Journal of Mathematical Economics 61 (2015) 211–220

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Equilibria in a class of aggregative location games

Tobias Harks [a,*], Max Klimm [b]

[a] Institut für Mathematik, Universität Augsburg, 86135 Augsburg, Germany

[b] Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany

ARTICLE INFO

Article history:
Received 4 January 2015
Received in revised form
21 September 2015
Accepted 22 September 2015

Keywords:
Pure Nash equilibrium
Aggregative game
Non-convex strategy set
Cournot oligopoly
Congestion game
Queuing game

ABSTRACT

Consider a multimarket oligopoly, where firms have a single license that allows them to supply exactly one market out of a given set of markets. How does the restriction to supply only one market influence the existence of equilibria in the game? To answer this question, we study a general class of aggregative location games where a strategy of a player is to choose simultaneously both a location out of a finite set and a non-negative quantity out of a compact interval. The utility of each player is assumed to depend solely on the chosen location, the chosen quantity, and the aggregated quantity of all other players on the chosen location. We show that each game in this class possesses a pure Nash equilibrium whenever the players' utility functions satisfy the assumptions *negative externality, decreasing marginal utility, continuity,* and *Location–Symmetry*. We also provide examples exhibiting that, if one of the assumptions is violated, a pure Nash equilibrium may fail to exist.

1. Introduction

We introduce a class of aggregative games that combines the characteristics of finite games, such as congestion games, and continuous games, such as Cournot oligopolies. As an example of a game in this class, consider a multimarket oligopoly in which each firm may offer a positive quantity on exactly one market only, but is free to choose the market out of a set of feasible markets. This situation arises, for instance, if governmental policies oblige each firm to be engaged in at most one market at a time, e.g., by issuing only a single license per firm. Mathematically, the restriction on a single positive quantity renders the strategy space to be non-convex. As a consequence, standard tools, such as fixed point theorems à la Kakutani, are not directly applicable to establish the existence of equilibria.

In this paper, we consider a general class of aggregative games that includes multimarket oligopolies with licenses discussed above as a special case. Formally, let A be a finite set of locations and $N = \{1, \dots, n\}$ be a finite set of players. Each player is associated with a non-empty subset $A_i \subseteq A$ of feasible locations and a non-empty and compact interval of non-negative quantities Q_i feasible to her. In a strategy profile, each player i chooses simultaneously both a feasible location $a_i \in A_i$ and a feasible quantity $q_i \in Q_i$. We require that the utility of each player depends solely on the location chosen, her own quantity, and the aggregated quantity of all other players choosing the same location. It is a useful observation that for such an aggregative game the utility of each player i in strategy profile $(a, q) = (a_1, \dots, a_n, q_1, \dots, q_n)$ can be represented by a set of indirect utility functions $v_{i,\sigma} : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$, $i \in N$, $\sigma \in A_i$ so that $u_i(a, q) = v_{i,a_i}(q_i, \ell_{-i,a_i}(a, q))$, where, for an arbitrary location $\sigma \in A$, we denote by $\ell_{-i,\sigma}(a, q) = \sum_{j \in N \setminus \{i\} : a_j = \sigma} q_j$ the aggregated quantity of all players except i on location σ.

We impose the following four assumptions on the player's indirect utility functions. The first assumption, called "Negative Externality", requires that the indirect utility of a player does not increase if the aggregated quantity of the other players on the same location increases. Informally, the second assumption "Decreasing Marginal Utility" requires that, for every player, the marginal indirect utility function exists and decreases if both the player's quantity and the aggregated quantity of the chosen location increase.[1] Third, we require that the indirect utility functions of each player are continuous. The last assumption is called "Location–Symmetry" and requires that, for each player i, we have $v_{i,\sigma} = v_{i,\tau}$ for all $\sigma, \tau \in A_i$.

We prove that aggregative games for which the indirect utility functions satisfy "Negative Externality", "Decreasing Marginal

* Corresponding author. Tel.: +31 43 3884630; fax: +31 43 3882000.
E-mail addresses: tobias.harks@math.uni-augsburg.de (T. Harks), klimm@math.tu-berlin.de (M. Klimm).

http://dx.doi.org/10.1016/j.jmateco.2015.09.006
0304-4068/

[1] As a consequence of this assumption, players will lower their quantities when their competitors raise their quantities. Thus, this assumption can also be seen as a variant of "strategic substitutes" (Bulow et al., 1985; Dubey et al., 2006).

Utility", "Continuity", and "Location–Symmetry" possess a pure Nash equilibrium. To prove this existence result, we devise an algorithm that computes a pure Nash equilibrium. Our algorithm relies on iteratively computing a *partial* equilibrium on every location separately using Kakutani's fixed point theorem. Here, a partial equilibrium is a strategy profile that is resilient against unilateral *quantity* deviations. Given a partial equilibrium, the algorithm selects a single player who can strictly improve and computes for this player a best reply. After such a best reply it recomputes a partial equilibrium and reiterates. We prove that a player-specific aggregated quantity vector of the partial equilibria lexicographically decreases in every iteration and, thus, the algorithm terminates after a finite number of iterations. A perhaps surprising property of our proof is that even though we iteratively recompute a partial equilibrium – using Kakutani's theorem as a black box – there is enough structure of such a partial equilibrium to prove that the algorithm terminates. For games with only two players, we prove that the assumption "Location–Symmetry" is not needed to guarantee the existence of an equilibrium, i.e., already "Negative Externality", "Decreasing Marginal Utility", and "Continuity" of the players' indirect utilities are sufficient to yield the existence of a pure Nash equilibrium. To demonstrate the usefulness of our results, we give concrete examples of games that fit into our model: restricted multimarket oligopolies, congestion games with variable quantities, and multiserver queuing games. For all these examples, to the best of our knowledge, we establish for the first time the existence of a pure Nash equilibrium.

1.1. Related work

Many works on the existence of Nash equilibria in strategic games impose strong assumptions on the topological properties of the players' strategy sets. Most prominently, Nash's famous existence result for equilibria in mixed extensions of finite games uses the fact that the mixed strategy set of each player is the simplex spanned by her pure strategies and, thus, a well-behaved convex and compact subset of some Euclidean space. The existence of a mixed equilibrium is then established via the fixed point theorems of Kakutani (cf. Nash, 1950a) or Brouwer (cf. Nash, 1950b).[2] For these fixed point arguments, however, the convexity of the (mixed) strategy sets is crucial. We are interested in pure Nash equilibria in this work, and, for our class of games, the strategy space of a player is not necessarily convex.

A strategically equivalent game to ours with convex strategy space can be obtained by taking the convex hull of the strategy space and assigning sufficiently low utility values for infeasible strategies. This method inevitably leads to games with *discontinuous* utility functions. There is a substantial body of literature on discontinuous games that identifies conditions under which an equilibrium exists (cf. Barelli and Meneghel, 2013, Bich, 2009, Carmona, 2009, 2011, Dasgupta and Maskin, 1986, McLennan et al., 2011, Reny, 1999 and Simon, 1987). To the best of our knowledge, the currently most general sufficient condition for the existence of an equilibrium is given by Barelli and Meneghel (2013) using the concept of *continuous security*. In Appendix B.2, we show that this condition is not satisfied for our class of games. It is also straightforward to show that our games are not supermodular (see Appendix B.3) which prevents us from the application of Tarski's fixed-point theorem or further comparative statics analysis (cf. Amir, 1996, Milgrom and Roberts, 1990; Milgrom and Shannon, 1994, Roy and Sabarwal, 2010, Tarski, 1955, Topkis, 1979, 1998 and Vives, 1990 for works in this field).

Closer to our work, Dubey et al. (2006) considered a class of games, for which the strategy set of each player is a compact and possibly non-convex subset of the non-negative real line, and the utility of each player depends only on her own strategy, and the sum of the others' strategies. They derived the existence of a pure Nash equilibrium assuming that there exists a selection from the best reply correspondence of each player, which is non-increasing or non-decreasing in the aggregated strategy of the other players.[3] This assumption is met, e.g., by Cournot oligopolies. Jensen (2010) generalized the work of Dubey et al. and Kukushkin as he allows for higher dimensional strategy sets. In Jensen's model, the utility of each player only depends on her own strategy and a one-dimensional aggregate of the strategies of the other players. In particular, the aggregate is independent from the own strategy. This is in contrast to our model where the utility of a player depends on the chosen location, her own quantity, and the aggregated quantity of all other players that choose the same location.

Very recently, Martimort and Stole (2011) presented several characterizations of equilibria in aggregative games which lead, however, not directly to existence results. Only for special cases (such as affine utilities) they establish sufficient conditions for the existence of an equilibrium.

An extended abstract of this paper appeared in the Proceedings of the 5th Workshop on Internet and Networks Economics (Harks and Klimm, 2011a).

1.2. Paper outline

The remainder of this paper is organized as follows. In Section 2, we introduce our basic model of an aggregative game and the assumptions that we impose on the players' utility functions. In Sections 3.1 and 3.2, we provide our main results for the existence of a pure Nash equilibrium in games with an arbitrary number of players and two-player games, respectively. In Section 3.3, we discuss how the assumptions "Continuity" (CON) and "Location–Symmetry" (LOC) can be weakened. In Section 4, we demonstrate the usefulness of our results by giving several applications that fit into our model. In Appendix A, we complement our results and show that if one of our assumptions on the indirect utility functions is violated, then there is a game without a pure Nash equilibrium. In Appendix B, we discuss the relationship of our existence result to known results for potential games, discontinuous games, and supermodular games.

