.

The new set $(U_i)_{i \in N}$ of utility functions defines a new g-scaled congestion game with variable demands G. We claim that G does not possess a PNE. For a contradiction, suppose (α, d) is a PNE of G.

We first show that it is without loss of generality to assume that $d_i \leq \tau_i^{\mathrm{r}}$ for all $i \in N$. Specifically, we show that if there is a PNE (α, d) of G, then there is also a PNE (α, d') with $d_i' \leq \tau_i^{\mathrm{r}}$ for all $i \in N$. To see this, note that $U_i(d_i) = U_i(\tau_i^{\mathrm{r}})$ for all $d_i \geq \tau_i^{\mathrm{r}}$ and all $i \in N$. If there is a resource $r \in \alpha_i$ with $c_r(\ell_r(\alpha, d)) > c_r(\ell_r(\alpha, d) - d_i + \tau_i^{\mathrm{r}})$, we derive that player i improves lowering her demand from d_i to τ_i^{r}, which contradicts the fact that (α, d) is a PNE. This implies that $c_r(\ell_r(\alpha, d)) = c_r(\ell_r(\alpha, d) - d_i + \tau_i^{\mathrm{r}})$ for all $r \in \alpha_i$ and setting $d_i' = \tau_i^{\mathrm{r}}$ we derive that $\pi_j(\alpha, d_i', d_{-i}) = \pi_j(\alpha, d)$ for all $j \in N$. From there, (α, d_{-i}, d_i') is also a PNE. Iterating this argument, we obtain a PNE (α, d') with $d_i' \leq \tau_i^{\mathrm{r}}$ for all $i \in N$.

Let (α, d') be such a PNE. We claim that $d_i' \geq \sigma_i^{\mathrm{r}}$ for all $i \in N$. Suppose there is $i \in N$ with $d_i' < \sigma_i^{\mathrm{r}}$ and consider the strategy where player i plays a demand of $d_i'' = \sigma_i^{\mathrm{r}}$ instead. We calculate

$$\pi_i(\alpha, d_i'', d_{-i}') - \pi_i(\alpha, d_i') = \frac{U_i^{\mathrm{r}}(d_i'') + \mu}{d_i''}(d_i'' - d_i') - g(d_i'') \sum_{r \in \alpha_i} c_r(\ell_r(\alpha, d') - d_i' + d_i'') + g(d_i') \sum_{r \in \alpha_i} c_r(\ell_r(\alpha, d'))$$

$$\geq \frac{\mu}{\sigma_{\max}}(d_i'' - d_i') - g(d_i'')L(d_i'' - d_i') - (g(d_i'') - g(d_i')) \sum_{r \in \alpha_i} c_r(\ell_r(\alpha, d'))$$

$$\geq (d_i'' - d_i)\left(\frac{\mu}{\sigma_{\max}} - g(\sigma_{\max}) \cdot L - M \cdot C_{\max}\right)$$

$$> 0.$$

Hence, player i improves, which contradicts the fact that (α, d') is a PNE. To finish the proof, assume there is a PNE (α, d') with $d_i \in [\sigma_i^r, \tau_i^r]$ for all $i \in N$. Using the fact that the new utility function on $[\sigma_i^r, \tau_i^r]$ equals the old utility function raised by the constant μ, we derive that (α, d') is also a PNE of G^r, and we reach a contradiction. □

Note that the utility functions constructed in the proof of Theorem 7 are not differentiable. We proceed to show that the result continues to hold if we assume that demands are unrestricted and all utility functions are smooth (i.e., infinitely often differentiable).

