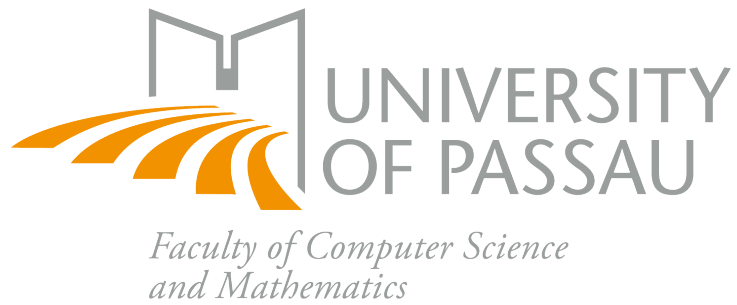




Lecture

Nonlinear Optimization

given in WS 2022/23, 2023/2024, 2025/2026

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Preface

These lecture notes were developed for optimization courses given at the University of Augsburg from 2015 to 2022. The contents of Chapters 3-8 are based on hand-written notes I took as a student at a lecture given by Helmut Maurer (University of Münster) in 2003. Sections 9.1–9.4 are loosely based on lecture notes of Hans-Joachim Oberle (Hamburg University). Section 9.5 on the Frank–Wolfe method is based on an article by Sebastian Pokutta (ZIB).

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Passau, October 2023

Prof. Dr. Tobias Harks

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Chapter 1

Introduction

1.1 Motivation and Terminology

We consider a normed vector space $(V, \|\cdot\|)$ for $K \subset V$ nonempty. We are given a function

$$f : K \rightarrow \mathbb{R}.$$

Our goal is to solve $\min\{f(x) \mid x \in K\}$. We sometimes write:

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{s.t.: } x \in K. \end{aligned} \tag{1.1}$$

An optimal solution of (1.1) is called global minimum.

Definition 1.1. $x^* \in K$ is a global minimum of f over K , if

$$f(x) \geq f(x^*) \text{ for all } x \in K.$$

If

$$f(x) > f(x^*) \text{ for all } x \in K, x \neq x^*,$$

we speak of a strict global minimum. Global maxima are defined analogously.

- infinite dimensional optimization (e.g., V is a function space, L_1 or L_p space.),
- finite dimensional optimization ($V = \mathbb{R}^n$),
- continuous optimization (real-valued variables, possibly with equality constraints),
- discrete optimization ($K \subseteq \mathbb{Z}^n$).

1.2 Examples and Applications

Example 1.2 (Optimal Supply). The goal is to buy an amount M of a certain commodity. We have the offers of n suppliers, where every supplier $i \in \{1, \dots, n\}$ has a maximum supply of M_i units of the commodity. The prices of the i -th supplier are given by a

function $f_i(x_i)$.

$$\begin{aligned} \min \quad & \sum_{i=1}^n f_i(x_i) \\ \text{s.t.} \quad & \sum_{i=1}^n x_i = M \\ & 0 \leq x_i \leq M_i, i = 1, \dots, n. \end{aligned}$$

Example 1.3 (Regression). An experiment shows the following data $(t_i, y_i), i = 1, \dots, m$. The hypothesis class under which this data is generated is given by a parameterized function $f(t, p)$. The goal is to choose parameters p in order to minimize the resulting error measured as:

$$\sum_{i=1}^m (y_i - f(t_i, p))^2$$

A more general **least-squares** problem is given as:

$$\begin{aligned} \min \quad & \sum_{i=1}^q \Phi_i(x)^2 \\ \text{s.t.} \quad & g_j(x) \leq 0, j = 1, \dots, r \\ & h_j(x) = 0, j = 1, \dots, s. \end{aligned}$$

Example 1.4 (Optimal control). We search for a control function that steers a car with minimal “effort” in a given time frame $[0, t_f]$ from A to B . We use Newton’s laws describing where $s(t)$ denotes the location at time t , $v(t)$ the speed at time t and $a(t)$ denotes the acceleration:

$$\dot{s}(t) = v(t), \dot{v}(t) = a(t).$$

Suppose we are in dimension 1 and there is a straight line between A and B of length d . The coordinate of A is normalized to $s = 0$ and $s = d$ for B . We need to satisfy $s(0) = 0, v(0) = 0, s(t_f) = d, v(t_f) = 0$.

The control function corresponds to $a(t)$, where a positive sign is acceleration and negative sign is deceleration. The effort accumulates quadratically in a :

$$\int_0^{t_f} a(t)^2 dt$$

We obtain the following optimal control problem:

$$\begin{aligned} \min \quad & \int_0^{t_f} a(t)^2 dt \\ & \dot{s}(t) = v(t) \\ & \dot{v}(t) = a(t) \\ & s(0) = 0, s(t_f) = d \end{aligned}$$

$$v(0) = 0, v(t_f) = 0.$$

1.3 Finite Dimensional Optimization

In this lecture, we consider only finite dimensional problems, that is, $V := \mathbb{R}^n$.

The set of points in K that attain the minimum is denoted by $\arg \min(f, K)$. We get

$$\arg \min(f, K) := \{x \in K \mid f(x) = \inf_{y \in K} f(y)\}.$$

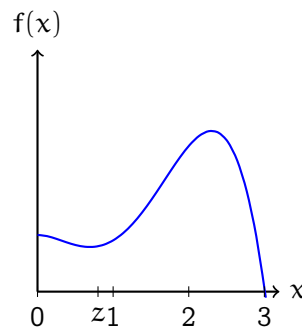


Figure 1.1: f attains on $K = [0, 3]$ its **global** minimum at $x^* = 3$. There is a further **local** minimum at z .

Definition 1.5. A point $x^* \in K$ is a local minimum of f over K , if there is $\rho > 0$ such that

$$f(x) \geq f(x^*) \text{ for all } x \in K \cap B_\rho(x^*),$$

where

$$B_\rho(x) = \{y \in \mathbb{R}^n \mid \|x - y\| < \rho\}$$

denotes the open ball around x with radius $\rho > 0$. If

$$f(x) > f(x^*) \text{ for all } x \in K \cap B_\rho(x^*), x \neq x^*,$$

we speak of a strict local minimum.

Usually K is represented via functional inequalities or equalities. In this case, we obtain:

$$K = \{x \in \mathbb{R}^n \mid h_i(x) = 0, i \in I_1 = \{1, \dots, m\}, g_j(x) \leq 0, j \in I_2 = \{1, \dots, p\}\}, \quad (1.2)$$

In differentiable optimization we usually assume smooth data; the minimax objectives listed below need not be differentiable.

We obtain the following classes of optimization problems:

- unrestricted optimization: $m = p = 0$
- restricted optimization: $m > 0$ or $p > 0$

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- linear optimization: f linear, g_j, h_i affine
- quadratic optimization: $f(x) = x^T A x + b^T x + c$, g_j, h_i affine
- convex optimization: f convex, g_j convex, h_i affine
- L_2 -problems: the function f has the form

$$f(x) = \sum_{i=1}^n w_i (f_i(x))^2,$$

with smooth functions f_i and weights $w_i > 0$.

- minimax problems: f has the form

$$f(x) = \max\{f_i(x), i = 1, \dots, m\}$$

with smooth functions f_i .

The following questions are the key drivers for the content of this lecture:

- When do optimal solutions exist?
- Are they unique?
- Can we derive useful necessary and sufficient optimality conditions?
- What about algorithms for solving such problems?
- What is the dependence of optimal solutions on problem parameters?

We recap a fundamental result due to Weierstrass.

Theorem 1.6 (Weierstrass). Let $K \subset \mathbb{R}^n$ be nonempty and compact and $f : K \rightarrow \mathbb{R}$ continuous. Then, there is $x^* \in K$ with

$$f(x^*) \leq f(x) \text{ for all } x \in K.$$

Proof. As f is continuous, the image $f(K)$ of the compactum K is bounded in $\mathbb{R}$ and the infimum

$$A := \inf\{f(x) | x \in K\} \in \mathbb{R}$$

exists. Hence, there is a sequence $x_n \in K, n \in \mathbb{N}$ with

$$\lim_{n \rightarrow \infty} f(x_n) = A.$$

As $x_n, n \in \mathbb{N}$ is bounded, we can use the theorem of Bolzano/Weierstrass giving a convergent subsequence $x_{n_k}, k \in \mathbb{N}$ with

$$\lim_{k \rightarrow \infty} x_{n_k} =: x^* \in K.$$

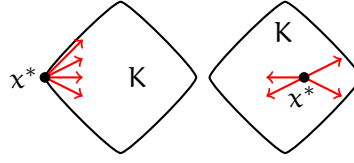


Figure 1.2: The red arcs represent feasible directions in $D_K(x^*)$.

With continuity of f we get

$$f(x^*) = \lim_{k \rightarrow \infty} f(x_{n_k}) = A.$$

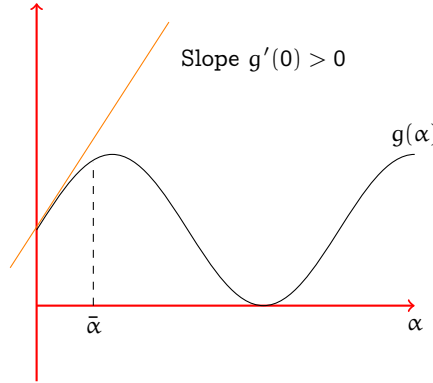
Thus, f attains at x^* its minimum over K . □

1.4 Classical Differentiable Optimization

1.4.1 Variational Inequalities

Definition 1.7 (Feasible Directions). Let $x \in K \subseteq \mathbb{R}^n$ with $K \neq \emptyset$. The vector $d \in \mathbb{R}^n$ is a feasible direction at x , if there is $\bar{\alpha} > 0$ such that $x + \alpha d \in K$ for all $0 \leq \alpha \leq \bar{\alpha}$.

We denote by $D_K(x)$ the set of feasible directions at x . It is easy to see that $D_K(x)$ is a cone with vertex at 0 (cf. 2.1), hence, $D_K(x)$ is known as the **cone of feasible directions**.



For continuous optimization problems (1.1) we obtain the following necessary optimality conditions.

Theorem 1.8 (Variational Inequality). Let $K \subseteq \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuously differentiable. Let x^* be a local minimum of f over K and $d \in D_K(x^*)$. Then

$$\nabla f(x^*)^T d \geq 0.$$

Proof. Since $d \in D_K(x^*)$, there is $\bar{\alpha} > 0$ such that $x^*(\alpha) := x^* + \alpha d \in K$ for all $0 \leq \alpha \leq \bar{\alpha}$. We define a 1-dimensional function $g(\alpha) := f(x^*(\alpha))$. For a local minimum x^* (w.r.t. $B_\rho(x^*)$), we have

$$g(\alpha) \geq g(0) \text{ for all } \alpha \in [0, \min\{\bar{\rho}, \bar{\alpha}\}],$$

where

$$\bar{\rho} := \sup\{\alpha \geq 0 \mid x^* + \alpha d \in B_{\rho/2}(x^*)\}.$$

Thus,

$$\lim_{\alpha \rightarrow +0} \frac{g(\alpha) - g(0)}{\alpha} \geq 0.$$

With the differentiability of f we get

$$0 \leq \lim_{\alpha \rightarrow +0} \frac{g(\alpha) - g(0)}{\alpha} = g'(0) = \nabla f(x^*)^\top d.$$

□

Theorem 1.9. Let $K \subseteq \mathbb{R}^n$ and $f : K \rightarrow \mathbb{R}$ be continuously differentiable. If $x^* \in \text{int}(K)$ is a local minimum of f over K , then:

$$\nabla f(x^*) = 0. \quad (1.3)$$

In particular, we have (1.3) for every local minimum of an unconstrained optimization problem.

Proof. With Theorem 1.8 we get for every $d \in D_K(x^*)$: $\nabla f(x^*)^\top d \geq 0$. With $x^* \in \text{int}(K)$, we get $D_K(x^*) = \mathbb{R}^n$. □

Remark 1.10. Note that the concept of feasible directions of Definition 1.7 is not useful for sets given by curved equality constraints. Here, we need **curved directions** leading to concepts of the **tangent cone** and **linearized cone** that we will see later.

1.4.2 Convex Optimization

We consider now a differentiable convex function f over a convex set $K \subset \mathbb{R}^n$.

Definition 1.11. A set $K \subset \mathbb{R}^n$ is convex, if for all $x, y \in K$ the segment between x and y lies in K , that is,

$$\lambda x + (1 - \lambda)y \in K \text{ for all } \lambda \in [0, 1].$$

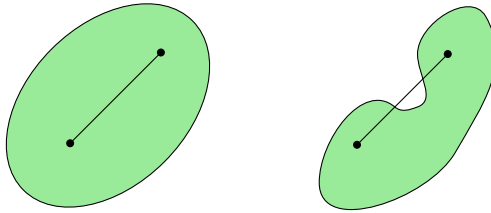


Figure 1.3: Left: convex set. Right: non-convex set.

Definition 1.12. Let $K \subset \mathbb{R}^n$ be convex. A function $f : K \rightarrow \mathbb{R}$ is convex, if for all $x, y \in K$ and $\lambda \in [0, 1]$ we have:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \quad (1.4)$$

f is strictly convex, if for all $x \neq y$ and $\lambda \in (0, 1)$ the above inequality is strict. The function f is called (strictly) concave, if $-f$ is (strictly) convex.

Theorem 1.13. Let $K \subset \mathbb{R}^n$ be convex, and let $f_1, f_2 : K \rightarrow \mathbb{R}$ be convex functions and let $\alpha > 0$. Then, the functions $\alpha f_1, f_1 + f_2$ and $\max\{f_1, f_2\}$ are convex over K .

Proof. Exercise. □

Differences, products and minima of convex functions are not always convex!

Definition 1.14. Let $K \subset \mathbb{R}^n$ be convex, and $f : K \rightarrow \mathbb{R}$. The set

$$\text{Epi}(f) = \{(x, \alpha) \in K \times \mathbb{R} : f(x) \leq \alpha\}$$

is the epigraph of f . For $\beta \in \mathbb{R}$, we term the set

$$L(f, \beta) = \{x \in K : f(x) \leq \beta\}$$

lower level set of f with level β .

Theorem 1.15. Let $K \subseteq \mathbb{R}^n$ and $f : K \rightarrow \mathbb{R}$. Then:

1. f is convex $\Leftrightarrow \text{Epi}(f)$ is convex.
2. f is convex $\Rightarrow L(f, \beta)$ is convex for all $\beta \in \mathbb{R}$. The reverse need not be true.

Proof. Exercise. □

For convex differentiable functions we obtain the following characterization:

Theorem 1.16. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $f \in C^1$. Then:

1. f is convex over the convex set $K \subseteq \mathbb{R}^n$ iff for all $x, y \in K$:

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x). \quad (1.5)$$

2. f is strictly convex $\Rightarrow (1.5)$ is strict for all $x \neq y \in K$.

Proof. We first show $\Leftarrow$ for the first statement. Assume (1.5) holds for all $x, y \in K$. Choose arbitrary $x, y \in K$ and $\lambda \in (0, 1)$. With convexity of K we get

$$z = \lambda x + (1 - \lambda)y \in K. \quad (1.6)$$

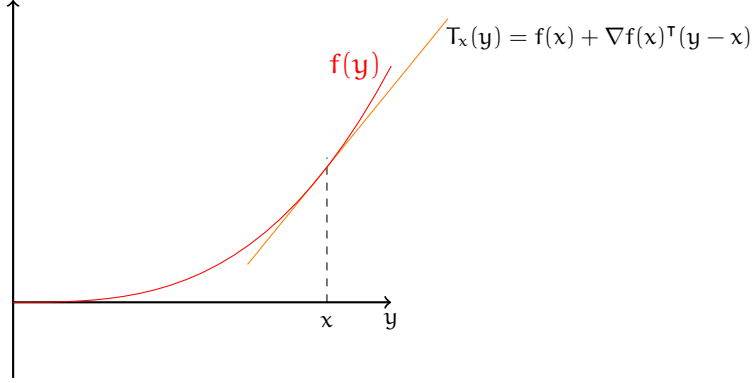


Figure 1.4: Illustration of inequality 1.5. $T_x(y)$ represents the tangent plane of f in x and we have $T_x(y) \leq f(y)$ for all $y \in K$.

With (1.5), we get for $x, y, z \in K$:

$$f(x) \geq f(z) + (x - z)^T \nabla f(z) \quad (1.7)$$

$$f(y) \geq f(z) + (y - z)^T \nabla f(z). \quad (1.8)$$

Multiply (1.7) with λ and (1.8) with $(1 - \lambda)$, add both inequalities and obtain:

$$\begin{aligned} \lambda f(x) + (1 - \lambda)f(y) &\geq f(z) + \left(\lambda(x - z) + (1 - \lambda)(y - z) \right)^T \nabla f(z) \\ &= f(z) + \left(\lambda x + (1 - \lambda)y - z \right)^T \nabla f(z) \\ &= f(z). \end{aligned}$$

With (1.6), the second expression of the second equation is 0. Thus, f is convex.

$\Rightarrow$: Let f be convex. We choose $x, y \in K$ and define $\psi : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\psi(\lambda) = (1 - \lambda)f(x) + \lambda f(y) - f((1 - \lambda)x + \lambda y).$$

With convexity of f we get for all $\lambda \in [0, 1]$ that $\psi(\lambda) \geq 0$. Moreover $\psi(0) = 0$. We compute the derivative of ψ at 0 and get

$$0 \leq \lim_{t \rightarrow 0^+} \frac{\psi(t) - \psi(0)}{t} = \psi'(0) = -f(x) + f(y) - \nabla f(x)^T(y - x).$$

The second statement is easy. □

We obtain a sufficient optimality criterion for convex optimization problems.

Theorem 1.17. Let $K \subset \mathbb{R}^n$ be convex and let $f : K \rightarrow \mathbb{R}$ be a differentiable convex function. Then, every local minimum of f over K is also a global Minimum.

Proof. Let x^* be a local minimum. With Theorem 1.8, we get for every $d \in D_K(x^*)$ the condition $\nabla f(x^*)^T d \geq 0$. Because K is convex, for any $y \in K$, we get $x^* + \lambda(y - x^*) =$

$\lambda y + (1 - \lambda)x^* \in K$ for all $\lambda \in [0, 1]$. Hence, $y - x^* \in D_K(x^*)$. We get

$$f(y) \geq f(x^*) + \nabla f(x^*)^T(y - x^*) \geq f(x^*),$$

where the first inequality follows from Theorem 1.16 and the second one from the variational inequality. $\square$

For unrestricted convex problems, we get the following implication.