2. The model

Let A be a finite set of locations and let $N = \{1, \dots, n\}$ be a finite set of players. For each player $i \in N$ we are given a closed interval $Q_i = [\alpha_i, \omega_i] \subseteq \mathbb{R}_{\geqslant 0}$ of feasible quantities and a subset $A_i \subseteq A$ of feasible locations. A *strategy* of player i is a tuple (a_i, q_i) where $a_i \in A_i$ is a feasible location and $q_i \in Q_i$ is a feasible quantity for player i. A *strategy profile* of the game is a tuple (a, q) where $a = (a_1, \dots, a_n)$ is the location profile and $q = (q_1, \dots, q_n)$ is the quantity profile. Note that for $|A_i| > 1$, embedding A_i into $\mathbb{N}$ renders the strategy space $S_i = A_i \times Q_i$ into a non-convex subset of $\mathbb{R}^2_{\geqslant 0}$. For a location $\sigma \in A$, and a strategy profile (a, q), we denote by $\ell_\sigma(a, q) = \sum_{j \in N: a_j = \sigma} q_j$ the aggregated quantity on location σ

[2] Further generalizations of Kakutani's fixed point theorem to strategy spaces that are non-empty, convex and compact subsets of Hausdorff locally convex topological vector spaces can be found in Debreu (1952), Fan (1952), and Glicksberg (1952).

[3] See also Kukushkin (1994, 2004) for related results on the existence of equilibria and the convergence of improvement dynamics in finite non-convex games satisfying strategic complementarities and substitutes.

and by $\ell_{-i,\sigma}(a,q) = \sum_{j \in N\setminus\{i\}: a_j=\sigma} q_j$ the aggregated quantity of all players except i on location σ.

Throughout this paper, we assume that games are *aggregative*, i.e., the utility of player i under strategy profile (a,q) depends solely on the location a_i, the quantity q_i chosen by player i, and the aggregated quantity of other players with the same location $\ell_{-i,a_i}(a,q)$. For each player i and each of her feasible locations $\sigma \in A_i$ we represent her utility by an *indirect utility function* $v_{i,\sigma} : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$. The utility of player i under strategy profile (a,q) is then defined as $u_i(a,q) = v_{i,a_i}\big(q_i, \ell_{-i,a_i}(a,q)\big)$. Given a strategy profile (a,q) a strategy $(a_i', q_i') \in S_i$ is a *better reply* of player i to (a,q) if $u_i(a_i', a_{-i}, q_i', q_{-i}) > u_i(a,q)$; it is a *best reply* if $u_i(a_i', a_{-i}, q_i', q_{-i}) \geqslant u_i(a_i'', a_{-i}, q_i'', q_{-i})$ for all $(a_i'', q_i'') \in S_i$. The strategy profile (a,q) is a *pure Nash equilibrium* if, for all players i, the strategy (a_i, q_i) is a best reply to (a,q).

We make the following four assumptions on the indirect utility functions $v_{i,\sigma}$ of player i and location $\sigma \in A_i$. The first assumption is called "Negative Externality" and requires that the indirect utility of every player increases as the aggregated quantity of other players with the same location decreases.

Assumption 1 (*Negative Externality (EXT)*). For all $i \in N$, $\sigma \in A_i$ and $x \in Q_i$, the indirect utility function $v_{i,\sigma}(x,\cdot)$ is non-increasing in the second entry, i.e., $v_{i,\sigma}(x,y) \geqslant v_{i,\sigma}(x,y')$ for all $y, y' \in \mathbb{R}_{\geqslant 0}$ with $y \leqslant y'$.

This assumption is natural when players compete over scarce resources to satisfy their quantity and has been made explicitly or implicitly in various contexts ranging from Cournot oligopolies (Cournot, 1838; Johari and Tsitsiklis, 2005) and Cournot oligopsonies (Naylor, 1994; Tirole, 1988) to traffic and communication networks (Beckmann et al., 1956; Haurie and Marcotte, 1985; Kelly et al., 1998), and biology (Milinsky, 1979).

The second assumption is called "Decreasing Marginal Utility" and requires that, for each player with a non-trivial interval of feasible quantities, the marginal indirect utility function exists and decreases if the player's quantity and the aggregated quantity of the chosen location increase.

Assumption 2 (*Decreasing Marginal Utility (DMU)*). For all $i \in N$ with $\alpha_i < \omega_i$ and $\sigma \in A_i$, the indirect utility function $v_{i,\sigma}(\cdot,\cdot)$ is piece-wise continuously differentiable in the first entry and satisfies

$$\partial_x^+ v_{i,\sigma}(x,y) > \partial_x^- v_{i,\sigma}(x',y') \tag{1}$$

for all $x \in [\alpha_i, \omega_i)$, $x' \in (\alpha_i, \omega_i]$ with $x \leqslant x'$ and $x + y \leqslant x' + y'$, where at least one of the two inequalities is strict.

In the above assumption, we denote by $\partial_x^+ v_{i,\sigma}$ and $\partial_x^- v_{i,\sigma}$ the right and left partial derivatives of $v_{i,\sigma}$ with respect to the first entry, respectively.

The assumption "Decreasing Marginal Utility" implies in particular that $v_{i,\sigma}$ is concave in the first entry.[4] Further, it requires that player i's marginal utility function strictly decreases if both the player's own quantity and the aggregated quantity on a location increases (and one of these quantities increases strictly). As a consequence, in a game that satisfies this assumption, players will lower their quantities when other players increase their quantities and, thus, "Decreasing Marginal Utility" can be interpreted as a form of "strategic substitutes" in the sense of Bulow et al. (1985) and Dubey et al. (2006). We remark that if for some $i \in N$ it holds that $\alpha_i = \omega_i$, then "Decreasing Marginal Utility" is by convention satisfied.

As a third assumption, we require that the indirect utility functions are continuous.

Assumption 3 (*Continuity (CON)*). For all $i \in N$ and $\sigma \in A_i$ the indirect utility function $v_{i,\sigma}$ is continuous.

It is worth noting that we need this assumption only to prove that, for each location profile $a = (a_1, \ldots, a_n)$, there are quantities $q = (q_1, \ldots, q_n)$ such that no player can improve her utility by unilaterally changing the demand while leaving the location fixed; see Proposition 5. While continuity (together with decreasing marginal utility) is clearly sufficient for the existence of such *partial equilibria* (as we term such strategy profiles), we may slightly generalize our results by replacing the continuity assumption with any other assumption that guarantees the existence of such partial equilibria. This is discussed more formally in Section 3.3.

The next assumption "Location–Symmetry" imposes that players have no a priori preferences over locations, i.e., each player's utility is solely defined by her own quantity and the aggregated quantity of the chosen location and *not* by the identity of the location itself.

Assumption 4 (*Location–Symmetry (LOC)*). For all $i \in N$, we have $v_{i,\sigma} = v_{i,\tau}$ for all $\sigma, \tau \in A_i$.

Note that "Location–Symmetry" does not require symmetry among players, i.e., we still allow $v_i \neq v_j$ for $i \neq j$. We only require that every player is indifferent between any two feasible locations as long as their own quantity and the aggregated quantity on these locations is equal. Clearly, "Location–Symmetry" is the most restrictive and controversial assumption. We show, however, that without Location–Symmetry, there are games without a pure Nash equilibrium. It should be noted, that for our main result to hold, we can slightly weaken "Location–Symmetry" by only requiring that the player-specific indirect utility functions are only equal up to location-specific shifts on the resources. This result is formally proven in Section 3.3.

3. Existence of pure Nash equilibria

We give two existence results for pure Nash equilibria in aggregative games. We first show that aggregative games satisfying the assumptions "Negative Externality" (EXT), "Decreasing Marginal Utility" (DMU), "Continuity" (CON), and "Location–Symmetry" (LOC) always possess a pure Nash equilibrium. For aggregative games with only two players, we prove the existence of a pure Nash equilibrium under the weaker assumption that only Continuity (CON), Negative Externality (EXT) and Decreasing Marginal Utility (DMU) are satisfied. In Appendix A, we show that if any of the assumptions is dropped, then there are instances without a pure Nash equilibrium.

To prove our results, we introduce the concept of a *partial equilibrium*. The strategy profile (a,q) is called a partial equilibrium, if $u_i(a,q) \geqslant u_i(a, q_i', q_{-i})$ for all $i \in N$ and $q_i' \in Q_i$. Using Kakutani's fixed point theorem, it follows that under assumptions CON and DMU for every location profile $a \in A$, there exists a partial equilibrium of the form (a,q).

Proposition 5. *Let G be an aggregative game for which the indirect utility functions satisfy CON and DMU. Then, for every location profile a, there exists a quantity profile $q = (q_1, \ldots, q_n)$ with $q_i \in Q_i$ for all $i \in N$ such that (a,q) is a partial equilibrium.*

[4] In telecommunication networks, strict concavity of the utility function in the quantity is justified by application-specific characteristics such as the rate-control algorithm used in common congestion control protocols (cf. Kelly et al., 1998; Shenker, 1995).

Proof. Let a be an arbitrary location profile. Consider the restricted game $\tilde{G}$ with $\tilde{A}_i = \{a_i\}$. In $\tilde{G}$, the strategy space of each player reduces to the convex and closed interval $Q_i \subseteq \mathbb{R}_{\geqslant 0}$. Using CON and DMU, the utility function of each player is continuous and concave. By Kakutani's fixed point theorem, a pure Nash equilibrium q of $\tilde{G}$ exists and, by construction, (a, q) is a partial equilibrium of G. □

Remark 6. The existence of a partial equilibrium can be derived under weaker assumptions than CON and DMU on the utility functions, e.g., those required in Jensen (2010) and Dubey et al. (2006).

The following lemma will be important throughout this paper. It expresses the first-order optimality conditions of a partial equilibrium. The proof is straightforward and left to the reader.

Lemma 7. *Let G be an aggregative game for which the indirect utility functions satisfy DMU, and let (a, q) be a partial equilibrium of G. Then, for all $i \in N$ with $\alpha_i < \omega_i$ the following conditions hold:*

1. $\partial_x^+ v_{i,a_i}\big(q_i, \ell_{-i,a_i}(a, q)\big) \leqslant 0$, *if* $q_i < \omega_i$.
2. $\partial_x^- v_{i,a_i}\big(q_i, \ell_{-i,a_i}(a, q)\big) \geqslant 0$, *if* $q_i > \alpha_i$.