Theorem 8. *Let G^r be a g-scaled congestion game with variable demands such that all utility functions U_i^r: $[\sigma_i^r, \tau_i^r] \to \mathbb{R}_{\geq 0}$ satisfy $\tau_i^r < \infty$. If G^r does not possess a PNE, then there is a corresponding g-scaled congestion game with unrestricted variable demands and smooth utility functions that does not possess a PNE as well.*

Proof. Let U_i denote the piecewise linear utility function of player i constructed in the proof of Theorem 7. For $i \in N$, we define the improvement function ζ_i: $S \to \mathbb{R}_{\geq 0}$ as the function that maps every strategy profile to the value by which player i can maximally improve her utility when switching to her best reply. Formally,

$$\zeta_i(s) = \max_{s_i' \in S_i}\{\pi_i(s_i', s_{-i}) - \pi_i(s)\}.$$

Note that for every player i, there is $\omega_i \in \mathbb{R}_{>0}$ such that U_i is constant on $[\omega_i, \infty]$. From there, we may effectively restrict the demand of every player i to $[0, \omega_i]$. Setting $\bar{S}_i = \{s_i = (\alpha_i, d_i) \in S_i\colon d_i \leq \omega_i\}$, we observe that $\max_{s_i' \in S_i} \pi_i(s_i, s_{-i}) = \max_{s_i' \in \bar{S}_i} \pi_i(s_i, s_{-i})$. Using that π_i is continuous and $\bar{S}$ is compact, the maximum is attained and thus, ζ is well defined.

Writing the strategy profile s as $s = (\alpha, d)$ and using that all private payoff functions are continuous in d, we observe that $\zeta_i(\alpha, d)$ is continuous in d as well. Next, we define ζ: $S \to \mathbb{R}_{\geq 0}$ as $\zeta(s) = \max_{i \in N} \zeta_i(s)$. As the maximum of finitely many continuous functions, ζ is continuous in d as well. As a consequence, $\epsilon = \min_{s \in \bar{S}} \zeta(s) = \min_{s \in S} \zeta(s)$ is attained and, since G does not admit a PNE, $\epsilon > 0$.

For any $\delta > 0$, the utility function U_i of every player i is infinitely often differentiable except on δ-balls around σ_i and τ_i. Ghomi [12] proved that a convex function can be approximated by a smooth convex function such that both functions comply in all regions where the original function is already smooth. Using this result, we can replace U_i by an approximation $\tilde{U}_i$: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is concave, smooth, and satisfies $U_i(x) = \tilde{U}_i$ for all $x \in [0, \sigma_i - \delta) \cup (\sigma_i + \delta, \tau_i - \delta) \cup (\tau_i + \delta, \infty)$. Using that U_i is continuous, we may choose $\delta > 0$ such that $|U_i(x) - \tilde{U}_i(x)| < \epsilon/2$ for all $x \in \mathbb{R}_{\geq 0}$. The new set of utility functions $(\tilde{U}_i)_{i \in N}$ defines a new g-scaled congestion game with variable demands $\tilde{G} = (N, S, \tilde{\pi})$. Because $\zeta(s) \geq \epsilon$ for all $s \in S$, there is for all $s \in S$ a player $i(s)$ and an alternative strategy $s_{i(s)}'(s) \in S_{i(s)}$ such that $\pi_{i(s)}(s) \leq \pi_{i(s)}(s_{i(s)}'(s), s) - \epsilon$. Using that $|\pi_i(s) - \tilde{\pi}_i(s)| < \epsilon/2$ for all $s \in S$ and $i \in N$, we derive that $\tilde{\pi}_{i(s)}(s) < \tilde{\pi}_{i(s)}(s_i'(s), s_{-i})$. Hence, the game G^ϵ does not possess a PNE. □

We have obtained the following characterization of consistency for congestion games with variable demands.

Theorem 9. *Let $\mathscr{C} \neq \varnothing$ be a set of continuous functions and g a scaling function. Then, the following are equivalent*:

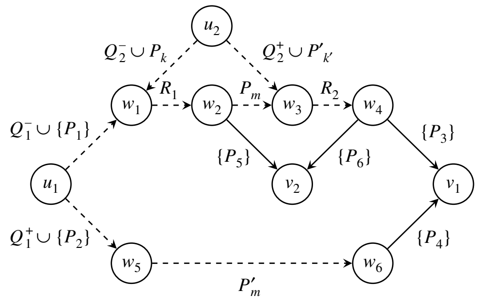
(1) *$\mathscr{C}$ is consistent for g-scaled congestion games with unrestricted variable demands and smooth utility functions.*