Corollary 1.18. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable convex function. Then, every $x^* \in \mathbb{R}^n$ with $\nabla f(x^*) = 0$ is a global minimum of the associated unrestricted optimization problem.

Chapter 2

Convexity and Separating Hyperplanes

2.1 Convex Sets and Cones

Definition 2.1. 1. For $M \subset \mathbb{R}^n$ we define

$$\text{co}(M) := \cap \{K \supset M \mid K \text{ convex}\}$$

as the convex hull of M . For $x^0, \dots, x^k \in \mathbb{R}^n$ we define

$$\text{co}(x^0, \dots, x^k) = \text{co}(\{x^0, \dots, x^k\}).$$

This set is known as the simplex spanned by the points $x^0, \dots, x^k$. If $x^1 - x^0, \dots, x^k - x^0$ are linearly independent, then the simplex is non-degenerate.

2. A subset $K \subset \mathbb{R}^n$ is a cone (with vertex at 0), if for all $x \in K$ the half-ray through x lies in K , i.e.

$$\alpha x \in K \text{ for all } \alpha \geq 0.$$

3. Let $K \subset \mathbb{R}^n$ and $x \in K$. The cone

$$K(x) := \{\alpha(y - x) \mid y \in K, \alpha > 0\} = \bigcup_{\alpha > 0} \alpha(K - x)$$

is termed the conic hull of K w.r.t. x .

See Fig. 2.1 for an illustration.

Definition 2.2. Let $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$.

1. The set $H = \{x \in \mathbb{R}^n : a^T x = b\}$ is called hyperplane.
2. The sets $H^- = \{x \in \mathbb{R}^n : a^T x \leq b\}$ and $H^+ = \{x \in \mathbb{R}^n : a^T x \geq b\}$ are Halfspaces.
3. Let A be a real-valued $m \times n$ matrix and $b \in \mathbb{R}^m$.
 $K = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is a polyhedron and $K = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ is a polyhedron in standard form.

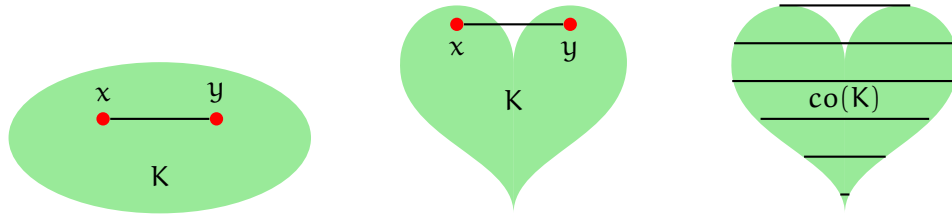


Figure 2.1: The first set is convex. The second heart is non-convex and the dashed set represents the convex hull.

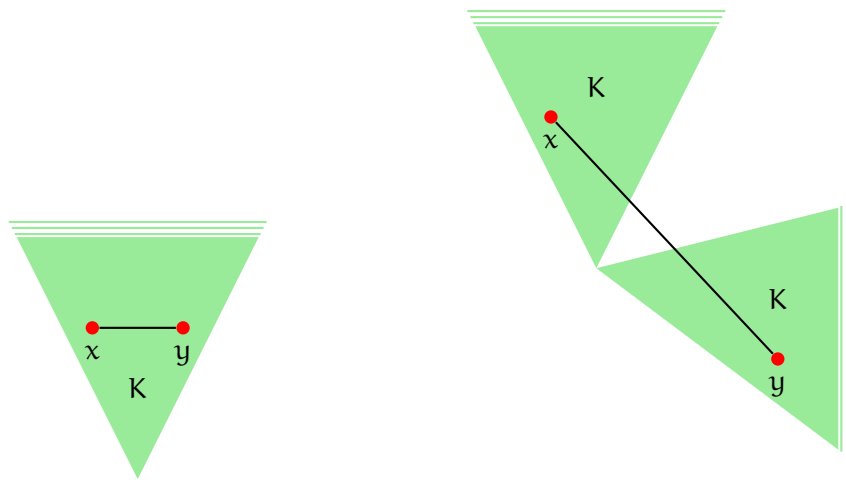


Figure 2.2: Left: convex cone. Right: non-convex cone.

2.2 Convex Combinations

Definition 2.3. Let $x^1, \dots, x^k \in \mathbb{R}^n$ and $\lambda_1, \dots, \lambda_k \in \mathbb{R}_{\geq 0}$ with $\lambda_1 + \dots + \lambda_k = 1$.

- The vector $\sum_{i=1}^k \lambda_i x^i$ is called convex combination of $x^1, \dots, x^k$.

Theorem 2.4.

1. The intersection of convex sets is convex.
2. Every polyhedron is convex.
3. A convex combination of finitely many points of a convex set lies in the respective set.
4. The convex hull of a set $K \subset \mathbb{R}^n$ is the set of all finitely generated convex combinations of points in K . The set of convex combinations of a finite point set is convex.

Proof. (1): Let $X_i, i \in I$ be convex sets and define $X := \bigcap_{i \in I} X_i$. For $x, y \in X$ we have $x, y \in X_i$ for all $i \in I$, hence $\lambda x + (1 - \lambda)y \in X_i$ for all $i \in I$ and, hence, $\lambda x + (1 - \lambda)y \in X$.

(2): A polyhedron is the intersection of finitely many convex halfspaces.

(3): We prove via induction over k , that every convex combination of k points in X lies in X . For $k = 1$ the statement is trivial and for $k = 2$ the statement follows from the convexity of X . For the step $k - 1 \rightarrow k$ consider a convex combination $\mu_1 x^1 + \dots + \mu_k x^k$. If $\mu_i = 0$ for some $i \in \{1, \dots, k\}$ we can use the induction hypothesis, hence, we can assume w.l.o.g. that $\mu_k \in (0, 1)$. Define

$$v_1 = \frac{\mu_1}{1 - \mu_k}, \dots, v_{k-1} = \frac{\mu_{k-1}}{1 - \mu_k} \geq 0, \sum_{l=1}^{k-1} v_l = 1.$$

Set

$$y := \sum_{l=1}^{k-1} v_l x^l$$

and observe that $y \in X$ follows by the induction hypothesis. We get

$$x = (1 - \mu_k)y + \mu_k x^k \in X,$$

because the case $k = 2$ was shown already.

(4): Let L be the set of convex combinations of points in K .

$L \subseteq \text{co}(K)$: With (1) the set $\text{co}(K)$ is convex, hence, all convex combinations of points in $\text{co}(K)$ lie again in $\text{co}(K)$. With the definition of the convex hull, we get $K \subseteq \text{co}(K)$, thus, $L \subseteq \text{co}(K)$.

$\text{co}(K) \subseteq L$: Let $x, y \in L$ with

$$x = \sum_{i=1}^k \alpha_i x^i \text{ and } y = \sum_{j=1}^l \beta_j y^j, \text{ where } \alpha_i, \beta_j \geq 0, \sum_{i=1}^k \alpha_i = 1, \sum_{j=1}^l \beta_j = 1.$$

For $\lambda \in [0, 1]$ we get

$$z = \lambda x + (1 - \lambda)y = \sum_{i=1}^k \lambda \alpha_i x^i + \sum_{j=1}^l (1 - \lambda) \beta_j y^j,$$

and thus $z \in L$ because

$$0 \leq \lambda \alpha_i \leq 1 \forall i, \quad 0 \leq (1 - \lambda) \beta_j \leq 1 \forall j \text{ and}$$

$$\sum_{i=1}^k \lambda \alpha_i + \sum_{j=1}^l (1 - \lambda) \beta_j = \lambda \sum_{i=1}^k \alpha_i + (1 - \lambda) \sum_{j=1}^l \beta_j = \lambda + 1 - \lambda = 1,$$

which shows that z is a convex combination of points in K . Thus L is convex. Obviously

$K \subseteq L$, since every $x^p \in K$ can be written as

$$x^p = \sum_{j \in J} \lambda_j x^j \text{ with } \lambda_p = 1 \text{ and } \lambda_j = 0 \text{ for } j \neq p.$$

Per definition we get $\text{co}(K) \subseteq L$, because L is a convex set containing K . $\square$

Corollary 2.5. The set of convex combinations of $x^1, \dots, x^k \in \mathbb{R}^n$ is the smallest (w.r.t. inclusion) convex subset of $\mathbb{R}^n$, which contains $x^1, \dots, x^k$.

Proof. Let X be the set of convex combinations of $x^1, \dots, x^k \in \mathbb{R}^n$. Define

$$Y := \text{co}(x^1, \dots, x^k) = \bigcap_{\substack{C \subseteq \mathbb{R}^n \\ C \text{ convex} \\ \{x^1, \dots, x^k\} \subset C}} C. \quad (2.1)$$

Y is well-defined because $\mathbb{R}^n$ is one candidate C . With Theorem 2.4(1.) Y is convex as intersection of convex sets. Y is also the smallest convex set containing $x^1, \dots, x^k$. Since X is convex (see Theorem 2.4(4.)) we get $Y \subseteq X$. Let $x \in X$. With the definition of X we get $x = \sum_{i=1}^k \lambda_i x^i$. As by assumption $x^1, \dots, x^k \in Y$ we get with Theorem 2.4(3.), that $x \in Y$. $\square$

Theorem 2.6 (Carathéodory). For $K \subset \mathbb{R}^n$, $\text{co}(K)$ is equal to the set of all convex combinations which require at most $(n + 1)$ points of K .

Proof. Let $x \in \text{co}(K)$. With Theorem 2.4 (4) there are $x^1, \dots, x^k \in K$ with

$$x = \sum_{i=1}^k \lambda_i x^i \text{ with } \lambda_i \geq 0 \text{ for all } i = 1, \dots, k \text{ and } \sum_{i=1}^k \lambda_i = 1.$$

If $k \leq n + 1$ we are done. If $k > n + 1$, then we show that for the representation of x we can ignore one of the k points: Define the $(k - 1)$ vectors $y^i = x^i - x^k, i = 1, \dots, k - 1$. For $k > n + 1$, the points y^i are linearly dependent, i.e., there are $\alpha_1, \dots, \alpha_{k-1}$ with $\alpha_j \neq 0$ for at least one $j \in \{1, \dots, k - 1\}$ with

$$\begin{aligned} & \sum_{i=1}^{k-1} \alpha_i y^i = 0 \\ \Leftrightarrow & \sum_{i=1}^{k-1} \alpha_i (x^i - x^k) = 0 \\ \Leftrightarrow & \sum_{i=1}^{k-1} \alpha_i x^i + \left(-\sum_{i=1}^{k-1} \alpha_i\right) x^k = 0. \end{aligned}$$

With $\alpha_k = -\sum_{i=1}^{k-1} \alpha_i$ we get

$$\sum_{i=1}^k \alpha_i x^i = 0 \text{ and } \sum_{i=1}^k \alpha_i = 0.$$

Because $\alpha_j \neq 0$ for at least one $j \in \{1, \dots, k-1\}$, the following value is well-defined:

$$i_0 \in \arg \min_{i: \alpha_i > 0} \frac{\lambda_i}{\alpha_i}, \quad \tau := \frac{\lambda_{i_0}}{\alpha_{i_0}}.$$

We get

$$\lambda_i - \frac{\lambda_{i_0}}{\alpha_{i_0}} \alpha_i \geq 0 \quad \forall i \text{ and } \sum_{i=1}^k \left(\lambda_i - \frac{\lambda_{i_0}}{\alpha_{i_0}} \alpha_i \right) = 1.$$

Moreover

$$x = \sum_{i=1}^k \lambda_i x^i = \sum_{i=1}^k \left(\lambda_i x^i - \frac{\lambda_{i_0}}{\alpha_{i_0}} \alpha_i x^i \right) = \sum_{i=1}^k \left(\lambda_i - \alpha_i \frac{\lambda_{i_0}}{\alpha_{i_0}} \right) x^i.$$

Here, we have $\lambda_{i_0} - \alpha_{i_0} \frac{\lambda_{i_0}}{\alpha_{i_0}} = 0$, and, hence, x can be represented as a convex combination of at most $k-1$ points. $\square$

2.3 Separating Hyperplanes

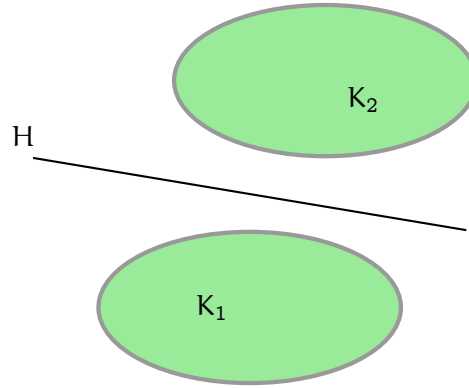


Figure 2.3: Illustration of the (proper) separation of two disjoint convex sets.

Definition 2.7. Two sets $K_1, K_2 \subset \mathbb{R}^n$ are called separable, if there are $c \in \mathbb{R}$ and row vector $\lambda \in \mathbb{R}^n, \lambda \neq 0$ with

$$\lambda x \leq c \leq \lambda y \text{ for all } x \in K_1, y \in K_2.$$

The hyperplane $H = \{x \in \mathbb{R}^n | \lambda x = c\}$ is called separating hyperplane (cf. Fig. 2.3); The sets K_1, K_2 are properly separable via H , if $K_1 \cup K_2$ is not contained in H . The hyperplane

H defines two halfspaces

$$H^+ = \{x \in \mathbb{R}^n | \lambda x \geq c\}, \quad H^- = \{x \in \mathbb{R}^n | \lambda x \leq c\}.$$

The sets K_1, K_2 are separable, if either $K_1 \subseteq H^+, K_2 \subseteq H^-$ or $K_1 \subseteq H^-, K_2 \subseteq H^+$.

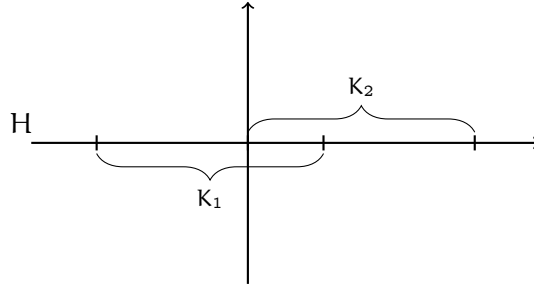


Figure 2.4: The two sets $K_1, K_2 \subset \mathbb{R}^2$ are separable but not properly. The separating hyperplane is given as $H = \{x \in \mathbb{R}^2 | x_2 = 0\}$.

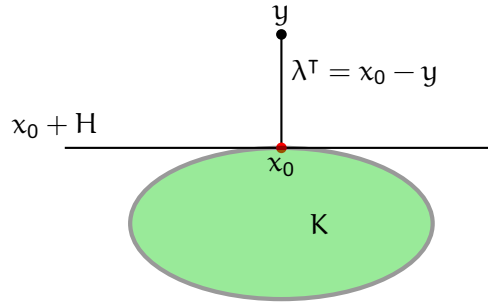


Figure 2.5: Illustration of the proof of Theorem 2.8. If $x_0 = y$ then $H + y$ is a separating supporting hyperplane.

Theorem 2.8. Let $K \subset \mathbb{R}^n$ be a non-empty, convex set and let $y \notin \overset{\circ}{K} := \text{int}(K)$. Then, $\{y\}$ and K are separable, i.e., there is a row vector $\lambda \in \mathbb{R}^n \setminus \{0\}$ with

$$\lambda y \leq \lambda x \text{ for all } x \in K.$$

If $\overset{\circ}{K} \neq \emptyset$ then $\{y\}$ and $\overset{\circ}{K}$ are properly separable and we get

$$\lambda y < \lambda x \text{ for all } x \in \overset{\circ}{K}.$$

Proof. 1. Case: $y \notin \bar{K}$, where $\bar{K}$ denotes the topological closure of K . With $\|x\|$ we denote as usual the Euclidean norm. Set

$$d := \inf_{x \in \bar{K}} \|x - y\| > 0.$$

The function $f(x) := \|y - x\|$ is continuous and attains on $\bar{K} \cap \{x \in \mathbb{R}^n \mid \|x - y\| \leq 2d\}$ its minimum (Theorem of Weierstrass). As $\bar{K}$ is closed, there is $x_0 \in \bar{K}$ with $d = \|y - x_0\|$. With convexity of $\bar{K}$ one can further show that the point x_0 is unique (cf. Fig. 2.5).

We show $\lambda := (x_0 - y)^\top \neq 0$ satisfies the conditions of the theorem. Let $x \in K$. With convexity of $\bar{K}$ we get

$$x_0 + \alpha(x - x_0) \in \bar{K} \text{ for } 0 \leq \alpha \leq 1.$$

Hence,

$$\|x_0 + \alpha(x - x_0) - y\|^2 \geq \|x_0 - y\|^2$$

and therefore

$$2\alpha(x_0 - y)^\top(x - x_0) + \alpha^2 \|x - x_0\|^2 \geq 0.$$

Division by $\alpha > 0$ yields for $\alpha \downarrow 0$

$$(x_0 - y)^\top(x - x_0) \geq 0$$

and therefore we get using $\lambda^\top = (x_0 - y)$ and $d = \|\lambda\|$

$$\lambda x \geq \lambda x_0 = \lambda y + d^2 > \lambda y.$$

Thus, $H := \{x \in \mathbb{R}^n \mid \lambda x = \lambda x_0\}$ is a hyperplane separating $\{y\}$ and K .