For a location $\sigma \in A$, we define the *active set* on location σ under strategy profile (a, q) as $N_\sigma(a, q) = \{i \in N : a_i = \sigma\}$. We need the following lemma.

Lemma 8 (*Uniqueness Lemma*)**.** *Let G be an aggregative game for which the indirect utility functions satisfy DMU, and let (a, q) and (a', q') be two partial equilibria of G. Then, the following holds:*

1. $\ell_\sigma(a, q) = \ell_\sigma(a', q')$ *for all* $\sigma \in A$ *with* $N_\sigma(a, q) = N_\sigma(a', q')$.
2. $\ell_\sigma(a, q) \leqslant \ell_\sigma(a', q')$ *for all* $\sigma \in A$ *with* $N_\sigma(a, q) \subseteq N_\sigma(a', q')$.

Proof. It suffices to prove 2. If, for a strategy profile (a, q), we have $N_\sigma(a, q) = \varnothing$, then $\ell_\sigma(a, q) = 0$ and there is nothing left to show. For the sake of a contradiction, suppose that there is a location $\sigma \in A$ with $\ell_\sigma(a, q) > \ell_\sigma(a', q')$ and $\varnothing \neq N_\sigma(a, q) \subseteq N_\sigma(a', q')$. This implies the existence of a player $i \in N_\sigma(a, q)$ with $q_i > q_i'$. In particular, we have $\omega_i \geqslant q_i > q_i' \geqslant \alpha_i$. The conditions of Lemma 7 for a partial equilibrium give $\partial_x^- v_{i,\sigma}\big(q_i, \ell_{-i,\sigma}(a, q)\big) \geqslant 0$ and $\partial_x^+ v_{i,\sigma}\big(q_i, \ell_{-i,\sigma}(a', q')\big) \leqslant 0$. We get

$$0 \geqslant \partial_x^+ v_{i,\sigma}\big(q_i', \ell_{-i,\sigma}(a', q')\big) \overset{\text{DMU}}{>} \partial_x^- v_{i,\sigma}\big(q_i, \ell_{-i,\sigma}(a, q)\big)$$

a contradiction. The strict inequality is based on the assumption DMU using $q_i' < q_i$ and $\ell_\sigma(a', q') < \ell_\sigma(a, q)$. □

3.1. Multiplayer games

We are now ready to prove the existence of a pure Nash equilibrium in aggregative games with an arbitrary number of players. We claim that the following iterative Contraction-Switching procedure converges to a pure Nash equilibrium after a finite number of iterations.

1. Start with an arbitrary strategy profile (a, q).
2. **Contraction phase**: Compute a partial equilibrium $(a, \tilde{q})$.
3. **Switching phase**: If there is a player i with a better reply to $(a, \tilde{q})$, pick a best reply (a_i', q_i'), set $(a, q) = (a_i', a_{-i}, q_i', \tilde{q}_{-i})$ and proceed with 2. Else, return $(a, \tilde{q})$.

In Step 2, we actually call an oracle that takes as input a strategy profile (a, q) and returns a partial equilibrium of the form $(a, \tilde{q})$. By Proposition 5, this is always possible.

In the following, we show that this procedure ends after finitely many steps (involving finitely many calls of the oracle) and outputs a pure Nash equilibrium. The following properties are the key to prove that the Contraction-Switching procedure terminates.

Lemma 9. *Let G be an aggregative game for which all indirect utility functions satisfy EXT, DMU, and LOC. Let (a, q) be a partial equilibrium of G, let (a_i', q_i') be a best reply of player i with $u_i(a_i', a_{-i}, q_i', q_{-i}) > u_i(a, q)$ and let $(a_i', a_{-i}, \tilde{q}_i, \tilde{q}_{-i})$ be a partial equilibrium. Then, the following properties hold:*

1. $\ell_{a_i'}(a_i', a_{-i}, q_i', q_{-i}) < \ell_{a_i}(a, q)$ *(Switching Property)*
2. $\ell_{a_i'}(a_i', a_{-i}, \tilde{q}_i, \tilde{q}_{-i}) \leqslant \ell_{a_i'}(a_i', a_{-i}, q_i', q_{-i})$ *(Contraction Property)*
3. $\ell_{a_i}(a_i', a_{-i}, \tilde{q}_i, \tilde{q}_{-i}) \leqslant \ell_{a_i}(a, q)$ *(Monotonicity Property).*

Proof. We begin proving the switching property. For the sake of a contradiction, assume $\ell_{a_i'}(a_i', a_{-i}, q_i', q_{-i}) \geqslant \ell_{a_i}(a, q)$. We distinguish the following three cases:

First case $q_i' > q_i$: As (a, q) is a partial equilibrium and $q_i < q_i' \leqslant \omega_i$, by Lemma 7, we have $0 \geqslant \partial_x^+ v_{i,a_i}\big(q_i, \ell_{-i,a_i}(a, q)\big)$. We calculate

$$\begin{aligned} 0 &\geqslant \partial_x^+ v_{i,a_i}\big(q_i, \ell_{-i,a_i}(a, q)\big) \\ &\overset{\text{DMU}}{>} \partial_x^- v_{i,a_i}\big(q_i', \ell_{-i,a_i'}(a_i', a_{-i}, q_i', q_{-i})\big) \\ &\overset{\text{LOC}}{=} \partial_x^- v_{i,a_i'}\big(q_i', \ell_{-i,a_i'}(a_i', a_{-i}, q_i', q_{-i})\big) \\ &\geqslant 0, \end{aligned}$$

which gives a contradiction. The second inequality follows from the assumption DMU. The last inequality stems from the facts that (a_i', q_i') is a best reply of player i and that $q_i' > q_i \geqslant \alpha_i$.

Second case $q_i' = q_i$: This implies $\ell_{-i,a_i'}(a_i', a_{-i}, q_i', q_{-i}) \geqslant \ell_{-i,a_i}(a, q)$, hence, using the assumptions LOC and EXT, we obtain

$$\begin{aligned} u_i(a_i', a_{-i}, q_i', q_{-i}) &= v_{i,a_i'}\big(q_i', \ell_{-i,a_i'}(a_i', a_{-i}, q_i', q_{-i})\big) \\ &\overset{\text{LOC}}{=} v_{i,a_i}\big(q_i', \ell_{-i,a_i'}(a_i', a_{-i}, q_i', q_{-i})\big) \\ &\overset{\text{EXT}}{\leqslant} v_{i,a_i}\big(q_i, \ell_{-i,a_i}(a, q)\big) \\ &= u_i(a, q). \end{aligned}$$

We derive that player i does not strictly improve her utility, a contradiction.

Third case $q_i' < q_i$: Observe that $\ell_{-i,a_i'}(a_i', a_{-i}, q_i', q_{-i}) > \ell_{-i,a_i}(a, q)$ as $q_i' < q_i$. Consider the strategy (a_i, q_i') of player i. We obtain

$$\begin{aligned} u_i(a_i, a_{-i}, q_i', q_{-i}) &= v_{i,a_i}\big(q_i', \ell_{-i,a_i}(a_i, a_{-i}, q_i', q_{-i})\big) \\ &= v_{i,a_i}\big(q_i', \ell_{-i,a_i}(a, q)\big) \\ &\overset{\text{LOC}}{=} v_{i,a_i'}\big(q_i', \ell_{-i,a_i}(a, q)\big) \\ &\overset{\text{EXT}}{\geqslant} v_{i,a_i'}\big(q_i', \ell_{-i,a_i'}(a_i', a_{-i}, q_i', q_{-i})\big) \\ &= u_i(a_i', a_{-i}, q_i', q_{-i}) \\ &> u_i(a, q), \end{aligned}$$

where the first inequality uses EXT and the second inequality uses the assumption that (a_i', q_i') is a better reply of player i. Thus, (a, q) is not a partial equilibrium and we reach a contradiction.

We proceed to prove the Contraction Property. For the sake of a contradiction, let us assume that $\ell_{a_i'}(a_i', a_{-i}, \tilde{q}_i, \tilde{q}_{-i}) > \ell_{a_i'}(a_i', a_{-i}, q_i', q_{-i})$. Then, at least one of the following two cases holds: Either $\tilde{q}_i > q_i'$ or there is another player $j \in N_{a_i'}(a_i', a_{-i}, \tilde{q}_i, \tilde{q}_{-i})\backslash\{i\}$ with $\tilde{q}_j > q_j$. If $\tilde{q}_i > q_i'$, we have $\partial_x^+ v_{i,a_i'}\big(q_i', \ell_{-i,a_i'}(a_i', a_{-i}, q_i', q_{-i})\big) \leqslant 0$ using the fact that (a_i', q_i') was a best reply of player i and that $q_i' < \tilde{q}_i \leqslant \omega_i$. By DMU, we then obtain

$$\begin{aligned} 0 &\geqslant \partial_x^+ v_{i,a_i'}\big(q_i', \ell_{-i,a_i'}(a_i', a_{-i}, q_i', q_{-i})\big) \\ &\overset{\text{DMU}}{>} \partial_x^- v_{i,a_i'}\big(\tilde{q}_i, \ell_{-i,a_i'}(a_i', a_{-i}, \tilde{q}_i, \tilde{q}_{-i})\big) \\ &\geqslant 0, \end{aligned}$$

a contradiction. The last inequality stems from the fact that $(a_i', a_{-i}, \tilde{q}_i, \tilde{q}_{-i})$ is a partial equilibrium and $\tilde{q}_i > q_i' \geqslant \alpha_i$.