(2) *Exactly one of the following two holds*:

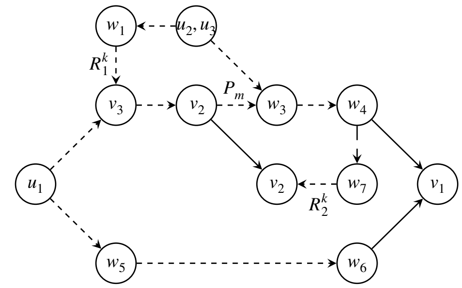
(a) *$\mathscr{C}$ only contains homogeneously exponential functions of type $c(x) = a_c e^{\phi x}$, where $a_c \in \mathbb{R}_{>0}$ may depend on c, and $\phi \in \mathbb{R}_{>0}$ must be equal for all $c \in \mathscr{C}$.*

(b) *$g(x) \equiv x$, and $\mathscr{C}$ only contains affine functions of type $c(x) = a_c x + b_c$, where $a_c \in \mathbb{R}_{>0}$ and $b_c \in \mathbb{R}_{\geq 0}$.*

6. Games on networks. In this section, we examine directed network congestion games, where the resources correspond to edges of a directed graph and the allowable subsets for a player correspond to the paths connecting a player-specific source and sink.



(a) Network congestion game with resource-dependent demands



(b) Network congestion game with variable demands

Figure 4. The network congestion game with resource-dependent demands and the network congestion game with variable demands constructed in the proofs of Lemmas 6 and 7, respectively. Solid lines correspond to single edges and dashed lines correspond to directed paths.

6.1. Network congestion games with resource-dependent demands. We first prove a variant of the generalized monotonicity lemma (Lemma 1) for games on directed networks. Note that this is a stronger result than Lemma 1 as we now require a network structure when constructing games without having a PNE.

Lemma 6 (Generalized Monotonicity Lemma for Directed Networks). *Let g be a scaling function and let c be a differentiable function. If c is consistent for g-scaled network congestion games with resource-dependent demands, then c satisfies the GMC for g.*

Proof. We show that if c does not satisfy the GMC for g, then there is a g-scaled network congestion game with resource-dependent demands with costs equal to c that does not possess a PNE. Let such c be given. Lemma 1 implies the existence of a two-player game G with costs equal to c that does not possess a PNE. The construction of G involves six mutually disjoint sets of resources $R_1, R_2, Q_1^-, Q_1^+, Q_2^-, Q_2^+ \in 2^R$ such that $\mathcal{A}_1 = \{R_1 \cup R_2 \cup Q_1^-, Q_1^+\}$, and $\mathcal{A}_2 = \{R_1 \cup Q_2^-, R_2 \cup Q_2^+\}$, where for $i \in \{1, 2\}$ always one of the two sets Q_i^- and Q_i^+ is empty. In addition, there are $x, y, \delta \in \mathbb{R}_{>0}$ such that the players' demands equal $d_{1,r} = x$ for all $r \in R$, $d_{2,r} = y$ for $r \in R_1 \cup Q_2^-$, and $d_{2,r} = y + \delta$, otherwise.

Because G is finite, there is $\rho > 0$ such that G does not possess a ρ-approximate PNE. To obtain a game with network structure, we slightly modify G by adding additional resources to each of the players' strategies without changing the equilibrium structure of the game. Let $k, k' \in \mathbb{N}_{>0}$ be such that

$$|(k+1)g(y)c(y) - (k'+1)g(y+\delta)c(y+\delta)| < \rho/2. \tag{24}$$

Such k, k' exist since $c(y) > 0$ and $c(y+\delta) > 0$. Let $m \in \mathbb{N}$ be such that

$$m \cdot g(y) \cdot \min\{c(y), c(x+y)\} > g(y)c(y). \tag{25}$$

We add $k+k'+2m+6$ new resources with cost function c. We pick $k+k'+2m$ of the new resources and partition them into subsets $P_k, P'_{k'}, P_m, P'_m$ with cardinalities $|P_k| = k$, $|P'_{k'}| = k'$, $|P_m| = m$, $|P'_m| = m$. The remaining 6 new resources are called $p_1, \ldots, p_6$. The demand of player 1 equals x for all new resources. Player 2 uses all new resources with demand y except for the resources contained in $P'_{k'} \cup \{p_6\}$, which she uses with demand $y + \delta$.