2. Case: $y \in \partial K = \bar{K} - \overset{\circ}{K}$.

For $y \in \partial K$ there is a sequence $\{y_k\}$, $y_k \notin \bar{K}$, with $y = \lim_{k \rightarrow \infty} y_k$. For y_k we can choose according to Case 1. a row vector $\lambda_k \neq 0$ with

$$\lambda_k y_k \leq \lambda_k x \text{ for all } x \in K.$$

W.l.o.g. we can set $\|\lambda_k\| = 1$ and hence we can assume that the bounded sequence $\{\lambda_k\}$ converges with $\lambda = \lim_{k \rightarrow \infty} \lambda_k$, $\|\lambda\| = 1$. Taking the limit on both sides yields

$$\lambda y \leq \lambda x \text{ for all } x \in K.$$

The statement

$$\lambda y < \lambda x \text{ for all } x \in \overset{\circ}{K}$$

follows immediately. □

In case $y \in \partial K$ we call the hyperplane supporting. From the separating hyperplane theorem 2.8 we get:

Theorem 2.9. Let $K \subset \mathbb{R}^n$ be a non-empty convex and closed cone and suppose $y \notin K$. Then, there is $\lambda \in \mathbb{R}^n \setminus \{0\}$ with

$$\lambda y < 0 \leq \lambda x \text{ for all } x \in K.$$

Proof. With $\bar{K} = K$ there is – using the first statement of Theorem 2.8 – a row vector

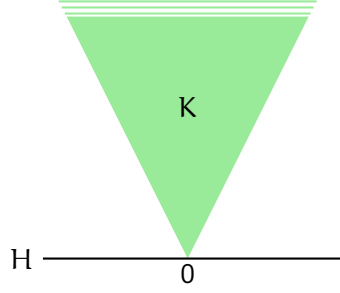


Figure 2.6: Separation of a convex cone via a hyperplane through 0.

$\lambda \in \mathbb{R}^n \setminus \{0\}$ such that for all $x \in K$ we have $\lambda y < \lambda x$. From $0 \in K$ we get $\lambda y < \lambda 0 = 0$. Suppose there is $x \in K$ with $\lambda x < 0$. Then,

$$\lambda(\alpha x) = \alpha(\lambda x) \rightarrow_{\alpha \rightarrow \infty} -\infty,$$

contradicting boundedness of λK via λy from below. Thus, we get

$$\forall x \in K : \lambda y < 0 \leq \lambda x.$$

□

We get as an implication the following theorem of the alternatives:

Theorem 2.10 (Lemma of Farkas). Let B a $k \times n$ matrix and $d \in \mathbb{R}^k$. Then, exactly one of the following statements is true

1. $Bx = d, x \geq 0$ admits a solution $x \in \mathbb{R}^n$.
2. $\lambda B \geq 0, \lambda d < 0$ admits a solution $\lambda \in \mathbb{R}^k$.

Proof. The cone

$$K := \{Bx | x \geq 0\} \subset \mathbb{R}^k$$

is nonempty and convex. It is also closed: each vector in K has a nonnegative representation using linearly independent columns of B (eliminate a dependent column while preserving nonnegativity). For a convergent sequence in K , choose a subsequence using the same independent columns. Their coefficients then converge by injectivity of the corresponding linear map, and the limiting coefficients remain nonnegative. Hence the limit belongs to K . Exactly one of the statements is true

(a) $d \in K$.

(b) $d \notin K$.

Statement (a) is statement (1) of the theorem. In case (b) we get with Theorem 2.8 the

existence of some $\lambda \in \mathbb{R}^k$ with

$$\lambda d < 0 \leq \lambda z \text{ for all } z \in K.$$

Also $\lambda Bx \geq 0$ for all $x \geq 0$, i.e., $\lambda B \geq 0$. This is Statement (2) of the theorem. Note that (a) and (1) (and (b) and (2)) are equivalent, and, thus, (1) and (2) cannot be true simultaneously. $\square$

Chapter 3

Nonlinear Optimization under Constraints

3.1 The Problem

We are given $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and a subset $S \subset \mathbb{R}^n$. The general optimization problem has the form:

$$\min \{f(x) | x \in S\}. \quad (3.1)$$

Recall that maximization

$$\max \{f(x) | x \in S\}$$

is equivalent to minimization

$$\min \{-f(x) | x \in S\}.$$

Definition 3.1 (Local Minimum). $\bar{x} \in S$ is called local minimum of (3.1), if there is an open neighbourhood $U \subset \mathbb{R}^n$ of $\bar{x}$ with

$$f(\bar{x}) \leq f(x) \text{ for all } x \in S \cap U.$$

$\bar{x} \in S$ is called strict local minimum of (3.1), if there is an open neighbourhood $U \subset \mathbb{R}^n$ of $\bar{x}$ with

$$f(\bar{x}) < f(x) \text{ for all } x \in S \cap U, x \neq \bar{x}.$$

Definition 3.2 (Global Minimum). $\bar{x} \in S$ is a global minimum of (3.1), if

$$f(\bar{x}) \leq f(x) \text{ for all } x \in S.$$

$\bar{x} \in S$ is strict global minimum of (3.1), if

$$f(\bar{x}) < f(x) \text{ for all } x \in S, x \neq \bar{x}.$$

Usually S is described by functional equations and inequalities. A general form reads as

$$S := \{x \in \mathbb{R}^n | g(x) \in K\}, \quad (3.2)$$

where $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $K \subset \mathbb{R}^m$ is convex. The problem (3.1) then reads as

$$\min \{f(x) | g(x) \in K\}. \quad (3.3)$$

Assume $K \neq \emptyset$ and let $k = \dim \text{aff}(K)$. Choose an invertible affine map $T : \mathbb{R}^m \rightarrow \mathbb{R}^m$ mapping $\text{aff}(K)$ onto $\mathbb{R}^k \times \{0_{m-k}\}$. Replacing g by $T \circ g$ and K by $T(K)$ preserves feasibility. In these coordinates we may write

$$T(K) = \tilde{K} \times \{0_{m-k}\}, \quad \tilde{K} \subset \mathbb{R}^k, \quad \text{int}(\tilde{K}) \neq \emptyset.$$

We rename $\tilde{K}$ as K and split the transformed mapping into its first k and last $m - k$ components. Accordingly, we can replace $g = (g_1, \dots, g_m)^\top$ with

$$g := (g_1, \dots, g_k)^\top \text{ and } h := (g_{k+1}, \dots, g_m)^\top.$$

The feasible set S is represented via a system of inclusions and equalities $g(x) \in K, \overset{\circ}{K} \neq \emptyset$ and $h(x) = 0$:

$$S := \{x \in \mathbb{R}^n | g(x) \in K, h(x) = 0, \overset{\circ}{K} \neq \emptyset\}.$$

The problem (3.3) is then equivalent to

$$\min \{f(x) | g(x) \in K, h(x) = 0, \overset{\circ}{K} \neq \emptyset\}. \quad (3.4)$$

3.2 Formulation for the Standard Cone

If we choose in (3.4) for K the standard cone

$$K := \mathbb{R}_-^k = \{x \in \mathbb{R}^k | x_i \leq 0, i = 1, \dots, k\},$$

we get the standard problem of nonlinear optimization:

$$\min \{f(x) | g(x) \leq 0, h(x) = 0\}. \quad (3.5)$$

The inequalities $g(x) \leq 0$ need to be component-wise valid. Equivalently

$$\begin{aligned} & \min f(x) \\ & \text{s.t.:} \\ & g_i(x) \leq 0, \quad i = 1, \dots, k \\ & g_j(x) = 0, \quad j = k + 1, \dots, m. \end{aligned} \quad (3.6)$$

For $k = 0$ there are only equality constraints. Such a problem is meaningful also when $m > n$, for example with redundant equations. The inequality $m \leq n$ is necessary only if the m constraint gradients are required to be linearly independent. Sign-constraints

$$x_i \geq 0, \quad i = 1, \dots, r \quad (r \leq n),$$

or variable bounds

$$a_i \leq x_i \leq b_i, \quad i = 1, \dots, r, \quad (r \leq n)$$

are modeled as inequalities.

We study differentiable optimization problems, thus, f and g are assumed to be continuously differentiable in an open neighbourhood of $\bar{x}$. The first derivatives $f'(x)$ or $g'(x)$ at $\bar{x}$ are given as

$$f'(\bar{x}) = \left(\frac{\partial f}{\partial x_1}(\bar{x}), \dots, \frac{\partial f}{\partial x_n}(\bar{x}) \right)$$

and the $m \times n$ -matrix

$$g'(\bar{x}) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(\bar{x}) & \cdot & \cdot & \cdot & \frac{\partial g_1}{\partial x_n}(\bar{x}) \\ \vdots & & & & \vdots \\ \frac{\partial g_m}{\partial x_1}(\bar{x}) & \cdot & \cdot & \cdot & \frac{\partial g_m}{\partial x_n}(\bar{x}) \end{bmatrix}.$$

We will derive in the following necessary and sufficient optimality conditions for the standard optimization problem.

Chapter 4

Tangent Cone and Regularity

In order to set up a theory of necessary and sufficient optimality conditions we introduce the **tangent cone** $T(S, \bar{x})$ of a set $S \subset \mathbb{R}^n$ at $\bar{x} \in S$.

Example 4.1. Let $S = \mathbb{R}^n$ and consider

$$\min\{f(x) | x \in S\}.$$

Let $\bar{x}$ be a local minimum of f over S , see Fig. 4.1. By definition of a local minimum of f if we move away from $\bar{x}$ along a **feasible direction** the objective may not decrease. Suppose we speak of linear feasible directions $d \in \mathbb{R}^n$ at $\bar{x} \in S$, that is, there is $\bar{\alpha} > 0$ with $x = \bar{x} + \alpha d \in S$ for all $0 \leq \alpha \leq \bar{\alpha}$.

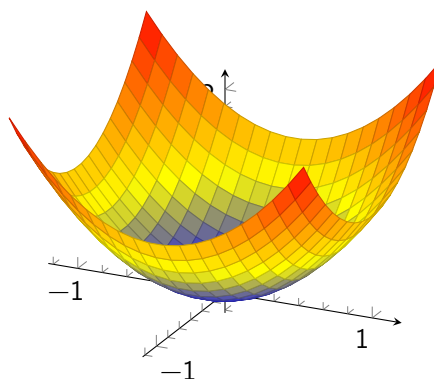


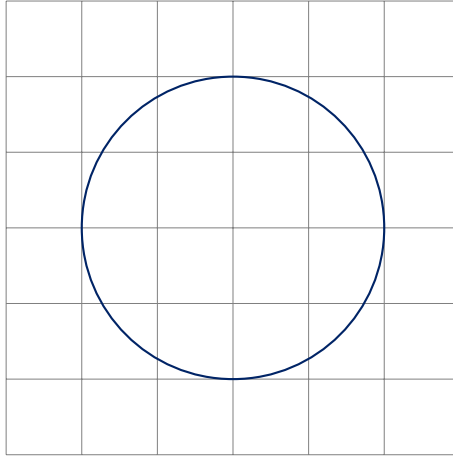
Figure 4.1: Example of a local min. at $\bar{x} = 0$.

For $\bar{x} \in \overset{\circ}{S}$, every $d \in \mathbb{R}^n$ is a feasible direction. For $\bar{x} \in \partial S$ the concept of linearly feasible directions is not enough.

Example 4.2. Let $S = \{x \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\}$ and consider

$$\min\{2x_1 + x_2^2 | x \in S\}.$$

Here $\overset{\circ}{S} = \emptyset$ and for any nonzero direction d , for no $\bar{x} \in S$, there is $\bar{\alpha} > 0$ with $x = \bar{x} + \alpha d \in S$



for all $0 \leq \alpha \leq \bar{\alpha}$. Hence, we need a more general concept: **infinitesimally feasible directions** at $\bar{x}$.

4.1 Motivation of the Theory

$$\begin{aligned} & \text{minimize } f(x) \\ & \text{s.t. } h_i(x) = 0, \quad i \in \{1, \dots, k\}, \\ & \quad g_j(x) \leq 0, \quad j \in \{k+1, \dots, m\}. \end{aligned} \tag{4.1}$$

with $x \in \mathbb{R}^n$ and smooth functions f, h_i, g_j for all i, j . Notation: $h = (h_1, \dots, h_k), g = (g_{k+1}, \dots, g_m)$

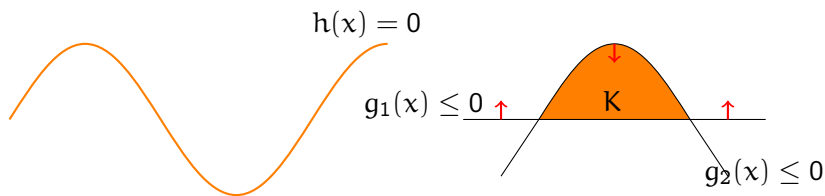


Figure 4.2: Illustration of S .

Let us now explain the idea of **infinitesimally feasible directions**. Consider the case $h(x) = 0$ as in Fig. 4.3. Under some assumptions (regularity as introduced later) the tangent plane $T_h(x) = \{v : \nabla h(x)^T \cdot v = 0\}$ contains the set of **infinitesimally feasible directions** at x (proof later).

Consider $f(x_1, x_2) = -x_1 - x_2$ with $-\nabla f(x_1, x_2) = (1, 1)$.

The vector $-\nabla f(x)$ describes a force that tends to decrease f . On the regular hypersurface $h(x) = 0$, a constraint force may act in either normal direction, so it has the form $-\lambda \nabla h(x)$ with $\lambda \in \mathbb{R}$. Equilibrium requires $\nabla f(x) + \lambda \nabla h(x) = 0$; there is no sign restriction on the multiplier of an equality constraint.

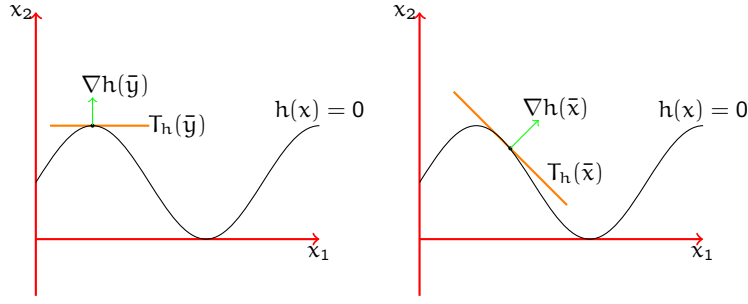


Figure 4.3: The set S is defined via $h(x) = 0$. The tangent plane is illustrated for $\bar{y}$ and $\bar{x}$.

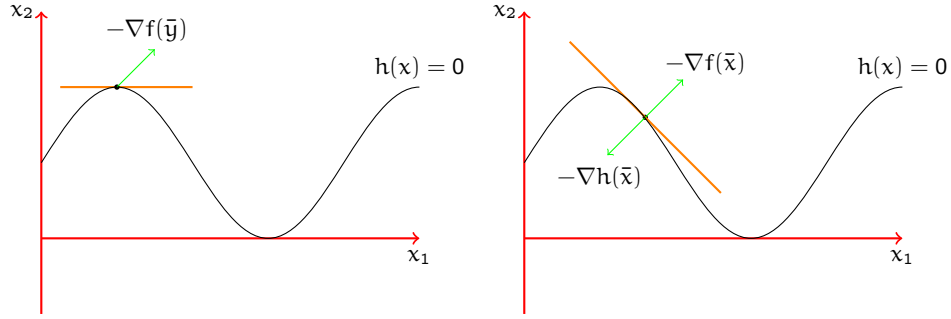


Figure 4.4: Gradients of h and f in a local minimum.

Formally, for a local minimum at $\bar{x}$ for any movement from $\bar{x}$ along v for some v in the tangent plane, $\nabla f(\bar{x})^\top v$ may only be nonnegative, that is, $\nabla f(\bar{x})^\top v \geq 0$. As with $v \in T_h(\bar{x})$ we get $-v \in T_h(\bar{x})$ we follow $\nabla f(\bar{x})^\top v = 0$. Hence, $\nabla f(\bar{x})^\top$ and $\nabla h(\bar{x})^\top$ are linearly dependent (without proof) and in $\bar{x}$ we have the condition

$$f'(\bar{x}) + \lambda h'(\bar{x}) = 0.$$

4.2 Tangent Cone and Variational Inequality

Definition 4.3 (Tangent Cone). The tangent cone $T(S, \bar{x})$ of $S \subset \mathbb{R}^n$ at $\bar{x} \in S$ is defined as:

$$T(S, \bar{x}) := \left\{ v \in \mathbb{R}^n \mid \exists x_i \in S, t_i > 0, \text{ with } \lim_{i \rightarrow \infty} t_i = 0, v = \lim_{i \rightarrow \infty} \frac{x_i - \bar{x}}{t_i} \right\}. \quad (4.2)$$

The definition $v \in T(S, \bar{x})$ is equivalent to

$$x_i = \bar{x} + t_i v + r_i, r_i \in \mathbb{R}^n, \lim_{i \rightarrow \infty} \frac{r_i}{t_i} = 0. \quad (4.3)$$

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With the Landau calculus, we have $r_i = o(t_i)$. In particular:

$$\lim_{i \rightarrow \infty} x_i = \bar{x}, \quad \lim_{i \rightarrow \infty} \frac{\|x_i - \bar{x}\|}{t_i} = \|v\|.$$

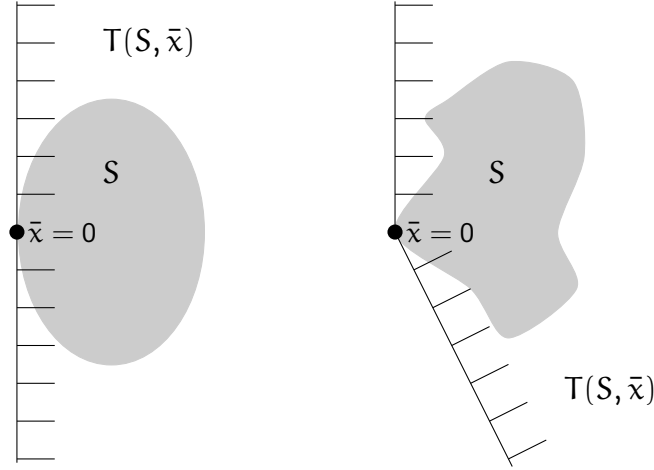


Figure 4.5: Tangent cone with vertex at 0.

Notation.

Let us recap some terminology for sets in $\mathbb{R}^n$. For $K \subset \mathbb{R}^n$ we use:

$\text{aff}(K)$: the smallest affine subspace in $\mathbb{R}^n$ containing K ,

$\text{span}(K)$: the smallest linear subspace in $\mathbb{R}^n$ containing K ,

$\text{int}(K) = \overset{\circ}{K}$: topological interior of K wrt. $\mathbb{R}^n$,

K^i : relative topological interior of K wrt. $\text{aff}(K)$,

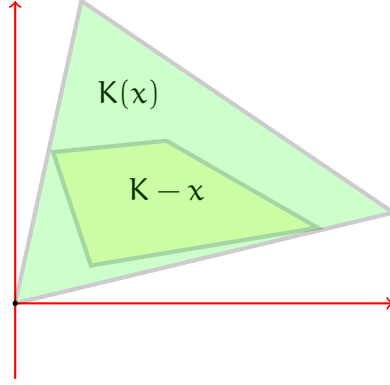
$\text{cl}(K) = \bar{K}$: topological closure of K ,

$\partial K = \bar{K} \setminus \overset{\circ}{K}$: boundary of K .