If there is, on the other hand, $j \in N_{a_i'}(a_i', a_{-i}, \tilde{q}_i, \tilde{q}_{-i})\setminus\{i\}$ with $\tilde{q}_j > q_j$, then we have $0 \geqslant \partial_x^+ v_{j,a_i'}\big(q_j, \ell_{-j,a_i'}(a, q)\big)$ as (a, q) was a partial equilibrium and $q_j < \tilde{q}_j \leqslant \omega_j$. We then get the same contradiction as for player i.

The monotonicity property follows directly from Lemma 8 (2). □

We are now ready to state and prove our main result.

Theorem 10. *For aggregative games, assumptions CON, EXT, DMU and LOC yield the existence of a pure Nash equilibrium.*

Proof. Using the previous lemmata, we proceed to prove that the Contraction-Switching procedure terminates for any given starting profile (a, q). First notice that there are only finitely many location profiles $a = (a_i)_{i\in N}$ as both the number of players and the number of locations is finite. We show that each possible location profile is visited at most once in the Contraction-Switching procedure.

To this end, we consider for a strategy profile (a, q), the vector $\mathcal{L}(a, q) = (\mathcal{L}_i(a, q))_{i\in N} = \big(\ell_{a_i}(a, q)\big)_{i\in N}$. We shall prove that $\mathcal{L}(a, q)$ strictly decreases with respect to the sorted lexicographical order $\prec_{\text{lex}}$ that is defined as follows: For two vectors $u, v \in \mathbb{R}^n_{\geqslant 0}$ we say that u is *sorted lexicographically* smaller than v, written $u \prec_{\text{lex}} v$, if there is an index $k \in \{1, \dots, n\}$ such that $u_{\pi(i)} = v_{\psi(i)}$ for all $i < k$ and $u_{\pi(k)} < v_{\psi(k)}$ where π and ψ are permutations that sort u and v non-increasingly, i.e., $u_{\pi(1)} \geqslant u_{\pi(2)} \geqslant \dots \geqslant u_{\pi(n)}$ and $v_{\psi(1)} \geqslant v_{\psi(2)} \geqslant \dots \geqslant v_{\psi(n)}$.

To see that $\mathcal{L}(a, q)$ decreases with respect to the sorted lexicographical order, let (a, q) be an arbitrary strategy profile and let (a_i', q_i') be a best response of player i as computed in Step 3 in the algorithm. Denote by $(a_i', a_{-i}, \tilde{q})$ a partial equilibrium. For every player $j \in N\setminus\big(N_{a_i}(a, q) \cup N_{a_i'}(a, q)\big)$ we have $\mathcal{L}_j(a, q) = \mathcal{L}_j(a_i', a_{-i}, q_i', q_{-i})$. The Switching Property proven in Lemma 9 ensures that the aggregated quantity on the new location a_i' stays strictly below that of the old location a_i, thus, $\mathcal{L}_i(a_i', a_{-i}, q_i', q_{-i}) < \mathcal{L}_i(a, q)$. The Contraction Property ensures that, after the new set of players on the new location a_i' settles to a partial equilibrium, the aggregated quantity will not increase, i.e., $\mathcal{L}_j(a_i', a_{-i}, \tilde{q}) \leqslant \mathcal{L}_j(a_i', a_{-i}, q_i', q_{-i})$ for all $j \in N_{a_i'}(a_i', a_{-i}, \tilde{q})$. This implies $\mathcal{L}_j(a_i', a_{-i}, \tilde{q}) < \mathcal{L}_i(a, q)$ for all $j \in N_{a_i'}(a_i', a_{-i}, \tilde{q})$. By the Monotonicity Property we have $\mathcal{L}_j(a_i', a_{-i}, \tilde{q}) \leqslant \mathcal{L}_i(a, q)$ for all $j \in N_{a_i}(a_i', a_{-i}, \tilde{q})$. Thus, the entry $\mathcal{L}_i(a, q)$ of player i strictly decreases to $\mathcal{L}_i(a_i', a_{-i}, \tilde{q})$ and none of the changed entries becomes larger than $\mathcal{L}_i(a, q)$, hence, the whole vector lexicographically decreases after every iteration of the Contraction-Switching procedure. This fact, together with the uniqueness of the aggregated quantity vector proven in Lemma 8, implies that the algorithm never visits the same location profile twice and, thus, terminates after finitely many steps. □

Remark 11. From a methodological point of view, the proof of Theorem 10, i.e., the convergence of the contraction-switching process, combines Kakutani's fixed-point theorem (which is in general applicable to strategy spaces that are non-empty, convex and compact subsets of Hausdorff locally convex topological vector spaces) with a combinatorial argument using a lexicographical potential function (which is usually applicable to finite discrete strategy spaces).

3.2. Two-player games

We now consider aggregative games with only two players. For this class of games we show that the assumptions CON, EXT and DMU are enough to guarantee the existence of a pure Nash equilibrium.

Theorem 12. *For two-player aggregative games, assumptions CON, EXT and DMU yield the existence of a pure Nash equilibrium.*

Proof. We shall prove that the following procedure computes a pure Nash equilibrium. Start with the empty strategy profile for player 2 (or simply remove player 2 from the game) and let player 1 choose a strategy (a_1, q_1) that maximizes her utility. Then, let player 2 enter the game and choose a best reply (a_2, q_2) to (a_1, q_1). If $a_1 \neq a_2$, we have reached a pure Nash equilibrium as EXT implies that player 1 has no interest in switching to location a_2. We proceed to analyze the remaining case $a_1 = a_2$. Let $\sigma = a_1$ and let $(\sigma, \sigma, \tilde{q}_1, \tilde{q}_2)$ be a partial equilibrium. We first show that $\tilde{q}_1 \leqslant q_1$. For the sake of a contradiction, suppose $\tilde{q}_1 > q_1$. Because $q_1 < \tilde{q}_1 \leqslant \omega_1$, we have $\partial_x^+ v_{1,\sigma}(q_1, 0) \leqslant 0$ as (σ, q_1) was a strategy maximizing the utility of player 1 assuming that player 2 is not present in the game. On the other hand, we have $\partial_x^- v_{1,\sigma}(\tilde{q}_1, \tilde{q}_2) \geqslant 0$ as $\tilde{q}_1 > q_1 \geqslant \alpha_1$ and $(\sigma, \sigma, \tilde{q}_1, \tilde{q}_2)$ is a partial equilibrium. Using DMU, we obtain $0 \geqslant \partial_x^+ v_{1,\sigma}(q_1, 0) > \partial_x^- v_{1,\sigma}(\tilde{q}_1, \tilde{q}_2) \geqslant 0$, a contradiction. Next, we show $u_2(\sigma, \sigma, \tilde{q}_1, \tilde{q}_2) \geqslant u_2(\sigma, \sigma, q_1, q_2)$. To see this, note that

$$\begin{aligned} u_2(\sigma, \sigma, \tilde{q}_1, \tilde{q}_2) &= v_{2,\sigma}(\tilde{q}_2, \tilde{q}_1) \geqslant v_{2,\sigma}(q_2, \tilde{q}_1) \\ &\overset{\text{EXT}}{\geqslant} v_{2,\sigma}(q_2, q_1) \\ &= u_2(\sigma, \sigma, q_1, q_2), \end{aligned}$$

where the first inequality uses the fact that $(\sigma, \sigma, \tilde{q}_1, \tilde{q}_2)$ is a partial equilibrium and the second inequality stems from the assumption EXT and the fact that $\tilde{q}_1 \leqslant q_1$.

Using that $u_2(\sigma, \sigma, \tilde{q}_1, \tilde{q}_2) \geqslant u_2(\sigma, \sigma, q_1, q_2)$ and that (σ, q_2) was a best reply of player 2, there is no better reply of player 2 to $(\sigma, \sigma, \tilde{q}_1, \tilde{q}_2)$. If player 1 does not want to deviate as well, we have reached a pure Nash equilibrium. Thus, the only remaining case is that player 1 deviates profitably to a best reply (a_1', q_1') with $a_1' \neq \sigma$. Then, let (σ, q_2') be a best reply of player 2 to (a_1', q_1'). Using EXT, we derive that player 2 does not want to leave location σ as (σ, q_2) was a best reply to (σ, q_1). Furthermore, as shown in Lemma 8, $q_2' \geqslant \tilde{q}_2$, which implies that player 1 does not want to switch back to location σ and we have reached a pure Nash equilibrium. □

3.3. Weakening of the assumptions

The Continuity assumption CON is only needed in Proposition 5 to establish the existence of partial equilibria. Thus, it is straightforward to replace it by the weaker assumption "Consistency" (CON') that is defined below.

Assumption 13 (*Consistency (CON')*)**.** For every location profile a, there exists a quantity profile $q = (q_1, \dots, q_n)$ with $q_i \in Q_i$ for all $i \in N$ such that (a, q) is a partial equilibrium.

The Location–Symmetry assumption LOC can be replaced by the weaker assumption LOC' that is defined below.

Assumption 14 (*Location–Symmetry' (LOC')*)**.** There is a non-negative vector $(t_\sigma)_{\sigma\in A} \in \mathbb{R}^{|A|}_{\geqslant 0}$ such that $v_{i,\sigma}(q_i, t_\sigma + y) = v_{i,\tau}(q_i, t_\tau + y)$ for all $i \in N$, $\sigma, \tau \in A_i$, $q_i \in Q_i$, $y \in \mathbb{R}_{\geqslant 0}$.

While in LOC it is required that for every player i, the indirect utility functions are all equal across locations, i.e., $v_{i,\sigma} = v_{i,\tau}$ for all $\sigma, \tau \in A_i$, the weaker assumption LOC' only requires that the indirect utility functions for two different locations must be equal *up to location-specific non-negative shifts* in the second entry of the indirect utility functions.

We proceed to show that we can replace CON by CON', and LOC by LOC' in Theorem 10 and still get the existence of a pure Nash equilibrium.