Consider the network game G^{dn} shown in Figure 4(a). The feasible allocations are equal to the sets of their respective (u_i, v_i)-paths, i.e.,

$$\mathcal{A}_1 = \{Q_1^- \cup \{p_1\} \cup R_1 \cup P_m \cup R_2 \cup \{p_3\}, Q_1^+ \cup \{p_2\} \cup P'_m \cup \{p_4\}\},$$
$$\mathcal{A}_2 = \{Q_2^- \cup P_k \cup R_1 \cup \{p_5\}, Q_2^+ \cup P'_{k'} \cup R_2 \cup \{p_6\}, Q_2^- \cup P_k \cup R_1 \cup P_m \cup R_2 \cup \{p_6\}\}.$$

Using (25), we observe that the third strategy $Q_2^- \cup P_k \cup R_1 \cup P_m \cup R_2 \cup \{p_6\}$ of player 2 is strictly dominated by her first strategy $Q_2^- \cup P_k \cup R_1 \cup \{p_5\}$. This implies that player 2 does not use her third strategy in any PNE. Compared to G, the payoff of player 1 is decreased by the constant $mg(x)c(x)$. The payoff of player 2 for her first allocation is decreased by $-(k+1)g(y)c(y)$ and the payoff of her second allocation is decreased by $-(k'+1)g(y+\delta)c(y+\delta)$. Using (24) and the fact that the initial game G does not possess a ρ-approximate PNE, we conclude that G^{dn} does not possess a PNE. □

We are ready to state our characterization theorem for network congestion games with resource-dependent demands.

THEOREM 10. *Let $\mathscr{C} \neq \varnothing$ be a set of continuous functions and let g be a scaling function. Then, the following are equivalent*:

(1) *$\mathscr{C}$ is consistent for g-scaled directed-network congestion games with resource-dependent demands.*

(2) *$\mathscr{C}$ is universally consistent for g-scaled directed-network congestion games with resource-dependent demands.*

(3) *At least one of the following two cases holds*:

(a) *$\mathscr{C}$ contains only constant functions.*

(a) *$g(x) \equiv x$, and $\mathscr{C}$ contains only affine functions.*

SKETCH OF PROOF. As shown in previous work (Harks and Klimm [16, Theorem 6.2]), every set of cost functions that is consistent for network weighted congestion games may only contain affine functions or only contain exponential functions. Given this observation and the fact that the GMC is also necessary for consistency for network games (Lemma 6), the result can be proven analogously to Theorem 6.

This characterization of consistency is even valid for games with three players.

6.2. Network congestion games with variable demands. We proceed to characterize the set of consistent cost functions for network congestion games with variable demands.

LEMMA 7. *Let g be a scaling function and c be a differentiable and convex cost function that is strictly positive on $\mathbb{R}_{>0}$ and has the property that c'/c is injective on $\mathbb{R}_{>0}$. If c is consistent for g-scaled directed-network congestion games with variable demands, then c satisfies the GMC for g.*

PROOF. Let g be an arbitrary scaling function and let c be a cost function with the demanded properties that is consistent for g-scaled congestion games with variable demands. For a contradiction, let us assume that c does not satisfy the GMC for g. The generalized monotonicity lemma for directed networks (Lemma 6) implies the existence of a g-scaled network congestion game with resource-dependent demands and costs equal c on all resources not possessing a PNE. We modify this construction to obtain a directed network congestion game with variable demands not possessing a PNE. First, we define the players' sets of feasible demands setting $\sigma_1 = \tau_1 = x$, $\sigma_2 = y$, and $\tau_2 = y + \delta$. Next, consider the network game shown in Figure 4(b). Because the cost player 2 experiences on the edge (w_2, v_2) is bounded by $\max_{z \in [y,\, y+\delta]} g(z)c(z)$ and the cost functions are strictly positive, we can make the path $P_{m'}$ sufficiently long such that the unique strategy of player 2 containing $P_{m'}$ is strictly dominated.