Conic hull: Let $K \subset \mathbb{R}^n$ and $x \in K$. The cone

$$K(x) := \{\alpha(y - x) | y \in K, \alpha > 0\} = \bigcup_{\alpha > 0} \alpha(K - x)$$

is called conical hull of K wrt. x . Per definition $K(x)$ corresponds to the conic hull of $K - x$.

Figure 4.6: Illustration of the conic hull $K(x)$.

Lemma 4.4. For every $S \subset \mathbb{R}^n$ and $\bar{x} \in S$ the set $T(S, \bar{x})$ is a closed cone with vertex at 0.

Proof. Obviously $T(S, \bar{x})$ is a cone with vertex at 0. It remains to show that $T(S, \bar{x})$ is closed. Let $v \in \text{cl}(T(S, \bar{x}))$ and $(v_l)_{l \in \mathbb{N}}$ a sequence with

$$v_l \rightarrow v, v_l \in T(S, \bar{x}), \text{ for which w.l.o.g. } \|v_l - v\| \leq \frac{1}{l}.$$

We need to show that $v \in T(S, \bar{x})$. Per definition of the tangent cone for every $l \in \mathbb{N}$ there are sequences $(x_{l,k})_{k \in \mathbb{N}}$ and $(t_{l,k})_{k \in \mathbb{N}}$ with

$$x_{l,k} \in S, t_{l,k} > 0, \text{ with } \lim_{k \rightarrow \infty} t_{l,k} = 0, v_l = \lim_{k \rightarrow \infty} \frac{x_{l,k} - \bar{x}}{t_{l,k}}. \quad (4.4)$$

Hence, for all $l \in \mathbb{N}$ there is an index k_l with

$$\left\| \frac{x_{l,k_l} - \bar{x}}{t_{l,k_l}} - v_l \right\| \leq \frac{1}{l}, \quad \|x_{l,k_l} - \bar{x}\| \leq \frac{1}{l} \text{ and } t_{l,k_l} \leq \frac{1}{l}.$$

For $l \rightarrow \infty$ we get $x_{l,k_l} \rightarrow \bar{x}$ and $t_{l,k_l} \rightarrow 0$. Moreover, with the triangle inequality we get

$$\left\| \frac{x_{l,k_l} - \bar{x}}{t_{l,k_l}} - v \right\| \leq \left\| \frac{x_{l,k_l} - \bar{x}}{t_{l,k_l}} - v_l \right\| + \|v_l - v\| \leq \frac{2}{l}.$$

It follows that $\frac{x_{l,k_l} - \bar{x}}{t_{l,k_l}} \rightarrow v$ and hence $v \in T(S, \bar{x})$. $\square$

For convex sets $K \subset \mathbb{R}^n$ we can easily compute the tangent cone.

Lemma 4.5. Let $K \subset \mathbb{R}^n$ be convex and let $\bar{x} \in K$. Then, $T(K, \bar{x})$ is the closure of the conical hull of K in $\bar{x}$, i.e.,

$$T(K, \bar{x}) = \text{cl}(\cup_{\alpha > 0} \alpha(K - \bar{x})) =: \overline{K(\bar{x})}.$$

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Proof. $T(K, \bar{x}) \supset \overline{K(\bar{x})}$: Let $x \in K$ and $\alpha > 0$. We need to show that $\alpha(x - \bar{x}) \in T(K, \bar{x})$. As K convex and $x \in K$, we get

$$\begin{aligned} x_i &= \bar{x} + \frac{\alpha}{i}(x - \bar{x}) \in K, \text{ for } i \geq \alpha \\ \Rightarrow \frac{x_i - \bar{x}}{1/i} &= \alpha(x - \bar{x}) \\ \Rightarrow \alpha(x - \bar{x}) &= \lim_{i \rightarrow \infty} \frac{x_i - \bar{x}}{1/i} = v \in T(K, \bar{x}). \end{aligned}$$

As $T(K, \bar{x})$ is closed, we get $T(K, \bar{x}) \supset \overline{K(\bar{x})}$.

$T(K, \bar{x}) \subset \overline{K(\bar{x})}$: Let $v \in T(K, \bar{x})$ with

$$x_i = \bar{x} + t_i v + r_i, r_i \in \mathbb{R}^n, \lim_{i \rightarrow \infty} \frac{r_i}{t_i} = 0.$$

Then,

$$\underbrace{\frac{x_i - \bar{x}}{t_i}}_{\in K(\bar{x})} = v + \frac{r_i}{t_i} \Rightarrow v = \lim_{i \rightarrow \infty} \frac{x_i - \bar{x}}{t_i} \in \overline{K(\bar{x})}.$$

□

We derive a fundamental necessary optimality criterion.

Theorem 4.6 (Variational Inequality). Let $\bar{x}$ be a local minimum of

$$\min \{f(x) | x \in S\}$$

Assume that f is differentiable at $\bar{x}$. Then,

$$f'(\bar{x})v \geq 0 \text{ for all } v \in T(S, \bar{x}).$$

Proof. Let $v \in T(S, \bar{x})$. With $x_i = \bar{x} + vt_i + r_i$ we get

$$\begin{aligned} f(x_i) &= f(\bar{x}) + f'(\bar{x})(x_i - \bar{x}) + o(\|x_i - \bar{x}\|) \\ &= f(\bar{x}) + f'(\bar{x})vt_i + o(t_i). \end{aligned}$$

We obtain

$$\lim_{i \rightarrow \infty} \frac{f(x_i) - f(\bar{x})}{t_i} = \lim_{i \rightarrow \infty} \left(f'(\bar{x})v + \frac{o(t_i)}{t_i} \right) = f'(\bar{x})v.$$

Since $\bar{x}$ is a local minimum, we get for i large enough

$$f(x_i) \geq f(\bar{x}).$$

Thus,

$$0 \leq \lim_{i \rightarrow \infty} \frac{f(x_i) - f(\bar{x})}{t_i} = f'(\bar{x})v.$$

□

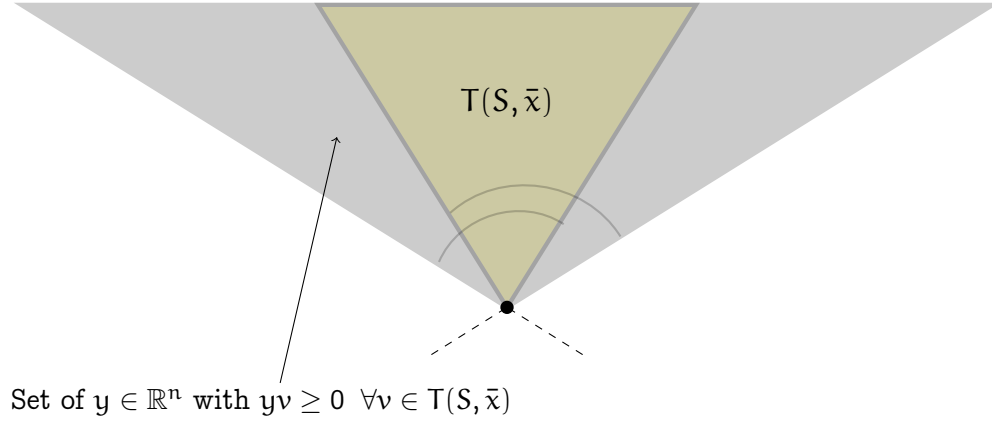


Figure 4.7: Set of vectors y for which $yv \geq 0$ for all $v \in T(S, \bar{x})$.

4.3 Linearized Cone

We consider the case that S is given as

$$S = \{x \in \mathbb{R}^n | g(x) \in K\},$$

where $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is C^1 and $K \subset \mathbb{R}^m$ is convex. As the tangent cone is hard to compute, we introduce the **linearized cone** of S in $\bar{x}$.

As motivation for that cone, let us linearize the inclusion $g(x) \in K$ in $\bar{x}$, i.e., for $v \in \mathbb{R}^n$ we consider

$$g(\bar{x}) + g'(\bar{x})v \in K,$$

hence

$$g'(\bar{x})v \in K - g(\bar{x}) \subset \bigcup_{\alpha > 0} \alpha(K - g(\bar{x})) =: K(g(\bar{x})).$$

This leads directly to the definition of the **linearized cone** of S in $\bar{x}$:

$$L(S, \bar{x}) = \{v \in \mathbb{R}^n | g'(\bar{x})v \in K(g(\bar{x}))\}. \quad (4.5)$$

| Lemma 4.7. $L(S, \bar{x})$ is a convex cone with vertex at 0.

Proof. We have $0 \in L(S, \bar{x})$, because $0 \in K(g(\bar{x}))$, which in turn follows from $g(\bar{x}) \in K$. The cone property $v \in L(S, \bar{x}) \Rightarrow \alpha v \in L(S, \bar{x})$ for all $\alpha \geq 0$ is also satisfied. We need to show convexity. Let $u, v \in L(S, \bar{x})$ and $\lambda \in (0, 1)$. For $w = \lambda u + (1 - \lambda)v$, we get

$$g'(\bar{x})w = \lambda \underbrace{g'(\bar{x})u}_{\in K(g(\bar{x}))} + (1 - \lambda) \underbrace{g'(\bar{x})v}_{\in K(g(\bar{x}))} \in K(g(\bar{x}))$$

where the last inclusion follows by convexity of $K(g(\bar{x}))$ (see exercise). $\square$

Exercise 4.8. Let $K \subset \mathbb{R}^m$ convex and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ a C^1 mapping. Let $x \in \mathbb{R}^n$ with $g(x) \in K$. Show that the conical hull of K w.r.t. $g(x)$, that is, $K(g(x)) = \cup_{\alpha > 0} \alpha(K - g(x))$, is convex.

Note that $L(S, \bar{x})$ depends not only on the set S but also on the chosen mapping g in order to represent S ; see the following example.

Example 4.9. Let $S := \{0\} \subset \mathbb{R}$, $\bar{x} = 0$. With $K = \{0\} \subset \mathbb{R}$, $g(x) = x^2$ we get that S can be represented as $S = \{x \in \mathbb{R} | g(x) \in K\}$. With $g'(\bar{x}) = 0$ we have

$$L(S, \bar{x}) = \{v \in \mathbb{R} | g'(\bar{x})v = 0\} = \mathbb{R}.$$

If we choose $g(x) = x$, we still get $S = \{x \in \mathbb{R} | g(x) \in K\}$. But with $g'(\bar{x}) = 1$ we get $L(S, \bar{x}) = \{0\}$.

Another special case is $K = \{0\}$. Here

$$S = \{x \in \mathbb{R}^n | g(x) = 0\}$$

is an equality-constrained set, which is locally a smooth manifold when $g'(\bar{x})$ has full row rank. Its linearized cone is the subspace

$$L(S, \bar{x}) = \{v \in \mathbb{R}^n | g'(\bar{x})v = 0\}.$$

Another important special case is the standard cone $K = \mathbb{R}_-^k \times \{0_{m-k}\}$. Here, S is given as

$$S = \left\{ x \in \mathbb{R}^n \left| \begin{array}{l} g_i(x) \leq 0, i = 1, \dots, k, \\ g_i(x) = 0, i = k + 1, \dots, m \end{array} \right. \right\}.$$

Let

$$I(\bar{x}) := \{i \in \{1, \dots, k\} | g_i(\bar{x}) = 0\}$$

be the set for which the inequalities $g_i(\bar{x}) \leq 0$ are active. We denote this set as the set of **active indices**.

We get that $K(g(\bar{x}))$ has the following form:

$$K(g(\bar{x})) = \left\{ y \in \mathbb{R}^m \left| \begin{array}{l} y_i \leq 0, i \in I(\bar{x}), \\ y_i = 0, i = k + 1, \dots, m \end{array} \right. \right\}$$

and hence the linearized cone can be computed as

$$L(S, \bar{x}) = \left\{ v \in \mathbb{R}^n \left| \begin{array}{l} g'_i(\bar{x})v \leq 0, i \in I(\bar{x}), \\ g'_i(\bar{x})v = 0, i = k + 1, \dots, m \end{array} \right. \right\}. \quad (4.6)$$

We obtain the following relationship between $T(S, \bar{x})$ and $L(S, \bar{x})$.

Lemma 4.10. If $K(g(\bar{x}))$ is closed, then

$$T(S, \bar{x}) \subset L(S, \bar{x}).$$

Proof. Let $v \in T(S, \bar{x})$ with

$$v = \lim_{i \rightarrow \infty} \frac{(x_i - \bar{x})}{t_i}, \quad x_i \in S, \quad t_i > 0.$$

As in the proof of Thm. 4.6 we get

$$g'(\bar{x})v = \lim_{i \rightarrow \infty} \frac{g(x_i) - g(\bar{x})}{t_i}.$$

With $x_i \in S \Rightarrow g(x_i) \in K$ and $t_i > 0$ using the definition of $K(g(\bar{x}))$ we get $\frac{g(x_i) - g(\bar{x})}{t_i} \in K(g(\bar{x}))$. Closedness of $K(g(\bar{x}))$ implies

$$g'(\bar{x})v \in K(g(\bar{x})), \quad \text{i.e. } v \in L(S, \bar{x}).$$

□

The reverse inclusion $L(S, \bar{x}) \subset T(S, \bar{x})$ is not valid in general as shown by the following example.

Example 4.11. Let $S := \{0\} \subset \mathbb{R}$, $\bar{x} = 0$, $T(S, \bar{x}) = \{0\}$.

1. With $K = \{0\} \subset \mathbb{R}$, $g(x) = x^2$ the set S is given as $S = \{x \in \mathbb{R} | g(x) \in K\}$. With $g'(\bar{x}) = 0$ we get

$$L(S, \bar{x}) = \{v \in \mathbb{R} | g'(\bar{x})v = 0\} = \mathbb{R}$$

and therefore $T(S, \bar{x}) \subsetneq L(S, \bar{x})$.

2. With $K = \{0\} \subset \mathbb{R}$, $g(x) = x$ we can represent S as $S = \{x \in \mathbb{R} | g(x) \in K\}$. With $g'(\bar{x}) = 1$ we get $L(S, \bar{x}) = \{0\}$ and therefore $T(S, \bar{x}) = L(S, \bar{x})$.

4.4 Regularity Conditions

For obtaining $L(S, \bar{x}) \subset T(S, \bar{x})$ one needs to impose additional conditions on g and K . Such conditions are known as **constraint qualifications** or **regularity conditions**. For a motivation, consider the case

$$S = \{x \in \mathbb{R}^n | g(x) = 0\}.$$

Definition 4.12. $\bar{x} \in S$ is called **regular**, if

$$\text{Im } g'(\bar{x}) = \mathbb{R}^m \tag{4.7}$$

where $\text{Im } g'(\bar{x})$ is the image of the linear mapping $x \mapsto g'(\bar{x})x$ for $x \in \mathbb{R}^n$.

$\bar{x}$ is regular iff the gradients $g'_i(\bar{x})^\top, i = 1, \dots, m$ are linearly independent, or, equivalently

$$g'(\bar{x})g'(\bar{x})^\top \text{ is non-singular.}$$

For later, we recap that exactly one of the following statements is true:

1. $\bar{x}$ is regular.
2. there is $\lambda \in \mathbb{R}^m, \lambda \neq 0$ with $\lambda g'(\bar{x}) = 0$.

Theorem 4.13. Let $S = \{x \in \mathbb{R}^n | g(x) = 0\}$ and let $\bar{x} \in S$ be regular. Then,

1. For $v \in \mathbb{R}^n$ with $g'(\bar{x})v = 0$ there is $\epsilon > 0$ and a curve $x : [-\epsilon, \epsilon] \rightarrow S$ with

$$x(0) = \bar{x}, \dot{x}(0) = \lim_{t \rightarrow 0} \frac{x(t) - \bar{x}}{t} = v.$$

2. We have $T(S, \bar{x}) = L(S, \bar{x}) = \{v \in \mathbb{R}^n | g'(\bar{x})v = 0\}$.

Proof. For (1): We define $F : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^m$ via

$$F(y, t) := g(\bar{x} + tv + g'(\bar{x})^\top y), \quad y \in \mathbb{R}^m, t \in \mathbb{R}.$$

We have $F(0_m, 0) = g(\bar{x}) = 0$ and the partial derivatives of F wrt. y read as

$$\frac{\partial F}{\partial y}(0_m, 0) = g'(\bar{x})g'(\bar{x})^\top.$$

This matrix is non-singular as $\bar{x}$ is regular. With the Implicit Function Theorem, there is $\epsilon > 0$ and a function $y : [-\epsilon, \epsilon] \rightarrow \mathbb{R}^m$ that is continuously differentiable in $t = 0$ with $y(0) = 0$ and

$$F(y(t), t) = 0 \text{ for } t \in [-\epsilon, \epsilon].$$

We obtain $\dot{y}(0) := \lim_{t \rightarrow 0} \frac{y(t) - y(0)}{t}$ and with the assumption $g'(\bar{x})v = 0$ we get

$$0 = \left. \frac{dF(y(t), t)}{dt} \right|_{t=0} = F_y(y(0), 0)\dot{y}(0) + F_t(y(0), 0) = g'(\bar{x})g'(\bar{x})^\top \dot{y}(0),$$

where we used $F_t(y(0), 0) = g'(\bar{x})v = 0$. Hence, $\dot{y}(0) = 0$. By setting

$$x(t) = \bar{x} + vt + g'(\bar{x})^\top y(t), \quad t \in [-\epsilon, \epsilon],$$

we obtain a curve $x(t)$ with the desired properties: $g(x(t)) = 0$, i.e. $x(t) \in S, x(0) = \bar{x}$ and $\dot{x}(0) = v$, because $\dot{y}(0) = 0$.