Corollary 15. *For aggregative games, assumptions EXT, DMU, CON', and LOC' yield the existence of a pure Nash equilibrium.*

Proof. As for the replacement of CON with CON' note that the Contraction-switching procedure is still well-defined and terminates as proven in Theorem 10.

As for the replacement of LOC with LOC', let G be an aggregative game satisfying CON', EXT, DMU and LOC' and, for each location $\sigma \in A$, let $t_\sigma \geqslant 0$ be as required in the definition of LOC'. For each $\sigma \in A$ we introduce an auxiliary player i_σ with $A_{i_\sigma} = \{\sigma\}$ and $Q_{i_\sigma} = \{t_\sigma\}$. For each player i of the original game G we define new indirect utility functions $\bar{v}_{i,\sigma}(x, y) = v_{i,\sigma}(x, y - t_\sigma)$ for all $\sigma \in A_i, x \in Q_i, y \in \mathbb{R}_{\geqslant 0}$. The auxiliary players have a constant utility function. Adding the auxiliary players, we obtain a new game $\bar{G}$ that satisfies CON', EXT, DMU and LOC and has the same utilities for the original players as in the original game. Theorem 10 establishes that the thus constructed game possesses a pure Nash equilibrium. □

4. Applications

We now present a series of examples that belong to the class of aggregative games introduced in this article.

Restricted multimarket oligopoly. Consider a multimarket oligopoly as proposed by Bulow et al. (1985), where a set of firms is engaged in a set of markets.[5] The strategy of firm i is to choose a production quantity $q_{i,\sigma}$ for each market σ. All markets are associated with an inverse demand function that maps the total production quantity on that market to the respective market price. The utility of each firm equals the profit from selling the produced goods on the markets minus the production costs. Under mild assumptions on the inverse demand and cost functions, the existence of an equilibrium follows from Kakutani's fixed point theorem (cf. Kakutani, 1941). There are situations, however, where firms cannot be engaged in all markets simultaneously. For instance, governmental restriction policies may oblige each firm to be engaged in at most one market at a time, e.g., by issuing a license per firm (cf. Stähler and Upmann, 2008). Alternatively, the firms' short-term assets may only suffice to serve one market. In these situations, a strategy of a player is to select one market and a production quantity for that market.

Let A be a set of markets each endowed with a non-increasing inverse demand function $P_\sigma, \sigma \in A$. In each strategy profile, each player chooses both a market a_i out of a player-specific set $A_i \subseteq A$ of feasible markets and a production quantity $q_i \in [\alpha_i, \omega_i]$ for the chosen market. Given a strategy profile (a, q), the utility of player i is then defined as $u_i(a, q) = P_{a_i}\big(\ell_{a_i}(a, q)\big)\, q_i - C_i(q_i)$. We call this class of games *restricted multimarket oligopolies*.

We obtain the following result as a corollary of Theorems 10 and 12.

Corollary 16. *Restricted multimarket oligopolies on markets with identical and continuous, non-increasing and strictly concave inverse demand functions and continuous, non-decreasing and convex production cost functions possess a pure Nash equilibrium.*

For games with two players a pure Nash equilibrium exists even if the inverse demand functions are not identical.

Proof. One can easily verify that assumptions CON, EXT and LOC are satisfied. Therefore, we here only check DMU. First note that by continuity and concavity/convexity of the market price/cost functions, for all $i \in N$ and $\sigma \in A$, the left and right derivatives of the indirect utility function $v_{i,\sigma}$ exist and have the form

$$\partial_x^- v_{i,\sigma}(x, y) = \partial^- P_\sigma(x + y) \cdot x + P_\sigma(x + y) - \partial^- C_i(x)$$
$$\partial_x^+ v_{i,\sigma}(x, y) = \partial^+ P_\sigma(x + y) \cdot x + P_\sigma(x + y) - \partial^+ C_i(x).$$

Let $i \in N$ and $\sigma \in A_i$ be arbitrary and let $x, x' \in Q_i$ and $y, y' \in \mathbb{R}_{\geqslant 0}$ be such that $x \leqslant x'$ and $x + y \leqslant x' + y'$, where at least one inequality is strict.

If the second inequality is strict, i.e., $x + y < x' + y'$, we obtain

$$\partial^+ P_\sigma(x + y) > \partial^- P_\sigma(x' + y'),$$

see Webster (1994, Theorem 5.1.3) for a reference. If, on the other hand, the first inequality is strict, i.e., $x < x'$, we use that

$$\partial^+ P_\sigma(x + y) \leqslant \partial^- P_\sigma(x + y) < 0$$

as P_σ is non-increasing and strictly concave. Thus, in both cases, we obtain

$$\partial^+ P_\sigma(x + y) \cdot x > \partial^- P_\sigma(x' + y') \cdot x'.$$

Together with the convexity of C_i and the assumption that P_σ is non-increasing, this implies that DMU is satisfied. The result then follows from Theorems 10 and 12. □

Congestion games with variable quantities. The class of congestion games is a well-studied class of games introduced by Rosenthal (1973). Because congestion games with weighted players and/or player-specific costs may fail to have pure Nash equilibria[6] many authors focused on singleton strategies. In such a game, a pure Nash equilibrium is guaranteed to exist even when players are weighted (Andelman et al., 2009; Even-Dar et al., 2007; Fotakis et al., 2002; Harks et al., 2012; Rozenfeld and Tennenholtz, 2006) or costs are player-specific (Konishi et al., 1997; Milchtaich, 1996). However, games with weighted players *and* player-specific costs need not possess a pure Nash equilibrium (Milchtaich, 1996).

Congestion games with weighted players and player-specific costs are a special case of the class of aggregative games introduced in this article. For each player i, the set of feasible quantities $q_i = \{q_i\}$ is contracted to a single point $q_i \in \mathbb{R}_{>0}$ and the indirect utility function is defined as $v_{i,a_i}(a, q) = -c_{i,a_i}(\ell_{a_i}(a, q))$, where $c_{i,a_i} : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is a player-specific cost function. It is not hard to see that this class of games satisfies EXT if all cost functions are non-decreasing and LOC if $c_{i,\sigma} = c_{i,\tau}$ for all $i \in N$ and $\sigma, \tau \in A_i$. The assumptions "Decreasing Marginal Utility" and DMU and "Consistency" (CON') are trivially satisfied as no player has a non-trivial interval of feasible quantities.

We obtain the following immediate corollary of Theorem 10.

Corollary 17. *Weighted congestion games with non-decreasing player-specific costs that are identical per player possess a pure Nash equilibrium.*

For two-player games, we can refer to Theorem 12 instead and, thus, the assumption that player-specific costs are identical per player can be dropped. This existence of a pure Nash equilibrium in two-player congestion games with player-specific costs has already been shown by Milchtaich (1996).

The assumption that the quantity of each player is *fixed* seems unrealistic for applications in which the quantity is reduced in reaction to high congestion. In congestion games with *variable* quantities, each player may adapt her quantity depending on the

[5] See also Topkis (1998, Section 4.4.3) for a generalization.

[6] See the counterexamples given in Libman and Orda (2001) for weighted congestion games and Milchtaich (1996, 2006) for games with player-specific costs.

level of congestion of the resources.[7] The incentive of each player i to use high quantities is stimulated by a player-specific reward function $U_i : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ that defines the reward received from the chosen quantity. Given a strategy profile (a, q), the utility of player i is defined as $u_i(a, q) = U_i(q_i) - c_{i,a_i}(\ell_{a_i}(a, q))$. The following observations establish the existence of a pure Nash equilibrium in games with strictly concave rewards and strictly convex costs as a direct corollary of Theorems 10 and 12. The proof uses similar arguments as in the proof of Corollary 16 and is omitted.

Corollary 18. *Congestion game with variable quantities and player-specific costs functions possess a pure Nash equilibrium if all reward functions are continuous, non-decreasing and concave and, for each player, the player-specific costs functions are continuous, non-decreasing, strictly convex and identical.*

For games with two players a pure Nash equilibrium exists even if the cost functions of each player i are not identical among the resources.

Queuing games

Consider m parallel $M/M/1$ queues served in a first-come-first-served fashion. There are n players with independent Poisson arrivals, where the arrival rates are denoted by $q_1, \ldots, q_n$. Every queue j has a single server with exponentially distributed service time with mean $1/\mu_j, \mu_j > 0$. A strategy of each player is to choose a single server a_i and to adjust her sending rate q_i to the queue of that server.[8]

For each player, there is a tradeoff between her sending rate and the average experienced delay. Assuming that the server does not drop packets the sending rate equals the throughput and, thus, the delay can be computed as

$$D_{a_i}(a, q) = \frac{1}{\mu_{a_i} - \sum_{j \in N: a_j = a_i} q_j} = \frac{1}{\mu_{a_i} - \ell_{a_i}(a, q)}.$$

For a player-specific parameter $\theta_i \in (0, 1]$ that trades off throughput against delay, the utility function of player i is defined as[9]

$$\begin{aligned} u_i(a, q) &= v_{i,a_i}\big(q_i, \ell_{-i,a_i}(a, q)\big) \\ &:= \frac{q_i^{\theta_i}}{D_{a_i}(a, q)} \\ &= q_i^{\theta_i}\left(\mu_{a_i} - \ell_{a_i}(a, q)\right). \end{aligned}$$

Korilis and Lazar (1995) showed the existence of a pure Nash equilibrium assuming a compact and convex strategy space in which rates may be fractionally assigned to servers. Our model imposes the combinatorial condition of using only a single server among a set of servers. The following simple consequence of Corollary 15 shows the existence of pure Nash equilibria under this condition.