Our goal is to enforce player 2 to use the demand y when on her left path $u_2 \rightarrow w_1 \rightarrow w_2 \rightarrow v_2$ and the demand $y + \delta$ when on her right path $u_2 \rightarrow w_3 \rightarrow w_4 \rightarrow v_2$. As in Lemma 5 this can be achieved by adding additional resources to every strategy of player 2, where the additional resources contained in the left path are used by an additional third player. Thus, for $k \in \mathbb{N}$, we add additional paths R_1^k, R_2^k containing k additional edges. We also add a third player associated with the source-sink pair (u_3, v_3) whose only strategy is to follow the paths $u_3 \rightarrow w_1 \rightarrow v_3$.

Along the same chain of reasoning as in Lemma 5, we can choose k large enough and an appropriate utility function of player 2 such that player 2 always uses the demand y when allocated on her left path and the demand $y + \delta$ when allocated on her right path. Then, using that the network congestion game with resource-dependent demands has no PNE implies that also the network congestion game with variable demands does not possess a PNE. □

We obtain the following result analogously to Theorem 6.

THEOREM 11. *Let $\mathscr{C} \neq \varnothing$ be a set of continuous functions and let g be a scaling function. Then, the following are equivalent*:

(1) *$\mathscr{C}$ is consistent for g-scaled directed-network congestion games with variable demands.*

(2) *Exactly one of the following two statements holds*:

(a) *$\mathscr{C}$ only contains homogeneously exponential functions of type $c(x) = a_c e^{\phi x}$, where $a_c \in \mathbb{R}_{>0}$ may depend on c, and $\phi \in \mathbb{R}_{>0}$ must be equal for all $c \in \mathscr{C}$;*

(b) *$g(x) \equiv x$, and $\mathscr{C}$ only contains affine functions of type $c(x) = a_c x + b_c$, where $a_c \in \mathbb{R}_{>0}$, $b_c \in \mathbb{R}_{\geq 0}$.*

7. Conclusions. We considered the fundamental problem of the existence of pure Nash equilibria and the AFIP in congestion games with variable demands. Several characterizations of the cost structure with respect to the existence of a PNE and the AFIP have been obtained. Since games with variable demands are general enough to closely capture many elements of practical applications, we are confident that our results help to understand the behavior of myopic play in real systems.

Although this paper addressed the existence of a PNE and the approximate finite improvement property with respect to the cost structure (without constraints on the strategy spaces and the utility functions), it is natural to ask for combinatorial properties of the strategy spaces that ensure the existence of pure Nash equilibria for general cost functions. In light of the positive result of Ackermann et al. [1] for weighted congestion games on matroids, particularly congestion games with variable demands where the set of feasible allocations of every player form the basis of a matroid are a promising avenue for future work. Alternatively, one can restrict the set of feasible utility functions (e.g., assume linear functions) and ask for the existence of a PNE. Also, as for weighted congestion games, the case of symmetric strategy spaces is not well understood.

Another interesting research direction is to investigate the prices of anarchy and stability in congestion games with variable demands. In particular, it would be very interesting to compare the prices of anarchy and stability of congestion games with variable demands and unrestricted demands with known results for weighted congestion games.

As shown in this paper, the concept of essential improvement moves may help to show the existence of pure Nash equilibria in games that do not admit a potential function. It would be interesting to see this technique being applied to further classes of games for which the FIP does not hold but where it is conjectured that a PNE exists (as, e.g., in weighted singleton congestion games with player-specific linear cost functions; see Gairing et al. [10] and Georgiou et al. [11]).

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