For (2): The claim follows from (1) and Lemma 4.10. □

We consider now the general case.

$$S = \{x \in \mathbb{R}^n | g(x) \in K\}, \quad K \text{ convex.} \tag{4.8}$$

Definition 4.14. Let S as in (4.8). $\bar{x} \in S$ is called regular, if

$$\text{Im } g'(\bar{x}) - K(g(\bar{x})) = \mathbb{R}^m. \quad (4.9)$$

This condition generalizes (4.7) and means geometrically that the linear subspace $\text{Im } g'(\bar{x})$ and the cone $K(g(\bar{x}))$ are transversal to each other.

In the following, we derive easier but equivalent conditions.

Lemma 4.15. Let K be convex. The following conditions are equivalent:

1. $\text{Im } g'(\bar{x}) - K(g(\bar{x})) = \mathbb{R}^m$,
2. $0 \in \text{int}(\text{Im } g'(\bar{x}) - K(g(\bar{x})))$,
3. $0 \in \text{int}(\text{Im } g'(\bar{x}) + g(\bar{x}) - K)$,

Proof. The proof is done in the following order: 1. $\Rightarrow$ 3. $\Rightarrow$ 2. $\Rightarrow$ 1.

1. $\Rightarrow$ 3. : The set $\text{Im } g'(\bar{x}) + g(\bar{x}) - K$ is convex and contains 0 because $g(\bar{x}) \in K$. Suppose $0 \notin \text{int}(\text{Im } g'(\bar{x}) + g(\bar{x}) - K)$. With the separating hyperplane theorem (Theorem 2.8) there is $\lambda \in \mathbb{R}^m, \lambda \neq 0$ with

$$\lambda (g'(\bar{x})v + g(\bar{x}) - y) \geq 0 \text{ for all } v \in \mathbb{R}^n, y \in K.$$

With the definition

$$K(g(\bar{x})) = \cup_{\alpha > 0} \alpha (K - g(\bar{x}))$$

we get

$$\lambda (g'(\bar{x})v - y) \geq 0 \text{ for all } v \in \mathbb{R}^n, y \in K(g(\bar{x})). \quad (4.10)$$

The assumption $\text{Im } g'(\bar{x}) - K(g(\bar{x})) = \mathbb{R}^m$ yields $\lambda = 0$ in contradiction to the choice of λ .

3. $\Rightarrow$ 2. : Follows from $K - g(\bar{x}) \subset K(g(\bar{x}))$.

2. $\Rightarrow$ 1. : Let $B_\epsilon(0) := \{y \in \mathbb{R}^m \mid \|y\| \leq \epsilon\}$. Per assumption there is $\epsilon > 0$ with

$$B_\epsilon(0) \subset \text{Im } g'(\bar{x}) - K(g(\bar{x})).$$

Since $\text{Im } g'(\bar{x}) - K(g(\bar{x}))$ is a cone, we get

$$\mathbb{R}^m = \cup_{\alpha \geq 0} \alpha B_\epsilon(0) \subset \text{Im } g'(\bar{x}) - K(g(\bar{x})),$$

implying 1. □

These conditions are due to S. Robinson (1976).

Using the proof of 1. $\Rightarrow$ 3. we get the following.

Corollary 4.16. $\bar{x} \in S$ is not regular if and only if there is $\lambda \in \mathbb{R}^m, \lambda \neq 0$ with

$$\lambda g'(\bar{x}) = 0, \lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})).$$

Exercise 4.17. Prove the statement of Corollary 4.16.

We specialize the conditions of Lemma 4.15 to the following case

$$S = \{x \in \mathbb{R}^n | g(x) \in K, h(x) = 0\}, \overset{\circ}{K} \neq \emptyset, K \text{ convex}, \quad (4.11)$$

with $g = (g_1, \dots, g_k)^\top, h = (g_{k+1}, \dots, g_m)^\top$. Then, 2. from 4.15 is equivalent to

$$\text{Im } h'(\bar{x}) = \mathbb{R}^{m-k} \text{ and there is } v \in \mathbb{R}^n \text{ with } h'(\bar{x})v = 0, g'(\bar{x})v \in \text{int}(K(g(\bar{x}))). \quad (4.12)$$

and 3. is equivalent to

$$\text{Im } h'(\bar{x}) = \mathbb{R}^{m-k} \text{ and there is } v \in \mathbb{R}^n \text{ with } h'(\bar{x})v = 0, g(\bar{x}) + g'(\bar{x})v \in \overset{\circ}{K}. \quad (4.13)$$

The conditions (4.12) and (4.13) are called **local Slater-Conditions**. For $K = \mathbb{R}_+^k$ we get from (4.12) the **Mangasarian-Fromovitz-Conditions** (cf. Mangasarian 1969).

Definition 4.18 (Mangasarian-Fromovitz). The gradients $g'_{k+1}(\bar{x})^\top, \dots, g'_m(\bar{x})^\top$ are linearly independent and there is $v \in \mathbb{R}^n$ with

$$\begin{aligned} g'_i(\bar{x})v &< 0, \quad i \in I(\bar{x}) \text{ (set of active indices)} \\ g'_i(\bar{x})v &= 0, \quad i = k+1, \dots, m. \end{aligned}$$

We now derive the central statement $L(S, \bar{x}) \subset T(S, \bar{x})$ under the aforementioned regularity conditions. We will do this for the case S being in the form (4.11).

Theorem 4.19. Let $\bar{x}$ be a regular point in

$$S = \{x \in \mathbb{R}^n | g(x) \in K, h(x) = 0\}, \overset{\circ}{K} \neq \emptyset, K \text{ convex}.$$

Then:

1. For $v \in \mathbb{R}^n$ with $h'(\bar{x})v = 0$, and $g(\bar{x}) + g'(\bar{x})v \in \overset{\circ}{K}$ there is $\epsilon > 0$ and a curve

$$x : [0, \epsilon] \rightarrow S$$

with

$$x(0) = \bar{x}, \dot{x}(0) = \lim_{t \downarrow 0} \frac{x(t) - \bar{x}}{t} = v.$$

2. We have $L(S, \bar{x}) \subset T(S, \bar{x})$.

Proof. For 1.: In case $k = m$ choose $\delta = 1$ and set $x(t) = \bar{x} + tv$ for $|t| \leq \delta$. If $k < m$ then using Thm. 4.13,(1.), there is $\delta > 0$ and a curve $x : [-\delta, \delta] \rightarrow \mathbb{R}^n$ with

$$h(x(t)) = 0 \text{ for } t \in [-\delta, \delta], \quad x(0) = \bar{x}, \quad \dot{x}(0) = v.$$

Using

$$g(\bar{x}) + g'(\bar{x})v \in \overset{\circ}{K}, \quad \lim_{t \rightarrow 0} \frac{g(x(t)) - g(\bar{x})}{t} = g'(\bar{x})v,$$

there is $\epsilon \leq \min\{\delta, 1\}$ with

$$g(\bar{x}) + \frac{g(x(t)) - g(\bar{x})}{t} \in K \text{ for } 0 < t \leq \epsilon.$$

As $g(\bar{x}) \in K$ and with convexity of K , we get

$$g(x(t)) = (1-t)g(\bar{x}) + t \left(g(\bar{x}) + \frac{g(x(t)) - g(\bar{x})}{t} \right) \in K \text{ for } 0 < t \leq \epsilon.$$

Thus, $x(t) \in S$ for all $0 \leq t \leq \epsilon$.

For 2.: We repeat the definition of $L(S, \bar{x})$:

$$L(S, \bar{x}) = \{v \in \mathbb{R}^n | g'(\bar{x})v \in K(g(\bar{x})), \quad h'(\bar{x})v = 0\}.$$

Let $v \in L(S, \bar{x})$. Per definition of $K(g(\bar{x}))$ there is $r > 0$ with

$$g'(\bar{x})v \in r(K - g(\bar{x})), \quad h'(\bar{x})v = 0.$$

This implies

$$g(\bar{x}) + g'(\bar{x})\frac{v}{r} \in K, \quad h'(\bar{x})v = 0.$$

Using regularity of $\bar{x}$ and consequently (4.13), there is $v_0 \in \mathbb{R}^n$ with

$$g(\bar{x}) + g'(\bar{x})v_0 \in \overset{\circ}{K}, \quad h'(\bar{x})v_0 = 0.$$

We use the convex combination

$$v_\alpha := (1 - \alpha)v_0 + \alpha\frac{v}{r}, \text{ for } 0 \leq \alpha < 1.$$

and get using the following statement

Exercise 4.20. Let K be convex and $x \in \overset{\circ}{K}, y \in \bar{K}$. Then,

$$(1 - \alpha)x + \alpha y \in \overset{\circ}{K}, \text{ for } 0 \leq \alpha < 1.$$

that

$$g(\bar{x}) + g'(\bar{x})v_\alpha \in \overset{\circ}{K}, \quad h'(\bar{x})v_\alpha = 0 \text{ for } 0 \leq \alpha < 1.$$

With part 1. we get

$$v_\alpha \in T(S, \bar{x}) \text{ for } 0 \leq \alpha < 1.$$

Using that $T(S, \bar{x})$ is closed, we get

$$v = \lim_{\alpha \rightarrow 1} rv_\alpha \in T(S, \bar{x}).$$

□

We give some examples.

Example 4.21. The set S is given as

$$g_1(x) = x_2 - x_1^3 \leq 0, \quad g_2(x) = x_2 \leq 0.$$

For $\bar{x} = 0$ we have $I(\bar{x}) = \{1, 2\}$.

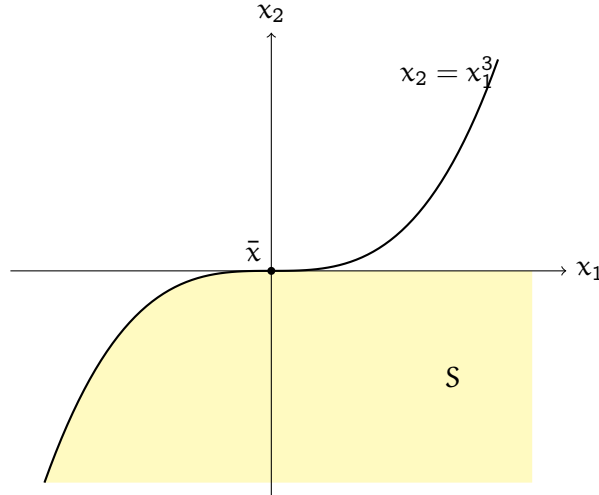


Figure 4.8: Set S .

The derivatives

$$g'_1(\bar{x}) = g'_2(\bar{x}) = (0, 1)$$

are not linearly independent. Yet, the point $\bar{x} = 0$ is regular in the sense of Definition 4.18, because for every $v \in \mathbb{R}^2$ with $v_2 < 0$ we get

$$g'_i(\bar{x})v = v_2 < 0, \quad i = 1, 2.$$

We obtain

$$T(S, \bar{x}) = L(S, \bar{x}) = \{v \in \mathbb{R}^2 | v_2 \leq 0\}$$

according to Thm. 4.19, (2.).

Let us slightly change that example.

Example 4.22. S is given as

$$g_1(x) = x_2 - x_1^3 \leq 0, \quad g_2(x) = -x_2 \leq 0.$$

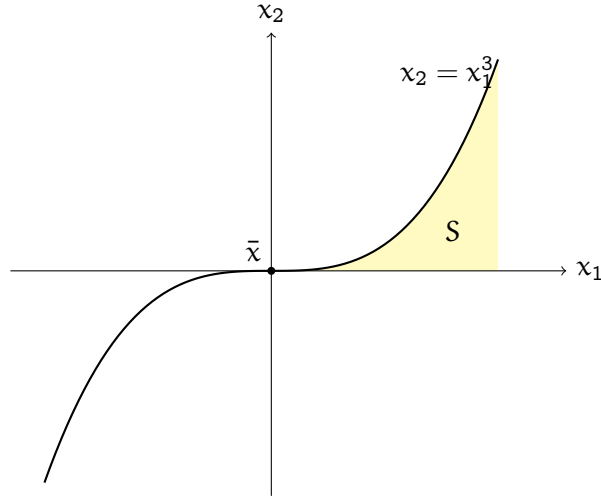


Figure 4.9: Set S .

For $\bar{x} = 0$ we get $I(\bar{x}) = \{1, 2\}$. The derivatives

$$g'_1(\bar{x}) = (0, 1) \text{ and } g'_2(\bar{x}) = (0, -1).$$

are linearly dependent. $\bar{x} = 0$ is not regular in the sense of Definition 4.18, since for no $v \in \mathbb{R}^2$, the conditions

$$g'_1(0)v = v_2 < 0, \text{ and } g'_2(0)v = -v_2 < 0$$

are satisfiable. We compute

$$T(S, \bar{x}) = \{v \in \mathbb{R}^2 | v_1 \geq 0, v_2 = 0\}$$

and with (4.6) we get

$$L(S, \bar{x}) = \{v \in \mathbb{R}^2 | v_2 = 0\}.$$

Here we have $T(S, \bar{x}) \subsetneq L(S, \bar{x})$.

Chapter 5

Necessary Optimality Conditions of First Order

$$\min \{f(x) \mid g(x) \in K\} \quad (5.1)$$

We assume that $f, g \in C^1$ and K is convex.

Let $\bar{x}$ be a local minimum of (5.1). Then, there is a neighbourhood $\mathcal{U}$ of $\bar{x}$ such that the following subsets of $\mathbb{R} \times \mathbb{R}^m$ defined as

$$\{(f(x) - f(\bar{x}), g(x)) \mid x \in \mathcal{U}\} \cap \{(r, y) \mid r < 0, y \in K\}$$

have an empty intersection. Thus, $(0, 0_m)$ lies on the boundary of

$$B = \{(f(x) - f(\bar{x}) + r, g(x) - y) \mid x \in \mathcal{U}, r \geq 0, y \in K\} \subset \mathbb{R} \times \mathbb{R}^m. \quad (5.2)$$

Let us linearize the set B , i.e., we replace $f(x)$ and $g(x)$ with their first order Taylor expansion (without rest term) in $\bar{x}$, hence,

$$f(\bar{x}) + f'(\bar{x})v \text{ and } g(\bar{x}) + g'(\bar{x})v \text{ with } v = x - \bar{x},$$

and we obtain the convex set

$$\tilde{B} = \{(f'(\bar{x})v + r, g'(\bar{x})v + g(\bar{x}) - y) \mid v \in \mathbb{R}^n, r \geq 0, y \in K\} \subset \mathbb{R} \times \mathbb{R}^m.$$

The conical hull of this set wrt. $(0, 0_m)$ is the following convex cone:

$$A = \{(f'(\bar{x})v + r, g'(\bar{x})v - y) \mid v \in \mathbb{R}^n, r \geq 0, y \in K(g(\bar{x}))\} \subset \mathbb{R} \times \mathbb{R}^m. \quad (5.3)$$

| Exercise 5.1. Show that indeed A is the conical hull of $\tilde{B}$ wrt. $(0, 0_m)$.

The set A can be interpreted as a convex approximation of the non-convex set B . The statement of the following theorem says that $(0, 0_m)$ lies on the boundary of A .

Theorem 5.2. Let $\bar{x}$ be a local minimum of (5.1). Then, the following statements are true.

1. Necessary Optimality Conditions of Fritz John:

There is $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m, (\lambda_0, \lambda) \neq (0, 0_m)$ with

$$\lambda_0 f'(\bar{x}) + \lambda g'(\bar{x}) = 0 \quad (5.4)$$

$$\lambda_0 \geq 0, \lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})). \quad (5.5)$$

2. Necessary Optimality Conditions of Karush-Kuhn-Tucker:

If $\bar{x}$ is regular, we get $\lambda_0 > 0$ in (1) and w.l.o.g. $\lambda_0 = 1$ holds. Thus, there is $\lambda \in \mathbb{R}^m$ with

$$f'(\bar{x}) + \lambda g'(\bar{x}) = 0 \quad (5.6)$$

$$\lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})). \quad (5.7)$$

Proof. Both statements are proven together.

1. Case: $\bar{x}$ is not regular. With Corollary 4.16 there is $\lambda \in \mathbb{R}^m, \lambda \neq 0$ with

$$\lambda g'(\bar{x}) = 0, \lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})).$$

The statement of the first part of the theorem follows with $\lambda_0 = 0$.

2. Case: Let $\bar{x}$ be regular. The variational inequality of Thm. 4.6 reads as:

$$f'(\bar{x})v \geq 0 \text{ for all } v \in T(S, \bar{x}), \text{ where } S := \{x \mid g(x) \in K\}.$$

With the regularity of $\bar{x}$, we get with Thm. 4.19 that

$$T(S, \bar{x}) \supset L(S, \bar{x}) = \{v \in \mathbb{R}^n \mid g'(\bar{x})v \in K(g(\bar{x}))\},$$

and hence

$$f'(\bar{x})v \geq 0 \text{ for all } v \in \mathbb{R}^n \text{ with } g'(\bar{x})v \in K(g(\bar{x})). \quad (5.8)$$

We consider the convex cone in (5.3):

$$A = \{(f'(\bar{x})v + r, g'(\bar{x})v - y) \mid v \in \mathbb{R}^n, r \geq 0, y \in K(g(\bar{x}))\} \subset \mathbb{R} \times \mathbb{R}^m.$$

Because of (5.8) we have that $(0, 0_m)$ lies on the boundary of A (it contains 0, but no point $(-a, 0_m)$ with $a > 0$) and with the separating hyperplane theorem there is $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m, (\lambda_0, \lambda) \neq 0$, with

$$\lambda_0(f'(\bar{x})v + r) + \lambda(g'(\bar{x})v - y) \geq 0 \text{ for all } v \in \mathbb{R}^n, r \geq 0, y \in K(g(\bar{x})). \quad (5.9)$$

For $r = 0$ and $y = 0$ we get

$$\lambda_0(f'(\bar{x})v) + \lambda g'(\bar{x})v \geq 0 \text{ for all } v \in \mathbb{R}^n$$

and therefore

$$\lambda_0(f'(\bar{x})) + \lambda g'(\bar{x}) = 0.$$

For $v = 0$ and $y = 0$ we get $\lambda_0 r \geq 0$ for all $r \geq 0$ and hence $\lambda_0 \geq 0$. Finally, (5.9) implies for $v = 0$ and $r = 0$

$$\lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})).$$

The case $\lambda_0 = 0$ contradicts the statement of Cor. 4.16 for regular points $\bar{x}$. Thus, $\lambda_0 > 0$ and w.l.o.g. $\lambda_0 = 1$.