Corollary 19. *Games on parallel M/M/1 queues possess a pure Nash equilibrium.*

Proof. We check that assumptions CON, EXT, DMU and LOC' are satisfied. Corollary 15 then implies the result. For CON and EXT this is trivial. As for DMU, let $\sigma \in A_i$ be arbitrary and let $x, x' \in Q_i$ and $y, y' \in \mathbb{R}_{\geqslant 0}$ with $x + y \leqslant x' + y'$ and $x \leqslant x'$, where one of the inequalities is strict. We calculate

$$\begin{aligned} \partial_x v_{i,\sigma}(x, y) &= \theta_i x^{\theta_i - 1}\left(\mu_\sigma - (x + y)\right) - x^{\theta_i} \\ &> \theta_i (x')^{\theta_i - 1}\left(\mu_\sigma - (x' + y')\right) - (x')^{\theta_i} \\ &= \partial_x v_{i,\sigma}(x', y), \end{aligned}$$

proving that DMU is satisfied.

Condition LOC' is satisfied as, for any two servers i, j, there are $t_i, t_j \geqslant 0$ with $1/(\mu_i + t_i) = 1/(\mu_j + t_j)$. $\square$

5. Conclusions

We presented a novel class of aggregative location games in which a strategy of a player consists of a finite choice (i.e., choosing one location out of a finite set of locations) and a continuous choice (i.e., choosing a non-negative quantity out of a compact interval) which renders the set of strategies non-convex. As our main result, we proved the existence of a Nash equilibrium provided that the players' utility functions satisfy the assumptions CON, EXT, DMU and LOC. We further demonstrated that our existence result has applications in different areas ranging from restricted multimarket Cournot oligopolies to multiserver queuing games.

To prove this result, we combined a combinatorial argument using a potential function with a fixed-point argument (that is usually applied to concave utility games with convex and compact strategy sets). We hope that this combination of tools from combinatorial and continuous convex games, respectively, may prove to be useful for other classes of games exhibiting both finite and continuous choices.

Several issues remain that deserve further attention. An interesting open question is to weaken our assumptions. In particular, the assumption DMU is closely related to the general concept of strategic substitutes/complements and it would be interesting if our existence result continues to hold for milder assumptions related to strategic substitutes or complements.

We have not addressed the complexity of computing an equilibrium in our model. It seems that our proof cannot directly be used to obtain an efficient (polynomial time) algorithm since we use as a subroutine Kakutani's fixed-point theorem which is (in general) as hard as computing a Brouwer fixed-point. The latter problem is known to be PPAD-hard, see Daskalakis et al. (2009). Also questions related to the (worst-case) quality of equilibria are left for future research.

Acknowledgments

We wish to thank Philippe Bich, Guilherme Carmona, R. Preston McAfee, and Andrew McLennan for useful conversations and pointers to the literature. Further comments of the attendees of 7th Workshop on Internet and Network Economics and the 13th SAET Conference on Current Trends in Economics are gratefully acknowledged. We are also grateful for numerous suggestions by four anonymous referees that helped improve the presentation of the paper.

Appendix A. Violation of assumptions

In this section, we show that if one of the assumptions EXT, DMU CON, and LOC is violated, the existence of a pure Nash equilibrium is not guaranteed anymore. In the following four sections, we construct aggregative games satisfying all assumptions but one and show that they do not possess an equilibrium.

[7] This class of games has been introduced in previous work (Harks and Klimm, 2011b), where we considered a model in which players choose a quantity and an arbitrary subset of locations (among a given set of subsets). We showed that only affine or certain exponential cost functions yield the existence of a pure Nash equilibrium. We did not study, however, the case of player-specific costs.

[8] Note that the case of *fractional* assignments to the servers has been well studied by previous works, cf. Korilis and Lazar (1995), Gai et al. (2011) and references therein. In our model every player selects a single server among a player-specific set of admissible servers. This requirement is crucial for time-critical applications, because splitting data streams across several servers leads to packets arriving out of order and packet jitter due to different server delays.

[9] As it is common in queuing theory, $u_i(a, q) = -\infty$, if $\ell_{a_i}(a, q) \geqslant \mu^{a_i}$.

A.1. Violation of negative externalities

There are two players $N = \{1, 2\}$ and two locations $A = \{\sigma, \tau\}$ feasible to both players. Both players have fixed quantities $Q_1 = \{1\}$, $Q_2 = \{2\}$. The players' indirect utility functions are defined as

$$v_{i,\kappa}(x, y) = \begin{cases} 1, & \text{if } y = 0, \\ 2, & \text{if } y = 1, \\ 0, & \text{if } y = 2, \end{cases}$$

for all $i \in N$, $\kappa \in A$. The indirect utility functions satisfy DMU, CON, and LOC, but not EXT, and are constructed such that player 2 prefers sharing a location with player 1 over being alone on a location, while player 1 prefers to be alone. Thus, the game has no pure Nash equilibria.

A.2. Violation of decreasing marginal utility

We now turn to the assumption DMU. It is well known that Cournot games (even with two players and a single market) may fail to possess a pure Nash equilibrium for (general) non-concave inverse demand functions. These games satisfy the assumptions CON, EXT and LOC, but not DMU. For the sake of completeness, we recall the example of such a game presented by Novshek (1985).

There are two players $N = \{1, 2\}$ competing in a single market σ. The inverse demand function P is defined as

$$P(x) = \begin{cases} 2 - x, & \text{if } x \in [0, 0.99] \\ \dfrac{8219}{8119} - \dfrac{19}{8119}x, & \text{if } x \in \left(0.99, \dfrac{100}{19}\right] \\ \dfrac{10,019}{19} - 100x, & \text{if } x \in \left(\dfrac{100}{19}, \dfrac{100,19}{19}\right] \\ 0, & \text{if } x \in \left(\dfrac{100,19}{19}, \infty\right) \end{cases}$$

and the cost functions of players 1 and 2 are defined as $C_1(x) = \frac{881}{800}x$ and $C_2(x) = \frac{381}{400}x$, respectively. The utility of each player i equals her own quantity multiplied with the price given by the inverse demand function, i.e., $u_i(a, q) = q_i P(q_1 + q_2)$. As there is only one market the game satisfies LOC. As the inverse demand function is non-increasing assumption EXT is satisfied. Novshek (1985) shows that a pure Nash equilibrium does not exist.

A.3. Violation of continuity

If we abandon continuity of the indirect utility functions, there exist examples in which a pure Nash equilibrium may fail to exist, even for the simple case of only one player. To see this, consider a single player i with a single location σ. The set of feasible quantities is given as $[\alpha_i, \omega_i] = [0, 1]$ and the indirect utility function equals $v_{i,\sigma}(x, y) = \sqrt{x}$, if $x < 1$, and 0, otherwise. The indirect utility function satisfies EXT, LOC and DMU, but violates CON. The game does not have a pure Nash equilibrium, since the player may always improve her utility by slightly raising her quantity (as long as the quantity is smaller than 1).

We proceed to construct a game violating LOC that has no pure Nash equilibrium. The game we derive involves three players. As shown in Theorem 12, having three players is actually necessary for a counterexample since games with two players (satisfying only DMU, EXT, and CON) always possess a pure Nash equilibrium.

A.4. Violation of Location–Symmetry

We are given three players $N = \{1, 2, 3\}$ and two locations $A = \{\sigma, \tau\}$ that are feasible to all players. The feasible quantity intervals are $Q_1 = [1, 3]$, $Q_2 = [1, 2]$ and $Q_3 = [1, 4]$. Let $\epsilon = 1/4$. The players' indirect utility functions are defined as follows:

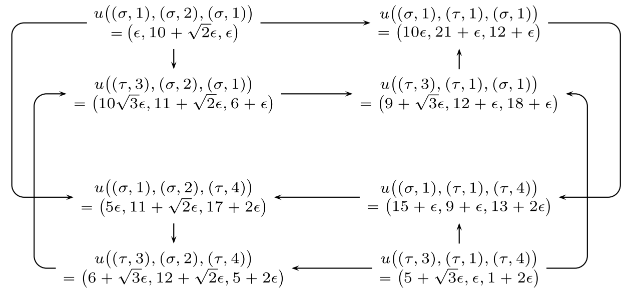

Fig. A.1. Finite subset of the strategy space of a game violating LOC as constructed in Appendix A.3. Arcs correspond to a better reply of a player. Note that for every strategy profile there is a player that has a better reply and, thus, the game does not have a pure Nash equilibrium.

$$v_{1,\kappa}(x, y) = \epsilon\sqrt{x} + \begin{cases} -3x + 18 - 5y, & \text{if } \kappa = \sigma, \\ x + 7 - y, & \text{if } \kappa = \tau, \end{cases}$$

$$v_{2,\kappa}(x, y) = \epsilon\sqrt{x} + \begin{cases} x + 10 - y, & \text{if } \kappa = \sigma, \\ -x + 22 - 3y, & \text{if } \kappa = \tau, \end{cases}$$

$$v_{3,\kappa}(x, y) = \epsilon\sqrt{x} + \begin{cases} -x + 19 - 6y, & \text{if } \kappa = \sigma, \\ x + 13 - 4y, & \text{if } \kappa = \tau, \end{cases}$$

The thus defined aggregative game satisfies EXT, DMU, and CON.

The marginal utility $\partial_x v_{1,a_1}(q_1, \ell_{-i,a_1}(a, q))$ of player 1 is always negative if $a_1 = \sigma$ and positive if $a_1 = \tau$. We conclude that for any pure Nash equilibrium (a^*, q^*) of G we have $q_1^* = 1$, if $a_1 = \sigma$ and $q_1^* = 3$, if $a_1 = \tau$. With the same arguments we derive that $q_2^* = 2$, if $a_2 = \sigma$, and $q_2^* = 1$, if $a_2 = \tau$ as well as $q_3^* = 1$ if $a_3 = \sigma$ and $q_3^* = 4$ if $a_3 = \tau$. This observation allows us to restrict the search space for a pure Nash equilibrium to 8 strategy profiles, i.e., one strategy profile for each of 8 possible location profiles a. One verifies easily that none of these is a pure Nash equilibrium, see Fig. A.1 for an illustration.