□

5.1 Lagrangian and Multipliers

The last theorem is known as the Lagrange-Multiplier rule. The vector $\lambda_i, i = 0, \dots, m$ is called Lagrange-Multiplier. The Lagrangian is defined as

$$L(x, \lambda_0, \lambda) := \lambda_0 f(x) + \lambda g(x), (\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m$$

and for regular points

$$L(x, \lambda) := f(x) + \lambda g(x), \lambda \in \mathbb{R}^m.$$

The last theorem then reads as:

1. John: $L_x(\bar{x}, \lambda_0, \lambda) = 0, \lambda_0 \geq 0, (\lambda_0, \lambda) \neq 0, \lambda(-y) \geq 0$ for all $y \in K(g(\bar{x}))$
2. KKT: $L_x(\bar{x}, \lambda) = 0, \lambda(-y) \geq 0$ for all $y \in K(g(\bar{x}))$,

where L_x denote the partial derivative of L wrt. x .

Exercise 5.3. Show that the following sets

$$\Lambda(\bar{x}) := \{\lambda \in \mathbb{R}^m | L_x(\bar{x}, \lambda) = 0, \lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x}))\} \quad (5.10)$$

are convex and closed.

If $\bar{x}$ is regular, then the above set is the set of Lagrange-Multipliers at $\bar{x}$.

Theorem 5.4. Let $\bar{x}$ be a local minimum of (5.1). Then, the following statements are equivalent.

1. $\bar{x}$ is regular.
2. $\Lambda(\bar{x}) \neq \emptyset$ and $\Lambda(\bar{x})$ is bounded.

Proof. 1. $\Rightarrow$ 2.: $\Lambda(\bar{x}) \neq \emptyset$ follows from Theorem 5.2, (2.).

Assumption: $\Lambda(\bar{x})$ is unbounded. Then, there is a sequence $\lambda_i \in \Lambda(\bar{x})$, $i \in \mathbb{N}$, with $\|\lambda_i\| \rightarrow \infty$ for $i \rightarrow \infty$. Per definition of $\Lambda(\bar{x})$ we have

$$f'(\bar{x}) + \lambda_i g'(\bar{x}) = 0, \lambda_i(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})).$$

With $\lambda_i \neq 0$ for i large enough, we get

$$\frac{1}{\|\lambda_i\|} f'(\bar{x}) + \frac{\lambda_i}{\|\lambda_i\|} g'(\bar{x}) = 0, \frac{\lambda_i}{\|\lambda_i\|}(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})). \quad (5.11)$$

As the boundary of $B_1(0)$ is compact in $\mathbb{R}^m$ we may assume that $\frac{\lambda_i}{\|\lambda_i\|} \rightarrow \lambda$ with $\|\lambda\| = 1$. Equation (5.11) yields for $i \rightarrow \infty$

$$\lambda g'(\bar{x}) = 0, \lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})).$$

With Corollary 4.16 we get a contradiction to the regularity at $\bar{x}$.

1. $\Leftarrow$ 2.: Let $\lambda_1 \in \Lambda(\bar{x})$.

Assumption: $\bar{x}$ is not regular.

With Corollary 4.16 there is $\lambda_2 \in \mathbb{R}^m$, $\lambda_2 \neq 0$ with

$$\lambda_2 g'(\bar{x}) = 0, \lambda_2(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})).$$

This implies $\lambda_1 + r\lambda_2 \in \Lambda(\bar{x})$ for all $r \geq 0$ in contradiction to the boundedness of $\Lambda(\bar{x})$. $\square$

We get the following implication.

Corollary 5.5. If $\bar{x}$ is a regular local minimum of (5.1), then $\Lambda(\bar{x})$ is a nonempty, compact and convex subset of $\mathbb{R}^m$.

Now we discuss uniqueness of the Lagrange-Multipliers.

Definition 5.6. $\bar{x}$ is called normal, if

$$\text{Im } g'(\bar{x}) - V = \mathbb{R}^m, \quad V := K(g(\bar{x})) \cap (-K(g(\bar{x}))). \quad (5.12)$$

Recall that $V = K(g(\bar{x})) \cap (-K(g(\bar{x})))$ is the largest linear subspace contained in $K(g(\bar{x}))$. With $K(g(\bar{x})) \supset V$ we get: $\bar{x}$ is normal $\Rightarrow \bar{x}$ is regular.

Theorem 5.7. If $\bar{x}$ is a normal local minimum of (5.1), then $\Lambda(\bar{x})$ is a singleton.

Proof. We get $\Lambda(\bar{x}) \neq \emptyset$, as $\bar{x}$ is regular. For $\lambda_1, \lambda_2 \in \Lambda(\bar{x})$ we show $\lambda_1 = \lambda_2$. Per definition of $\Lambda(\bar{x})$ we have

$$f'(\bar{x}) + \lambda_i g'(\bar{x}) = 0, \quad \lambda_i(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})) \quad (i = 1, 2).$$

The vector $\lambda := \lambda_1 - \lambda_2$ satisfies

$$\lambda g'(\bar{x}) = 0, \quad \lambda y = 0 \text{ for all } y \in V.$$

With condition (5.12) in Definition 5.6 (note $\text{Im } g'(\bar{x}) - V = \mathbb{R}^m$) we get $\lambda = 0$ and therefore $\lambda_1 = \lambda_2$. $\square$

5.2 Specialization for the Standard Cone

Exercise 5.8. Show that if the set K in formulation 5.1 is a convex cone, we get $K(g(\bar{x})) = K + \mathbb{R} g(\bar{x})$.

$$\lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x}))$$

is in this case equivalent to

$$\lambda(-y) \geq 0 \text{ for all } y \in K, \quad \lambda g(\bar{x}) = 0. \quad (5.13)$$

The equation $\lambda g(\bar{x}) = 0$ is called complementarity conditions.

Exercise 5.9. Show that for $K(g(\bar{x})) = K + \mathbb{R} g(\bar{x})$ we get that

$$\lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x}))$$

is equivalent to

$$\lambda(-y) \geq 0 \text{ for all } y \in K, \quad \lambda g(\bar{x}) = 0.$$

Now we consider the standard problem of nonlinear optimization.

$$\begin{aligned} \min \{f(x) \mid & g_i(x) \leq 0, \ i = 1, \dots, k, \\ & g_i(x) = 0, \ i = k+1, \dots, m\} \end{aligned} \quad (5.14)$$

This problem is obtained as special case of 5.1 by setting $K = \mathbb{R}_-^k \times \{0_{m-k}\}$. Recall:

$$I(\bar{x}) := \{i \in \{1, \dots, k\} \mid g_i(\bar{x}) = 0\}$$

$$J(\bar{x}) := I(\bar{x}) \cup \{k+1, \dots, m\}$$

For the Lagrange-Multipliers $\lambda = (\lambda_1, \dots, \lambda_m)$ we get from (5.13)

$$\lambda_i \geq 0 \text{ for all } i \in I(\bar{x}), \quad \lambda_i = 0 \text{ for all } i \notin J(\bar{x}).$$

We compute

$$K(g(\bar{x})) = \{y \in \mathbb{R}^m \mid y_i \leq 0, \ i \in I(\bar{x}), \ y_i = 0, \ i = k+1, \dots, m\}$$

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and the linear subspace V in (5.12) is given by

$$V = \{y \in \mathbb{R}^m | y_i = 0 \text{ for all } i \in J(\bar{x})\}.$$

Corollary 5.10. $\bar{x}$ is normal for 5.14 if and only if the gradients

$$g'_i(\bar{x}) \text{ } i \in J(\bar{x}) \text{ are linearly independent.} \quad (5.15)$$

Theorem 5.11. Let $\bar{x}$ be a local minimum of (5.14). Then, there is $\lambda_0 \geq 0$ and $\lambda \in \mathbb{R}^m$ with $(\lambda_0, \lambda) \neq (0, 0_m)$, such that:

1. $L_x(\bar{x}, \lambda_0, \lambda) = \lambda_0 f'(\bar{x}) + \lambda g'(\bar{x}) = \lambda_0 f'(\bar{x}) + \sum_{i=1}^m \lambda_i g'_i(\bar{x}) = 0 \in \mathbb{R}^n$.
2. $\lambda_i = 0$ for $i \notin J(\bar{x})$, i.e., $i \in \{1, \dots, k\}$ with $g_i(\bar{x}) < 0$.
3. $\lambda_i \geq 0$ for $i \in I(\bar{x})$.

We have $\lambda_0 > 0$, if $\bar{x}$ regular. Then, w.l.o.g. $\lambda_0 = 1$ (divide Lagrangian by $\lambda_0 > 0$). If $\bar{x}$ is normal, then $\lambda \in \mathbb{R}^m$ is unique.

Every feasible $\bar{x}$ with multipliers (λ_0, λ) satisfying the conditions of Thm. 5.11 are called critical point. It turns out that not every critical point is a local minimum of (5.14). In particular, for problems 5.11 with equations, the required conditions for the minimization

$$\min\{f(x) | g(x) = 0\}$$

and maximization variant

$$\max\{f(x) | g(x) = 0\}$$

coincide.

Example 5.12.

$$\begin{aligned} \min \{ & f(x) = x_1 + x_2 \\ & g(x) = x_1^2 + x_2^2 - 2 = 0. \} \end{aligned}$$

Every point of S defined as

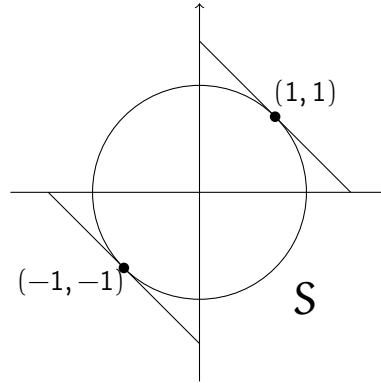
$$S := \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 - 2 = 0\}$$

is regular. The necessary KKT-conditions are

$$f'(\bar{x}) + \lambda g'(\bar{x}) = (1 + 2\lambda\bar{x}_1, 1 + 2\lambda\bar{x}_2) = 0.$$

This implies $\lambda \neq 0$. Together with $g(\bar{x}) = 0$ we get

$$\begin{aligned} \bar{y} &= (1, 1)^T, \lambda = -1/2 \\ \bar{x} &= (-1, -1)^T, \lambda = 1/2 \end{aligned}$$

Figure 5.1: Set S and critical points.

Obviously $\bar{y}$ is a local maximum, while $\bar{x}$ is a local minimum. With the [sufficient optimality conditions](#) that come up in the next section, we can show this formally.

We slightly modify the example.

Example 5.13.

$$\begin{aligned} \min \{ & f(x) = x_1 + x_2 \\ & g(x) = x_1^2 + x_2^2 - 2 \leq 0. \} \end{aligned}$$

Every point of the set S defined as

$$S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 - 2 \leq 0\}$$

is regular. The necessary KKT-conditions read as

$$f'(\bar{x}) + \lambda g'(\bar{x}) = (1 + 2\lambda\bar{x}_1, 1 + 2\lambda\bar{x}_2) = 0, \quad \lambda \geq 0, \quad \lambda g(\bar{x}) = 0.$$

We get the unique solution

$$\bar{x} = (-1, -1)^T, \quad \lambda = 1/2$$

Hence, the sign constraint $\lambda \geq 0$ sorts out the solution $\bar{y}$.

We give another example.

Example 5.14.

$$\begin{aligned} \min \{ & f(x) = -x_2 \\ & g_1(x) = x_1^2 + x_2 \leq 0 \\ & g_2(x) = -x_1^2 + x_2 \leq 0. \} \end{aligned}$$

$\bar{x} = (0, 0)^T$ is the global minimum. We get $I(\bar{x}) = \{1, 2\}$ and

$$g'_1(0, 0) = (0, 1), \quad g'_2(0, 0) = (0, 1)$$

are linearly dependent, thus, $\bar{x}$ is not normal. But $\bar{x}$ is regular, as it satisfies the

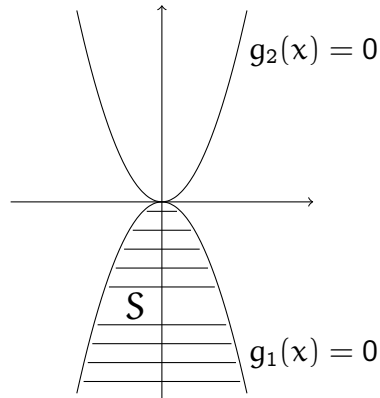


Figure 5.2: Set S and critical points.

Mangasarian–Fromovitz conditions for $\mathbf{v} = (0, -1)^\top$.

$$g_1'(0, 0)\mathbf{v} = g_2'(0, 0)\mathbf{v} = (0, 1) \cdot (0, -1)^\top = -1 < 0.$$

The necessary KKT-conditions read as

$$\mathbf{f}'(\bar{\mathbf{x}}) + \lambda \mathbf{g}'(\bar{\mathbf{x}}) = (0, -1 + \lambda_1 + \lambda_2) = 0, \quad \lambda_1 \geq 0, \lambda_2 \geq 0.$$

The set of multipliers is given by

$$\Lambda(\bar{\mathbf{x}}) = \{(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \mid \lambda_1 + \lambda_2 = 1\}.$$

This set is convex and compact in compliance with Corollary 5.5.

Chapter 6

Second-Order Necessary and Sufficient Optimality Conditions

$$\min \{f(x) \mid g(x) \in K\} \quad (6.1)$$

where we assume $f, g \in C^2$.

The Hessian matrix of f in $\bar{x}$ is denoted by

$$f''(\bar{x}) = \left(\frac{\partial^2 f(\bar{x})}{\partial x_i \partial x_j} \right)_{i,j=1,\dots,n}.$$

The Hessian matrix of the Lagrange function reads as

$$L_{xx}(\bar{x}, \lambda_0, \lambda) = \lambda_0 f''(\bar{x}) + \lambda g''(\bar{x}) = \lambda_0 f''(\bar{x}) + \sum_{i=1}^m \lambda_i g_i''(\bar{x}),$$

where the Hessian tensor of g in x is defined as

$$g''(x) := (g_1'', \dots, g_m'') \text{ with } \lambda g''(x) := \sum_{i=1}^m \lambda_i g_i''(x).$$

The linearized cone of S in $\bar{x}$ is per definition in (4.5) given as

$$L(S, \bar{x}) = \{v \in \mathbb{R}^n \mid g'(\bar{x})v \in K(g(\bar{x}))\}.$$

We now derive second order conditions using the Hessian matrix of the Lagrangian

$$\lambda_0 f''(\bar{x}) + \lambda g''(\bar{x})$$

wrt. the convex cone

$$\begin{aligned} C &:= \{v \in \mathbb{R}^n \mid f'(\bar{x})v \leq 0, \ g'(\bar{x})v \in K(g(\bar{x}))\} \\ &= \{v \in \mathbb{R}^n \mid f'(\bar{x})v \leq 0\} \cap L(S, \bar{x}). \end{aligned} \quad (6.2)$$

The set C represents the set of vectors v of the linearized cone $L(S, \bar{x})$ for which $f'(\bar{x})v \leq 0$. This is the **critical cone**; the inequality alone does not imply actual descent. At a KKT point, $f'(\bar{x})v = 0$ for every $v \in C$.

Theorem 6.1. Let $\bar{x} \in S$ and $K(g(\bar{x}))$ closed.

1. Second Order Necessary Conditions:

If $\bar{x}$ is a local minimum of (6.1), then, for every $v \in C$ there is $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m$, $(\lambda_0, \lambda) \neq 0$ with

- (a) $\lambda_0 \geq 0$, $\lambda(-y) \geq 0$ for all $y \in K(g(\bar{x}))$.
- (b) $\lambda_0 f'(\bar{x}) + \lambda g'(\bar{x}) = 0$
- (c) $v^T(\lambda_0 f''(\bar{x}) + \lambda g''(\bar{x}))v \geq 0$

2. Second Order Sufficient Conditions:

Suppose for every $v \in C \setminus \{0\}$ there exists $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m$, $(\lambda_0, \lambda) \neq 0$ with

- (a) $\lambda_0 \geq 0$, $\lambda(-y) \geq 0$ for all $y \in K(g(\bar{x}))$.
- (b) $\lambda_0 f'(\bar{x}) + \lambda g'(\bar{x}) = 0$
- (c) $v^T(\lambda_0 f''(\bar{x}) + \lambda g''(\bar{x}))v > 0$.

Then, there is $\epsilon > 0$ and a constant $c > 0$ with

$$f(x) \geq f(\bar{x}) + c \|x - \bar{x}\|^2 \text{ for all } x \in S \text{ with } \|x - \bar{x}\| \leq \epsilon. \quad (6.3)$$

In particular, $\bar{x}$ is a strict local minimum of (6.1).

3. If $\bar{x}$ is regular, then $\lambda_0 > 0$, i.e., w.l.o.g. $\lambda_0 = 1$ can be chosen in (1.) and (2.). If $\bar{x}$ is normal, then $\lambda_0 = 1$ and λ in (1.) and (2.) is unique and independent of $v \in C$.

Proof. For 1: The proof is similar to that for the first order conditions, see Lempio and Zowe (1981) for a complete proof.

For 2: We assume that (6.3) is false. Then, there is a sequence $\{x_i\} \subset S$ with

$$x_i \neq \bar{x}, \quad \lim_{i \rightarrow \infty} x_i = \bar{x}.$$

and

$$f(x_i) < f(\bar{x}) + \frac{1}{i} \|x_i - \bar{x}\|^2. \quad (6.4)$$

We set $v_i := x_i - \bar{x} \neq 0$. The boundary of the unit ball is compact and hence we can assume w.l.o.g. that

$$v := \lim_{i \rightarrow \infty} \frac{v_i}{\|v_i\|} \quad (\text{with } \|v\| = 1).$$

With (6.4) we get

$$f'(\bar{x})v = \lim_{i \rightarrow \infty} \frac{f(x_i) - f(\bar{x})}{\|x_i - \bar{x}\|} \leq 0,$$

and with the closedness of $K(g(\bar{x}))$ we get

$$g'(\bar{x})v = \lim_{i \rightarrow \infty} \frac{g(x_i) - g(\bar{x})}{\|x_i - \bar{x}\|} \in K(g(\bar{x})).$$

Per definition of the cone (6.2) we get $v \in C \setminus \{0\}$. For such v there is per assumption $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m$, $(\lambda_0, \lambda) \neq 0$, satisfying (a), (b), (c). The Taylor expansion in $\bar{x}$ of second order of f and g yields together with (6.4):

$$f'(\bar{x})v_i + \frac{1}{2}v_i^T f''(\bar{x})v_i + o(\|v_i\|^2) = f(x_i) - f(\bar{x}) \leq \frac{1}{i} \|v_i\|^2 \quad (6.5)$$

$$g'(\bar{x})v_i + \frac{1}{2}v_i^T g''(\bar{x})v_i + o(\|v_i\|^2) = g(x_i) - g(\bar{x}) \in K(g(\bar{x})). \quad (6.6)$$

Multiplying (6.5) with λ_0 and (6.6) with λ , we get via addition of both equalities and considering (a), (b) the inequality

$$\frac{1}{2}v_i^T (\lambda_0 f''(\bar{x}) + \lambda g''(\bar{x}))v_i + o(\|v_i\|^2) \leq \frac{\lambda_0}{i} \|v_i\|^2.$$

(We use in particular (a), i.e., $g(x_i) - g(\bar{x}) \in K(g(\bar{x}))$ and hence $\lambda(g(x_i) - g(\bar{x})) \leq 0$.) The division by $\|v_i\|^2$ yields in the limit $i \rightarrow \infty$ the inequality

$$\frac{1}{2}v^T (\lambda_0 f''(\bar{x}) + \lambda g''(\bar{x}))v \leq 0$$

in contradiction to (c).

For 3: If $\bar{x}$ is regular, we get $\lambda_0 > 0$ from Thm. 5.2(2.). If $\bar{x}$ is normal, we get the statement from Thm. 5.7. $\square$

The cone C in (6.2) can also be described without using f . Suppose that $\bar{x}$ is normal. If $\bar{x}$ is a local minimum, then there is a unique $\lambda \in \mathbb{R}^m$ with

$$f'(\bar{x}) + \lambda g'(\bar{x}) = 0, \quad \lambda(-y) \geq 0 \text{ for } y \in K(g(\bar{x})). \quad (6.7)$$

For $v \in C$ we get per definition

$$f'(\bar{x})v \leq 0, \quad \lambda(-y) \geq 0 \text{ for } y \in K(g(\bar{x})).$$

In conjunction with (6.7) we get

$$f'(\bar{x})v = -\lambda g'(\bar{x})v \leq 0, \quad \lambda(-g'(\bar{x})v) \geq 0.$$

Hence,

$$f'(\bar{x})v = 0 \text{ and } \lambda g'(\bar{x})v = 0 \text{ for all } v \in C.$$

Thus, C has the form

$$C = \{v \in \mathbb{R}^n \mid \lambda g'(\bar{x})v = 0, \quad g'(\bar{x})v \in K(g(\bar{x}))\}. \quad (6.8)$$

6.1 Specialization to the Standard Cone

For $K = \{0\}$ we get

$$C = L(S, \bar{x}) = \{v \in \mathbb{R}^n | g'(\bar{x})v = 0\}.$$

For the standard problem we get using $I(\bar{x})$ and $J(\bar{x}) = I(\bar{x}) \cup \{k+1, \dots, m\}$ (see Section 5) for the multiplier $\lambda \in \mathbb{R}^m$:

$$\lambda_i \geq 0 \text{ for all } i \in I(\bar{x}), \lambda_i = 0 \text{ for all } i \notin J(\bar{x}).$$

Using

$$K(g(\bar{x})) = \{y \in \mathbb{R}^m | y_i \leq 0, i \in I(\bar{x}), y_i = 0, i = k+1, \dots, m\}$$

C has the form

$$\begin{aligned} C = \{v \in \mathbb{R}^n | & g'_i(\bar{x})v \leq 0, i \in I(\bar{x}), \lambda_i = 0, \\ & g'_i(\bar{x})v = 0, i \in I(\bar{x}), \lambda_i > 0, \\ & g'_i(\bar{x})v = 0, i = k+1, \dots, m\}. \end{aligned} \quad (6.9)$$

Altogether we obtain the following conditions for problem (6.1).

Theorem 6.2. Let $\bar{x} \in S$ be normal, i.e., the gradients $g'_i(\bar{x})^\top, i \in J(\bar{x})$ are linearly independent.

1. Second-order necessary conditions:

If $\bar{x}$ is a local minimum (6.1), then there is a unique $\lambda_i \in \mathbb{R}, i \in J(\bar{x})$, with

- (a) $\lambda_i \geq 0$ for $i \in I(\bar{x})$,
- (b) $f'(\bar{x}) + \sum_{i \in J(\bar{x})} \lambda_i g'_i(\bar{x}) = 0$
- (c) $v^\top (f''(\bar{x}) + \sum_{i \in J(\bar{x})} \lambda_i g''_i(\bar{x}))v \geq 0$ for all $v \in C$ with C as in (6.9).

2. Second-order sufficient conditions:

Suppose there are $\lambda_i \in \mathbb{R}, i \in J(\bar{x})$ with

- (a) $\lambda_i \geq 0$ for $i \in I(\bar{x})$,
- (b) $f'(\bar{x}) + \sum_{i \in J(\bar{x})} \lambda_i g'_i(\bar{x}) = 0$
- (c) $v^\top (f''(\bar{x}) + \sum_{i \in J(\bar{x})} \lambda_i g''_i(\bar{x}))v > 0$ for all $v \in C \setminus \{0\}$ with C as in (6.9).

Then, there is $\epsilon > 0$ and $c > 0$ with

$$f(x) \geq f(\bar{x}) + c \|x - \bar{x}\|^2 \text{ for all } x \in S \text{ with } \|x - \bar{x}\| \leq \epsilon. \quad (6.10)$$

In particular, $\bar{x}$ is a strict local minimum of (6.1).

6.2 Examples

We revisit the example 5.12

$$\begin{aligned} \min \{ & f(x) = x_1 + x_2 \\ & g(x) = x_1^2 + x_2^2 - 2 = 0. \} \end{aligned}$$

which has the two critical points

$$\begin{aligned} \bar{x}_1 &= (1, 1)^T \text{ with } \lambda_1 = -1/2 \\ \bar{x}_2 &= (-1, -1)^T \text{ with } \lambda_2 = 1/2. \end{aligned}$$

The Hessian matrix of the Lagrangian in those points is

$$f''(\bar{x}) + \lambda g''(\bar{x}) = \lambda \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

For $\bar{x}_2$ and λ_2 we get that the matrix is pos. def. on $\mathbb{R}^2$, hence, the conditions of Thm. 6.2(2.) are satisfied with the subspace

$$C = \{v \in \mathbb{R}^2 | v_2 = -v_1\}.$$

Thus, $\bar{x}_2$ is a strict local minimum.

For $\bar{x}_1$ and λ_1 the matrix is negative definite, and hence $\bar{x}_1$ is a strict local maximum.

We give another example.

$$\begin{aligned} \min \{ & f(x) = x_1^2 + x_2 \\ & g_1(x) = x_1^2 + x_2^2 - 9 \leq 0, \\ & g_2(x) = x_1 + x_2 - 1 \leq 0. \} \end{aligned}$$

As candidate for a local minimum consider $\bar{x} = (0, -3)^T$, for which $g_1(\bar{x}) = 0$ and $g_2(\bar{x}) < 0$. Hence, $I(\bar{x}) = \{1\}$. The point $\bar{x}$ is normal and the KKT-conditions in Thm. 5.11 has with $\lambda_2 = 0$ the solution $\lambda_1 = 1/6$. The cone C in (6.9) is the subspace

$$\begin{aligned} C &= \{v \in \mathbb{R}^2 | 2\bar{x}_1 v_1 + 2\bar{x}_2 v_2 = 0\} \\ &= \{v \in \mathbb{R}^2 | v_2 = 0\} \end{aligned}$$

The Hessian matrix of the Lagrangian

$$f''(\bar{x}) + \lambda_1 g_1''(\bar{x}) = \begin{pmatrix} 2(1 + \lambda_1) & 0 \\ 0 & 2\lambda_1 \end{pmatrix}.$$

is positive definite on $\mathbb{R}^2$ and in particular

$$v^T(f''(\bar{x}) + \lambda_1 g_1''(\bar{x}))v = 2v_1^2(1 + \lambda_1) > 0$$

for all $v \in \mathbb{C} \setminus \{0\}$. With Thm. 6.2(2.), we get that $\bar{x} = (0, -3)^T$ is a strict local minimum.

Chapter 7

Sensitivity Analysis

We consider the standard problem:

$$\begin{aligned} \min \{f(x) \mid & g_i(x) \leq 0, \ i = 1, \dots, k, \\ & g_i(x) = 0, \ i = k + 1, \dots, m\} \end{aligned} \quad (7.1)$$

Suppose that the right-hand side (7.1) is perturbed:

$$\begin{aligned} \min \{f(x) \mid & g_i(x) \leq \epsilon_i, \ i = 1, \dots, k, \\ & g_i(x) = \epsilon_i, \ i = k + 1, \dots, m\} \end{aligned} \quad (7.2)$$

Problem (7.2) depends on the parameter $\epsilon := (\epsilon_1, \dots, \epsilon_m) \in \mathbb{R}^m$. For $\epsilon = 0$ the perturbed problem (7.2) becomes (7.1) which we denote as the unperturbed problem.

More generally for $\epsilon \in \mathbb{R}^p, p \geq 1$, we obtain:

$$\begin{aligned} \min \{f(x, \epsilon) \mid & g_i(x, \epsilon) \leq 0, \ i = 1, \dots, k, \\ & g_i(x, \epsilon) = 0, \ i = k + 1, \dots, m\} \end{aligned} \quad (7.3)$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^m$ are mappings with certain differentiability assumptions to be specified later and we allow even that f and g depend in a nonlinear way on ϵ .

For $K = \mathbb{R}_+^k \times \{0_{m-k}\}$, we obtain:

$$\min_{x \in \mathbb{R}^n} \{f(x, \epsilon) \mid g(x, \epsilon) \in K\}. \quad (7.4)$$

The feasible set (7.4) is given as

$$S(\epsilon) := \{x \in \mathbb{R}^n \mid g(x, \epsilon) \in K\}, \ \epsilon \in \mathbb{R}^p. \quad (7.5)$$

The optimal value function of (7.4) is denoted as:

$$\begin{aligned} w : \mathbb{R}^p &\rightarrow \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\} \text{ defined as} \\ w(\epsilon) &:= \inf_{x \in \mathbb{R}^n} \{f(x, \epsilon) | g(x, \epsilon) \in K\}, \epsilon \in \mathbb{R}^p. \end{aligned} \quad (7.6)$$

7.1 Local Sensitivity Analysis

Under which conditions can we embed a local minimum $x(0)$ of the unperturbed problem into a continuously differentiable family of perturbed local minima $x(\epsilon)$. W.l.o.g., we use the following notation:

$$\begin{aligned} I(\bar{x}) &= \{i \in \{1, \dots, k\} | g_i(\bar{x}, 0) = 0\} = \{1, \dots, k_0\}, \\ J(\bar{x}) &= \{1, \dots, k_0, k+1, \dots, m\}, \\ m_0 &:= |J(\bar{x})| = m + k_0 - k. \end{aligned} \quad (7.7)$$

Theorem 7.1. Assume that f and g are jointly C^2 in (x, ϵ) on a neighbourhood of $(\bar{x}, 0)$. Let $\bar{x} \in S(0)$ be a local minimum of (7.4). Let $\bar{x}$ be normal, i.e., the gradients $g_{ix}(\bar{x}, 0)^T$ are linearly independent for $i \in J(\bar{x})$. Assume there are uniquely determined $\bar{\lambda}_i, i \in J(\bar{x})$ such that the second order sufficient optimality conditions of Thm. 6.1, (2) are satisfied with

1. $\bar{\lambda}_i > 0$ for $i = 1, \dots, k_0$,
2. $f_x(\bar{x}, 0) + \sum_{i \in J(\bar{x})} \bar{\lambda}_i g_{ix}(\bar{x}, 0) = 0$,
3. $v^T \left(f_{xx}(\bar{x}, 0) + \sum_{i \in J(\bar{x})} \bar{\lambda}_i g_{ixx}(\bar{x}, 0) \right) v > 0$ for all $v \neq 0$ with

$$v \in C = \{v \in \mathbb{R}^n | g_{ix}(\bar{x}, 0)v = 0, i \in J(\bar{x})\}.$$

Then there is a neighbourhood $E \subset \mathbb{R}^p$ of $\epsilon = 0$ and continuously differentiable functions $x : E \rightarrow \mathbb{R}^n, \lambda_i : E \rightarrow \mathbb{R}, i \in J(\bar{x})$, with:

1. $x(0) = \bar{x}, \lambda_i(0) = \bar{\lambda}_i, i \in J(\bar{x})$,
2. for all $\epsilon \in E$: $x(\epsilon), \lambda_i(\epsilon), i \in J(\bar{x})$ satisfy the conditions of Thm. 6.1(2.) for the perturbed problem (7.4). In particular, $x(\epsilon)$ is a local minimum of (7.4).

Proof. We define

$$G : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^{m_0}, m_0 = m + k_0 - k,$$

via

$$G(x, \epsilon) = (g_1(x, \epsilon), \dots, g_{k_0}(x, \epsilon), g_{k+1}(x, \epsilon), \dots, g_m(x, \epsilon))^T.$$

Per construction, $\bar{x}$ is a strict local minimum of

$$\min\{f(x, 0) | G(x, 0) = 0\}.$$

The wanted function $x(\epsilon)$ of local minima to (7.4) should be C^1 , thus, they should satisfy

$$g_i(x(\epsilon), \epsilon) < 0 \text{ for } i \notin J(\bar{x}), \|\epsilon\| \text{ small enough.}$$

Hence, $x(\epsilon)$ should be a local minimum of the perturbed problem with equality constraints

$$\min\{f(x, \epsilon) \mid G(x, \epsilon) = 0\}. \quad (7.8)$$

The Lagrangian therefore is constructed as

$$L(x, v, \epsilon) = f(x, \epsilon) + vG(x, \epsilon), \quad v \in \mathbb{R}^{m_0}. \quad (7.9)$$

$x(\epsilon)$ and the multiplier $v(\epsilon) = (\lambda_i(\epsilon))_{i \in J(\bar{x})}$ needs to solve

$$F(x, v^T, \epsilon) := \begin{pmatrix} L_x(x, v, \epsilon)^T \\ G(x, \epsilon) \end{pmatrix} = 0, \quad (7.10)$$

where

$$F : \mathbb{R}^n \times \mathbb{R}^{m_0} \times \mathbb{R}^p \rightarrow \mathbb{R}^n \times \mathbb{R}^{m_0}.$$

Note that v^T as the argument of F appears as a column vector. For $\epsilon = 0$ we get per assumption $F(\bar{x}, \bar{v}^T, 0) = 0$ with $\bar{v} := (\bar{\lambda}_i)_{i \in J(\bar{x})}$ and F is C^1 in a neighbourhood of $(\bar{x}, \bar{v}^T, 0)$. The Jacobi-matrix of F wrt. (x, v^T) in $(\bar{x}, \bar{v}^T, 0)$ is given by the $(n + m_0) \times (n + m_0)$ matrix

$$A_0 := \frac{\partial F}{\partial (x, v^T)}(\bar{x}, \bar{v}^T, 0) = \begin{pmatrix} L_{xx}(\bar{x}, \bar{v}, 0) & G_x(\bar{x}, 0)^T \\ G_x(\bar{x}, 0) & 0 \end{pmatrix}. \quad (7.11)$$

In order to apply the implicit function theorem, we will show that A_0 is non-singular. Let $(v, w) \in \mathbb{R}^n \times \mathbb{R}^{m_0}$ with

$$A_0 \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} L_{xx}(\bar{x}, \bar{v}, 0) v + G_x(\bar{x}, 0)^T w \\ G_x(\bar{x}, 0) v \end{pmatrix} = 0. \quad (7.12)$$

Multiplying the equation with $(v, 0)^T$ from the left and using $G_x(\bar{x}, 0)v = 0$, we get

$$v^T L_{xx}(\bar{x}, \bar{v}, 0)v = 0.$$

The assumption of the Thm. yield $v = 0$. The equation (7.12) reduces to

$$G_x(\bar{x}, 0)^T w = 0.$$

Using that $\bar{x}$ is normal, the matrix $G_x(\bar{x}, 0)^T$ has rank m_0 and therefore $w = 0$. Thus, A_0 is non-singular.