Appendix B. Relationship to Other Models

The following remarks discuss the relationship of our model to potential games (Monderer and Shapley, 1996), supermodular games (Topkis, 1979, 1998) and games with discontinuous utilities (Reny, 1999; Bich, 2009; Carmona, 2009, 2011; McLennan et al., 2011; Barelli and Meneghel, 2013).

B.1. Relationship to potential games

We proceed to show that, although the proof of Theorem 10 uses the fact that a certain lexicographical order decreases, the games considered in this work are not potential games, in general.[10]

To illustrate this fact, we give an example of such a game involving three players $N = \{1, 2, 3\}$ with feasible quantities $Q_1 = Q_2 = Q_3 = [1, 5]$ and two locations $A = \{\sigma, \tau\}$ feasible to all players. Let $\epsilon = 1/4$. For all $\kappa \in A$ the players' indirect utility functions $v_{i,\kappa}$ are defined as

$$v_{1,\kappa}(x, y) = \epsilon\sqrt{x} + \min\{x, 3\} - \max\{0, y - 2\},$$
$$v_{2,\kappa}(x, y)$$

[10] A game is a potential game if it admits a generalized ordinal potential function, cf. Monderer and Shapley (1996) for a definition.

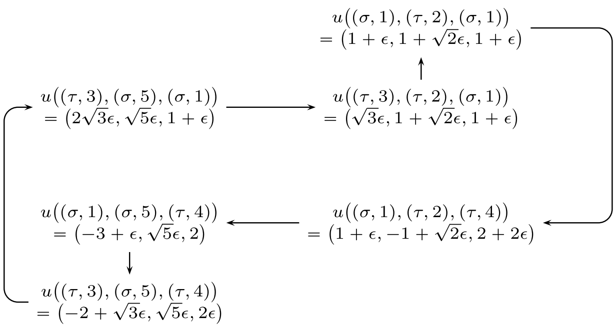

Fig. B.2. Cycle of better replies in an aggregative game satisfying CON, EXT, DMU and LOC as constructed in Appendix B.1.

$$= \epsilon\sqrt{x} + \begin{cases} \dfrac{x}{2} - \max\{0, 2y-10\}, & \text{if } x \in [0,2], \\ \dfrac{x-2}{3} + 1 - \max\{0, 2y-10\}, & \text{if } x \in (2,5], \\ 2 - \max\{0, 2y-10\}, & \text{if } x > 5, \end{cases}$$

$v_{3,\kappa}(x, y)$

$$= \epsilon\sqrt{x} + \begin{cases} x - \max\{0, 2y-12\}, & \text{if } x \in [0,1], \\ \dfrac{x-1}{3} + 1 - \max\{0, 2y-12\}, & \text{if } x \in (1,4], \\ 2 - \max\{0, 2y-12\}, & \text{if } x > 4. \end{cases}$$

The game satisfies EXT, DMU, CON, and LOC. Consider the closed sequence of strategy profiles

$$\begin{aligned}\gamma = \Big(&\big((\sigma,1),(\tau,2),(\sigma,1)\big), \big((\sigma,1),(\tau,2),(\tau,4)\big), \\ &\big((\sigma,1),(\sigma,5),(\tau,4)\big), \big((\tau,3),(\sigma,5),(\tau,4)\big), \\ &\big((\tau,3),(\sigma,5),(\sigma,1)\big), \big((\tau,3),(\tau,2),(\sigma,4)\big), \\ &\big((\sigma,1),(\tau,2),(\sigma,1)\big)\Big).\end{aligned}$$

One verifies easily that along this sequence there is always one player that improves switching to the next strategy profile, cf. Fig. B.2. As a consequence, this game does not possess a potential.

B.2. Relationship to games with convex strategy spaces and discontinuous utility functions

In this section, we discuss the relationship of our class of games to certain classes of games involving convex strategy spaces and discontinuous utility functions (cf. Barelli and Meneghel, 2013, Bich, 2009, Carmona, 2009, 2011 and McLennan et al., 2011). The players' strategic behavior in our class of games can be reproduced by a game with convex strategies and discontinuous utility functions by simply replacing the players' non-convex strategy sets by their convex hull and assigning a sufficiently low number to the players' utilities for strategies outside the players' original strategy space. In this section, we show that prevailing existence results (in particular the notion of "continuous security" introduced by Barelli and Meneghel, 2013) do not apply to the games obtain obtained from such a transformation, and, thus, our existence result is independent from those in the literature on discontinuous games.

To this end, consider the following game with two players $N = \{1, 2\}$ and two locations $A = \{\sigma, \tau\}$ feasible to both players. The set of feasible quantities to both players is $Q_1 = Q_2 = \{1\}$ and their utilities are defined by the following indirect utility functions:

$$v_{1,\kappa}(x, y) = 1, \qquad v_{2,\kappa}(x, y) = 1 - y$$

for $x = 1$ and all $y \in \mathbb{R}_{\geqslant 0}$ and $\kappa \in A$. When convexifying the players' strategies, we obtain a new game with strategy sets S_1 and S_2. The set of strategies S_i of each player i is identified with the interval $[0, 1]$, where strategy $s_i = 0$ corresponds to using location σ, while strategy $s_i = 1$ corresponds to location τ. If $s_i \in (0, 1)$, we consider player i as "not in the game" and assign her a utility of -1. We claim that the convexified game with strategies sets $S_1 = S_2 = [0, 1]$ is not continuously secure at $(0, 0) \in S_1 \times S_2$ in the sense of Barelli and Meneghel (2013, Definition 2.1). Note that $(0, 0)$ is not an equilibrium as player 2 may improve by deviating to location τ. For the sake of a contradiction, suppose that $(0, 0)$ is continuously secure, i.e., there is an open neighborhood V of $(0, 0)$ and, for each player i, a value $\alpha_i \in \mathbb{R}$ and a non-empty valued upper hemicontinuous correspondence $\phi : V \to S$ with

1. $\phi_i(y) \subseteq B_i(\alpha_i, y)$ for every $i \in \{1, 2\}$ and every $y \in V$,
2. for each $y \in V$ there is $i \in \{1, 2\}$ with $y_i \notin \operatorname{conv}(B_i(\alpha_i, y))$,

where $B_i(\alpha_i, y) = \{y_i \in S_i : u_i(y_i, x_{-i}) \geqslant \alpha_i\}$. First, note that $\alpha_1 \leqslant 1$ as $B_1(\alpha_1, (0, 0)) = \varnothing$, otherwise. This implies $S_1 = \operatorname{conv}(B_1(\alpha_1, (0, 0)))$. From $B_2(\alpha_2, (0, 0)) \neq \varnothing$ and $\operatorname{conv}(B_2(\alpha_2, (0, 0))) \neq S_2$ we derive that $\alpha_2 \in (0, 1]$ and it is without loss of generality to assume $\alpha_2 = 1/2$.

Next, let $\epsilon > 0$ be such that $(\epsilon, 0) \in V$. Since $S_1 = \operatorname{conv}(B_1(\alpha_1, (0, 0))) = \operatorname{conv}(B_1(\alpha_1, (\epsilon, 0)))$ it is necessary that $0 \notin \operatorname{conv}(B_2(\alpha_2, (\epsilon, 0)))$, but we have $\operatorname{conv}(B_2(\alpha_2, (\epsilon, 0))) = \operatorname{conv}(\{0, 1\}) = S_2$, which is a contradiction. We derive that the game is not continuously secure. However, the original (not convexified game) satisfies all our assumptions and, thus, has a pure Nash equilibrium.

B.3. Relationship to supermodular games

We show that the existence results of Topkis (1979, 1998) for supermodular games are unrelated to ours. Let G be an aggregative game and let A_i and Q_i denote the set of locations and quantities available to each player i, respectively. Under the additional assumption that $A \subset \mathbb{R}$ the joint strategy space $\times_{i\in N}(A_i \times q_i)$ forms a lattice with respect to the usual component-wise order. We proceed to show that, nonetheless, the games considered in this work are not supermodular games, in general, and, thus, the existence results of Topkis (1979, 1998) do not apply. In fact, we show that for an aggregative game as considered in this paper to be contained in the class of supermodular games the indirect utility function of each player i has to additionally satisfy the law of constant differences, i.e., $v_{i,\sigma}(x, y) - v_{i,\sigma}(x', y) = v_{i,\sigma}(x, y') - v_{i,\sigma}(x', y')$ for all $x, x' \in Q_i$, $y, y' \in \mathbb{R}_{\geqslant 0}$, and $\sigma \in A_i$.

To see this, consider a three-player game with $N = \{1, 2, 3\}$, $A = \{1, 2\}$, $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{1, 2\}$, and $Q_i = [0, 1]$ for all $i \in N$. Let $x, x' \in [0, 1]$ with $x < x'$ be arbitrary and consider the strategies $s_3 = (2, x)$ and $s'_3 = (1, x')$ as well as the component-wise minimum and maximum of these two strategies $s_3^{\min} = (1, x)$ and $s_3^{\max} = (2, x')$, respectively. The supermodularity[11] of the third player's utility function requires

$$u_3(s_3, s_{-3}) + u_3(s'_3, s_{-3}) \leqslant u_3(s_3^{\min}, s_{-3}) + u_3(s_3^{\max}, s_{-3}) \qquad \text{(B.1)}$$

for all $s_{-3} \in S_{-3}$. For arbitrary $y, y' \in [0, 1]$ consider the strategies $s_1 = (1, y')$ and $s_2 = (2, y)$ of players 1 and 2, respectively. From (B.1), we obtain

$$\begin{aligned} &v_{3,2}\big(x, \ell_{-3,2}(s_3, s_{-3})\big) + v_{3,1}\big(x', \ell_{-3,1}(s'_3, s_{-3})\big) \\ &\quad \leqslant v_{3,1}\big(x, \ell^1_{-3}(s_3^{\min}, s_{-3})\big) + v_{3,2}\big(x', \ell^2_{-3}(s_3^{\max}, s_{-3})\big), \end{aligned}$$

or, equivalently,

$$v_{3,2}(x, y) + v_{3,1}(x', y') \leqslant v_{3,1}(x, y') + v_{3,2}(x', y). \qquad \text{(B.2)}$$

[11] See Topkis (1998, Section 2.6) for a definition.

Considering the strategies $s'_1 = (1, y)$ and $s'_2 = (2, y')$ instead, we obtain analogously to (B.2) that

$$v_{3,2}(x, y') + v_{3,1}(x', y) \leqslant v_{3,1}(x, y) + v_{3,2}(x', y'). \tag{B.3}$$

Recall that $v_{3,1} = v_{3,2}$ by LOC. Then, inequalities (B.2) and (B.3) imply $v_{3,\sigma}(x, y) - v_{3,\sigma}(x', y) = v_{3,\sigma}(x, y') - v_{3,\sigma}(x', y')$ for all $\sigma \in \{1, 2\}$, as claimed. Assuming that the players' indirect utility functions have constant differences, the existence of a pure Nash equilibrium follows from Topkis (1979, 1998).

References

Amir, R., 1996. Cournot oligopoly and the theory of supermodular games. Games Econom. Behav. 15 (2), 132–148.

Andelman, N., Feldman, M., Mansour, Y., 2009. Strong price of anarchy. Games Econom. Behav. 65 (2), 289–317.

Barelli, P., Meneghel, I., 2013. A note on the equilibrium existence problem in discontinuous games. Econometrica 81 (2), 813–824.

Beckmann, M., McGuire, C., Winsten, C., 1956. Studies in the Economics and Transportation. Yale University Press, New Haven, CT, USA.

Bich, P., 2009. Existence of pure Nash equilibria in discontinuous and non quasiconcave games. Internat. J. Game Theory 38 (3), 395–410.

Bulow, J., Geanakoplos, J., Klemperer, P., 1985. Multimarket oligopoly: Strategic substitutes and complements. J. Polit. Econ. 93 (3), 488–511.

Carmona, G., 2009. An existence result for discontinuous games. J. Econom. Theory 144 (3), 1333–1340.

Carmona, G., 2011. Understanding some recent existence results for discontinuous games. Econom. Theory 48, 31–45.

Cournot, A., 1838. Recherches Sur Les Principes Mathematiques De La Theorie De La Richesse. Hachette, Paris, France.

Dasgupta, P., Maskin, E., 1986. The existence of equilibrium in discontinuous economic games, I: Theory. Rev. Econom. Stud. 53 (1), 1–26.

Daskalakis, C., Goldberg, P.W., Papadimitriou, C.H., 2009. The complexity of computing a Nash equilibrium. SIAM J. Comput. 39 (1), 195–259.

Debreu, G., 1952. A social equilibrium existence theorem. Proc. Natl. Acad. Sci. USA 38, 886–893.

Dubey, P., Haimanko, O., Zapechelnyuk, A., 2006. Strategic complements and substitutes, and potential games. Games Econom. Behav. 54 (1), 77–94.

Even-Dar, E., Kesselman, A., Mansour, Y., 2007. Convergence time to Nash equilibrium in load balancing. ACM Trans. Algorithms 3 (3), 1–21.

Fan, K., 1952. Fixed point and minmax theorems in locally convex topological linear spaces. Proc. Natl. Acad. Sci. USA 38, 121–126.

Fotakis, D., Kontogiannis, S., Koutsoupias, E., Mavronicolas, M., Spirakis, P., 2002. The structure and complexity of Nash equilibria for a selfish routing game. In: Widmayer, P., Triguero, F., Morales, R., Hennessy, M., Eidenbenz, S., Conejo, R. (Eds.), Proc. 29th Internat. Colloquium on Automata, Languages and Programming. In: LNCS, vol. 2380. pp. 123–134.

Gai, Y., Liu, H., Krishnamachari, B., 2011. A packet dropping mechanism for efficient operation of $M/M/1$ queues with selfish users. In: Proc. 30th IEEE Internat. Conf. on Computer Comm., pp. 2687–2695.

Glicksberg, I., 1952. A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points. Proc. Amer. Math. Soc. 3, 170–174.

Harks, T., Klimm, M., 2011a. Demand allocation games: Integrating discrete and continuous strategy spaces. In: Chen, N., Elkind, E., Koutsoupias, E. (Eds.), Proc. 7th Internat. Workshop on Internet and Network Econom.. In: LNCS, vol. 7090. pp. 194–205.

Harks, T., Klimm, M., 2011b. Congestion games with variable demands. In: K. Apt, (Ed.), Proc. 13th Conf. Theoret. Aspects of Rationality and Knowledge, pp. 111–120.

Harks, T., Klimm, M., Möhring, R., 2012. Strong equilibria in games with the lexicographical improvement property. Internat. J. Game Theory 42 (2), 461–482.

Haurie, A., Marcotte, P., 1985. On the relationship between Nash–Cournot and Wardrop equilibria. Networks 15, 295–308.

Jensen, M., 2010. Aggregative games and best-reply potentials. Econom. Theory 43, 45–66.

Johari, R., Tsitsiklis, J., 2005. Efficiency loss in Cournot games. Technical Report, LIDS-P-2639. Laboratory for Information and Decision Systems, MIT.

Kakutani, S., 1941. A generalization of Brouwer's fixed point theorem. Duke Math. J. 8 (3), 457–458.

Kelly, F., Maulloo, A., Tan, D., 1998. Rate control in communication networks: Shadow prices, proportional fairness, and stability. J. Oper. Res. Soc. 49, 237–252.

Konishi, H., Le Breton, M., Weber, S., 1997. Equilibria in a model with partial rivalry. J. Econom. Theory 72 (1), 225–237.

Korilis, Y., Lazar, A., 1995. On the existence of equilibria in noncooperative optimal flow control. J. ACM 42 (3), 584–613.

Kukushkin, N., 1994. A fixed-point theorem for decreasing mappings. Econom. Lett. 46, 23–26.

Kukushkin, N., 2004. Best response dynamics in finite games with additive aggregation. Games Econom. Behav. 48 (1), 94–110.

Libman, L., Orda, A., 2001. Atomic resource sharing in noncooperative networks. Telecommun. Syst. 17 (4), 385–409.

Martimort, D., Stole, L., 2011. Aggregate representations of aggregate games. MPRA Paper 32871, University Library of Munich, Germany, June.

McLennan, A., Monteiro, P., Tourky, R., 2011. Games with discontinuous payoffs: A strengthening of Reny's existence theorem. Econometrica 79 (5), 1643–1664.

Milchtaich, I., 1996. Congestion games with player-specific payoff functions. Games Econom. Behav. 13 (1), 111–124.

Milchtaich, I., 2006. The equilibrium existence problem in finite network congestion games. In: Mavronicolas, M., Kontogiannis, S. (Eds.), Proc. 2nd Internat. Workshop on Internet and Network Econom.. In: LNCS, vol. 4286. pp. 87–98.

Milgrom, P., Roberts, J., 1990. Rationalizability, learning, and equilibrium in games with strategic complementarities. Econometrica 58, 1255–1277.

Milgrom, P., Shannon, C., 1994. Monotone comparative statics. Econometrica 62 (1), 157–180.

Milinsky, M., 1979. An evolutionary stable feeding strategy in sticklebacks. Z. Tierpsychol. 51, 36–40.

Monderer, D., Shapley, L., 1996. Potential games. Games Econom. Behav. 14 (1), 124–143.

Nash, J., 1950a. Equilibrium points in n-person games. Proc. Natl. Acad. Sci. USA 36, 48–49.

Nash, J., 1950b. Non-cooperative games (Ph.D. thesis). Princeton.

Naylor, R., 1994. Pay discrimination and imperfect competition in the labor market. J. Econometrics 60 (2), 177–188.

Novshek, W., 1985. On the existence of Cournot equilibrium. Rev. Econom. Stud. 52 (1), 85–98.

Reny, P., 1999. On the existence of pure and mixed strategy Nash equilibria in discontinuous games. Econometrica 67 (5), 1029–1056.

Rosenthal, R., 1973. A class of games possessing pure-strategy Nash equilibria. Internat. J. Game Theory 2 (1), 65–67.

Roy, S., Sabarwal, T., 2010. Monotone comparative statics for games with strategic substitutes. J. Math. Econom. 46 (5), 793–806.

Rozenfeld, O., Tennenholtz, M., 2006. Strong and correlated strong equilibria in monotone congestion games. In: Mavronicolas, M., Kontogiannis, S. (Eds.), Proc. 2nd Internat. Workshop on Internet and Network Econom.. In: LNCS, vol. 4286. pp. 74–86.

Shenker, S., 1995. Fundamental design issues for the future Internet. IEEE J. Sel. Areas Commun. 13, 1176–1188.

Simon, L., 1987. Games with discontinuous payoffs. Rev. Econom. Stud. 54, 569–597.

Stähler, F., Upmann, T., 2008. Market entry regulation and international competition. Rev. Int. Econ. 16 (4), 611–626.

Tarski, A., 1955. A lattice-theoretical fixpoint theorem and its applications. Pacific J. Math. 5 (2), 285–309.

Topkis, D., 1979. Equilibrium points in nonzero n-person submodular games. SIAM J. Control Optim. 17, 773–787.

Tirole, J., 1988. The Theory of Industrial Organization. MIT Press, Cambridge, MA, USA.

Topkis, D., 1998. Supermodularity and Complementarity. Princeton University Press, Princeton, NJ, USA.

Vives, X., 1990. Nash equilibrium with strategic complementarities. J. Math. Econom. 19 (3), 305–321.

Webster, R., 1994. Convexity. Oxford University Press, Oxford, UK.