We can apply the implicit function theorem on the system of equations (7.10) and get

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the existence of a neighbourhood $E \subset \mathbb{R}^p$ of $\epsilon = 0$ and C^1 functions $x : E \rightarrow \mathbb{R}^n, v(\epsilon) = (\lambda_i(\epsilon))_{i \in J(\bar{x})} : E \rightarrow \mathbb{R}^{m_0}$, with:

$$\begin{aligned} F(x(\epsilon), v(\epsilon)^T, \epsilon) &= 0 \text{ for all } \epsilon \in E, \\ x(0) &= \bar{x}, \quad v(0) = \bar{v}. \end{aligned} \tag{7.13}$$

For completing the proof, we need to verify that $x(\epsilon)$ and $(\lambda_i(\epsilon))_{i \in J(\bar{x})}$ satisfy the second order sufficient optimality conditions. Because of

$$\begin{aligned} \lambda_i(0) &= \bar{\lambda}_i > 0, \quad i = 1, \dots, k_0, \\ g_i(x(0), 0) &= g_i(\bar{x}, 0) < 0, \text{ for } i = k_0 + 1, \dots, k \end{aligned}$$

and the continuity of the functions, we can choose E small enough, such that for all $\epsilon \in E$ we get

$$\begin{aligned} \lambda_i(\epsilon) &> 0, \quad i = 1, \dots, k_0, \\ g_i(x(\epsilon), \epsilon) &< 0, \text{ for } i = k_0 + 1, \dots, k \end{aligned}$$

and hence $J(\bar{x}) = J(\bar{x})$ for all $\epsilon \in E$. With (7.10) and (7.13) we get that $x(\epsilon) \in S(\epsilon)$ for all $\epsilon \in E$. Moreover, the KKT-conditions are satisfied:

$$L_x(x(\epsilon), v(\epsilon), \epsilon) = 0 \text{ for all } \epsilon \in E.$$

Again using continuity, we can choose E small enough such that for all $\epsilon \in E$:

$$v^T \left(f_{xx}(x(\epsilon), \epsilon) + \sum_{i \in J(\bar{x})} \lambda_i(\epsilon) g_{ixx}(x(\epsilon), \epsilon) \right) v > 0$$

for all $v \neq 0$ with

$$v \in C = \{v \in \mathbb{R}^n | g_{ix}(x(\epsilon), \epsilon)v = 0, \quad i \in J(\bar{x})\}.$$

The matrix $G_x(x(\epsilon), \epsilon)$ has rank m_0 implying that $x(\epsilon)$ is normal. Altogether, $x(\epsilon)$ is a strict local minimum of the perturbed problem (7.4). $\square$

Corollary 7.2. For the functions $x : E \rightarrow \mathbb{R}^n$, $v(\epsilon) = (\lambda_i(\epsilon))_{i \in J(\bar{x})} : E \rightarrow \mathbb{R}^{m_0}$ appearing in Thm. 7.1, the following statements are true:

1. With the non-singular $(n + m_0) \times (n + m_0)$ matrix A_0 and the $(n + m_0) \times p$ matrix

$$B_0 = \begin{pmatrix} L_{x\epsilon}(\bar{x}, \bar{v}, 0) \\ G_\epsilon(\bar{x}, 0) \end{pmatrix}$$

we can compute $\dot{x}(0)$ and $\dot{v}(0)$ as

$$\begin{pmatrix} \dot{x}(0) \\ \dot{v}(0)^\top \end{pmatrix} = -A_0^{-1} B_0.$$

2. Generalized shadow-price formula:

$$\left. \frac{d}{d\epsilon} f(x(\epsilon), \epsilon) \right|_{\epsilon=0} = L_\epsilon(\bar{x}, \bar{v}, 0) = f_\epsilon(\bar{x}, 0) + \bar{v} G_\epsilon(\bar{x}, 0).$$

Here $\dot{x}(0)$ and $\dot{v}(0)^\top$ denote the Jacobians with respect to ϵ (of sizes $n \times p$ and $m_0 \times p$). The upper block $L_{x\epsilon}$ of B_0 is the derivative of the column $L_x^\top$ with respect to ϵ . The shadow-price formula describes the local value $f(x(\epsilon), \epsilon)$; it describes the global value $w(\epsilon)$ only when this branch remains globally optimal.

Proof. For (1): The differentiation of (7.13) yields with (7.10) and (7.11):

$$\left. \frac{d}{d\epsilon} F(x(\epsilon), v(\epsilon)^\top, \epsilon) \right|_{\epsilon=0} = A_0 \begin{pmatrix} \dot{x}(0) \\ \dot{v}(0)^\top \end{pmatrix} + B_0 = 0.$$

For (2): From $G(x(\epsilon), \epsilon) = 0$ we get

$$0 = \left. \frac{d}{d\epsilon} G(x(\epsilon), \epsilon) \right|_{\epsilon=0} = G_x(\bar{x}, 0) \dot{x}(0) + G_\epsilon(\bar{x}, 0).$$

Together with

$$f_x(\bar{x}, 0) = -\bar{v} G_x(\bar{x}, 0)$$

we get

$$\begin{aligned} \left. \frac{d}{d\epsilon} f(x(\epsilon), \epsilon) \right|_{\epsilon=0} &= f_x(\bar{x}, 0) \dot{x}(0) + f_\epsilon(\bar{x}, 0) \\ &= -\bar{v} G_x(\bar{x}, 0) \dot{x}(0) + f_\epsilon(\bar{x}, 0) \\ &= \bar{v} G_\epsilon(\bar{x}, 0) + f_\epsilon(\bar{x}, 0) \\ &= L_\epsilon(\bar{x}, \bar{v}, 0). \end{aligned}$$

7.2 Application to Real-Time Optimization

Part (1) of the Corollary allows to represent the solution $x(\epsilon)$ of the perturbed problem via a Taylor-expansion at the unperturbed solution :

$$\begin{pmatrix} x(\epsilon) \\ v(\epsilon)^T \end{pmatrix} \approx \begin{pmatrix} \bar{x} \\ \bar{v}^T \end{pmatrix} + \begin{pmatrix} \dot{x}(0) \\ \dot{v}(0)^T \end{pmatrix} \epsilon. \quad (7.14)$$

The formula 2. becomes for (7.2):

$$\left. \frac{d}{d\epsilon_i} f(x(\epsilon), \epsilon) \right|_{\epsilon=0} = \begin{cases} -\bar{\lambda}_i & \text{for } i \in J(\bar{x}) \\ 0, & \text{for } i \notin J(\bar{x}). \end{cases} \quad (7.15)$$

This formula shows how to interpret the Lagrange multipliers as shadow prices.

The formula (7.14) allows for an application in the area of **real-time optimization**. Suppose we compute **offline** a solution of the unperturbed problem. If the system data changes, a new solution can be computed in **real-time** (without resolving the equation system) via the approximation (7.14).

Example 7.3.

$$\begin{aligned} \min \quad & f(x, \epsilon) = -(0.5 + \epsilon)\sqrt{x_1} - (0.5 - \epsilon)x_2 \\ & x_1 + x_2 \leq 1, \\ & x_1 \geq 0.1, \\ & x_2 \geq 0. \end{aligned}$$

for $\epsilon = 0$ the assumptions of Thm. 7.1 are satisfied with

$$\begin{aligned} \bar{x} &= (0.25, 0.75), \bar{x} \text{ is normal} \\ J(\bar{x}) &= \{1\}, G_1(x_1, x_2) = x_1 + x_2 - 1 \\ \bar{v} &= \bar{\lambda}_1 = 0.5 > 0. \end{aligned}$$

The Lagrangian (7.9) is given by

$$L(\bar{x}, \bar{v}, \epsilon) = -(0.5 + \epsilon)\sqrt{\bar{x}_1} - (0.5 - \epsilon)\bar{x}_2 + \bar{v}(\bar{x}_1 + \bar{x}_2 - 1).$$

The sufficient conditions of 2. order are valid, because

$$\begin{aligned} L_{xx}(\bar{x}, \bar{v}, 0) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ C &= \{v \in \mathbb{R}^2 \mid G_x(\bar{x})v = v_1 + v_2 = 0\} \\ v^T L_{xx}(\bar{x}, \bar{v}, 0)v &= v_1^2 > 0 \text{ for all } v \in C \setminus \{0\}. \end{aligned}$$

The formula gives

$$A_0 = \begin{pmatrix} L_{xx} & G_x^T \\ G_x & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} L_{x\epsilon} \\ G_\epsilon \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{v}(0) \end{pmatrix} = -A_0^{-1}B_0 = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}.$$

Hence, the approximation of first order yields

$$\begin{pmatrix} x_1(\epsilon) \\ x_2(\epsilon) \\ v(\epsilon) \end{pmatrix} \approx \begin{pmatrix} 0.25 \\ 0.75 \\ 0.5 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \epsilon.$$

For $\epsilon = 0.05$ we get

approximation : $(0.35, 0.65, 0.45)$

exactvalue(rounded) : $(0.373, 0.627, 0.45)$.

Chapter 8

Duality

$$\begin{aligned} \min \{f(x) \mid & g_i(x) \leq 0, i = 1, \dots, k, \\ & g_i(x) = 0, i = k + 1, \dots, m\} \end{aligned} \quad (8.1)$$

with equivalent representation $\min\{f(x) \mid x \in S\}$, where $S = \{x \in \mathbb{R}^n \mid g(x) \in K\}$ and $K = \mathbb{R}_-^k \times \{0_{m-k}\}$.

Consider the Lagrangian:

$$L(x, \lambda) := f(x) + \lambda g(x), \lambda \in \mathbb{R}_+^k \times \mathbb{R}^{m-k}.$$

We define the Lagrangian-Dual:

$$\begin{aligned} \mu : \mathbb{R}^m &\rightarrow \mathbb{R} \cup \{-\infty\} \\ \mu(\lambda) &= \inf_{x \in \mathbb{R}^n} L(x, \lambda) = \inf_{x \in \mathbb{R}^n} \{f(x) + \lambda g(x)\}. \end{aligned}$$

We assume $\mu(\lambda) = -\infty$, if $L(x, \lambda)$ is not bounded from below on $\mathbb{R}^n$.

Theorem 8.1. The effective domain $R := \{\lambda \in \mathbb{R}^m : \mu(\lambda) > -\infty\}$ is convex, and μ is concave on R .

Proof. Let $\lambda_1, \lambda_2 \in R$ and let $\alpha \in [0, 1]$. We obtain:

$$\begin{aligned} \mu(\alpha\lambda_1 + (1 - \alpha)\lambda_2) &= \inf_{x \in \mathbb{R}^n} \{f(x) + (\alpha\lambda_1 + (1 - \alpha)\lambda_2)g(x)\} \\ &\geq \inf_{y \in \mathbb{R}^n} \{\alpha f(y) + \alpha\lambda_1 g(y)\} + \inf_{z \in \mathbb{R}^n} \{(1 - \alpha)f(z) + ((1 - \alpha)\lambda_2)g(z)\} \\ &= \alpha\mu(\lambda_1) + (1 - \alpha)\mu(\lambda_2). \end{aligned}$$

□

8.1 Dual Problem and Weak Duality

Let $p^* = \inf\{f(x) \mid x \in S\}$ be the optimal value of (8.1). We show that for every Lagrange multiplier, i.e., $\lambda_i \geq 0, i = 1, \dots, k$, every value of $\mu(\lambda)$ yields a lower bound on p^* .

Theorem 8.2. For $\lambda \in \mathbb{R}^m$ with $\lambda_i \geq 0, i = 1, \dots, k$ and $\bar{x} \in S$, we have:

$$\mu(\lambda) \leq f(\bar{x}). \quad (8.2)$$

In particular, $\mu(\lambda) \leq p^*$.

Proof. Let $\bar{x} \in S$, i.e. $g_i(\bar{x}) \leq 0, i = 1, \dots, k$ and $g_i(\bar{x}) = 0$ for $i = k+1, \dots, m$. Then,

$$\sum_{i=1}^k \lambda_i g_i(\bar{x}) + \sum_{i=k+1}^m \lambda_i g_i(\bar{x}) \leq 0.$$

We get

$$L(\bar{x}, \lambda) = f(\bar{x}) + \sum_{i=1}^k \lambda_i g_i(\bar{x}) + \sum_{i=k+1}^m \lambda_i g_i(\bar{x}) \leq f(\bar{x})$$

and obtain

$$\mu(\lambda) = \inf_{x \in \mathbb{R}^n} L(x, \lambda) \leq L(\bar{x}, \lambda) \leq f(\bar{x}).$$

□

For every Lagrange multiplier λ , the corresponding value $\mu(\lambda)$ yields a lower bound on p^* . The dual problem maximizes this lower bound:

$$\max\{\mu(\lambda) \mid \lambda \in \mathbb{R}^m, \lambda_i \geq 0, i = 1, \dots, k\} \quad (8.3)$$

This problem is termed **dual problem** while the original problem (8.1) is the **primal problem**.

We say that λ with $\lambda_i \geq 0, i = 1, \dots, k$ is **dual feasible**, if $\mu(\lambda) > -\infty$. We denote with λ^* the optimal Lagrange multiplier.

Remark 8.3. Problem (8.3) is a convex optimization problem, because the objective is concave and the feasible region is convex. This property is independent on whether or not the primal is convex.

Let $d^* := \sup\{\mu(\lambda) : \lambda_i \geq 0, i = 1, \dots, k\}$ be the dual optimal value. Thm. 8.2 implies weak duality:

$$d^* \leq p^*.$$

Note that weak duality also holds, if d^* and p^* are infinite. If $p^* = -\infty$, then $d^* = -\infty$ and the dual is infeasible. Otherwise, if $d^* = \infty$, we get $p^* = \infty$, hence the primal problem is infeasible. The difference $p^* - d^*$ is known as **duality gap**.

8.2 Strong Duality and Saddle Points

If

$$d^* = p^*,$$

we say that **strong duality** holds.

Theorem 8.4. Let x^* be feasible for (8.1) and λ^* feasible for (8.3). Suppose that these feasible points attain the same objective value:

$$f(x^*) = \mu(\lambda^*).$$

Then:

1. x^* and λ^* are globally optimal for (8.1) and (8.3).
2. $\lambda_i^* g_i(x^*) = 0$ for all $i = 1, \dots, m$.

Proof. For 1., weak duality gives, for every primal feasible x and every dual feasible λ ,

$$\mu(\lambda) \leq f(x^*) = \mu(\lambda^*) \leq f(x).$$

Thus both given points are globally optimal.

For 2. we observe that for all $i > k$, the feasibility of x^* yields $g_i(x^*) = 0$ and therefore $\lambda_i^* g_i(x^*) = 0$ is implied. For $i \leq k$ we consider the inequality:

$$f(x^*) = \mu(\lambda^*) \leq L(x^*, \lambda^*) = f(x^*) + \lambda^* g(x^*).$$

We get $\lambda^* g(x^*) \geq 0$. On the other hand $\lambda^* g(x^*) \leq 0$, and hence we get $\lambda^* g(x^*) = 0$. As for every $i \leq k$ we have $\lambda_i^* \geq 0$ and $g_i(x^*) \leq 0$, the claim follows for all $i \leq k$. $\square$

We define a **saddle point**.

Definition 8.5. We are given a problem of the form (8.1). Let $(\bar{x}, \bar{\lambda})$ satisfy:

1. $\bar{\lambda}_i \geq 0, i = 1, \dots, k$
2. $L(\bar{x}, \lambda) \leq L(\bar{x}, \bar{\lambda})$ for all $\lambda \in \mathbb{R}^m$ with $\lambda_i \geq 0, i = 1, \dots, k$.
3. $L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda})$ for all $x \in \mathbb{R}^n$.

Then, $(\bar{x}, \bar{\lambda})$ is called saddle point for (8.1). The conditions are called **saddle point conditions**.

Theorem 8.6 (saddle point theorem). Let $(\bar{x}, \bar{\lambda})$ be a saddle point for (8.1). Then strong duality holds for $\bar{x}$ and $\bar{\lambda}$.

Proof. Condition 2. implies

$$f(\bar{x}) + v g(\bar{x}) \leq f(\bar{x}) + \bar{\lambda} g(\bar{x}) \text{ for all } v \in \mathbb{R}^m \text{ with } v_i \geq 0, i = 1, \dots, k.$$

After basic manipulations, we get

$$(v - \bar{\lambda})g(\bar{x}) \leq 0 \text{ for all } v \in \mathbb{R}^m \text{ with } v_i \geq 0, i = 1, \dots, k. \quad (8.4)$$

Inserting $v_i = \bar{\lambda}_i$ for $i = 1, \dots, k$, we get

$$\sum_{i=k+1}^m (v_i - \bar{\lambda}_i)g_i(\bar{x}) \leq 0 \text{ for all } v_i \in \mathbb{R}, i = k+1, \dots, m. \quad (8.5)$$

This implies $g_i(\bar{x}) = 0$ for all $i = k+1, \dots, m$. Hence, inequality (8.4) reduces to

$$\sum_{i=1}^k (v_i - \bar{\lambda}_i)g_i(\bar{x}) \leq 0 \text{ for all } v_i \geq 0, i = 1, \dots, k. \quad (8.6)$$

This condition implies $g_i(\bar{x}) \leq 0$ for all $i = 1, \dots, k$. Thus, $\bar{x} \in S$. Moreover, (8.6) yields

$$\bar{\lambda}_i g_i(\bar{x}) = 0, \text{ for all } i = 1, \dots, k.$$

Exercise 8.7. Show that (8.6) implies the following:

1. $g_i(\bar{x}) \leq 0$ for all $i = 1, \dots, k$.
2. $\bar{\lambda}_i g_i(\bar{x}) = 0$, for all $i = 1, \dots, k$.

Altogether, we obtain

$$f(\bar{x}) = f(\bar{x}) + \sum_{i=1}^k \bar{\lambda}_i g_i(\bar{x}) = f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}) = L(\bar{x}, \bar{\lambda}).$$

The condition 3. yields

$$L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}) \text{ for all } x \in \mathbb{R}^n.$$

Hence,

$$f(\bar{x}) = L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}) \text{ for all } x \in \mathbb{R}^n,$$

which in turn implies

$$f(\bar{x}) \leq \inf_{x \in \mathbb{R}^n} L(x, \bar{\lambda}) = \mu(\bar{\lambda}).$$

With weak duality, we get $f(\bar{x}) = \mu(\bar{\lambda})$. □

Exercise 8.8. Show that for any saddle point $(\bar{x}, \bar{\lambda})$ of (8.1) the KKT-conditions of Thm. 5.2 are satisfied.

8.3 Strong Duality for Convex Problems

We consider a [convex optimization problem](#):

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \quad i = 1, \dots, k \\ & Ax = b, \\ & x \in S, \end{aligned} \tag{8.7}$$

where $S \subset \mathbb{R}^n$ convex, A being a $(m - k) \times n$ matrix of full rank and f and g_i convex for $i = 1, \dots, k$. We assume that the following regularity conditions are satisfied ([Slater](#)): there is $\bar{x} \in \text{int}(S)$ with

$$g_i(\bar{x}) < 0, \quad i = 1, \dots, k, \quad A\bar{x} = b.$$

For this problem the set S remains part of the minimization domain:

$$\mu(\lambda, \nu) := \inf_{x \in S} \{f(x) + \lambda^T g(x) + \nu^T (Ax - b)\}, \quad \lambda \geq 0, \quad \nu \in \mathbb{R}^{m-k}.$$

Here full rank of A means full [row](#) rank.

Theorem 8.9 (Strong Duality). Consider (8.7) and assume that the Slater-regularity conditions are satisfied. Then, $d^* = p^*$.

Proof. The Slater point makes the primal feasible. If $p^* = -\infty$, weak duality gives $d^* = -\infty$. We therefore assume $p^* \in \mathbb{R}$. We define

$$\begin{aligned} A_1 &\subseteq \mathbb{R}^k \times \mathbb{R}^{m-k} \times \mathbb{R} \\ A_1 &= \{(u, v, t) \mid \exists x \in S, g_i(x) \leq u_i, i = 1, \dots, k, A_j x - b_j = v_j, j = 1, \dots, m - k, f(x) \leq t\} \end{aligned}$$

Note that A_1 is convex. We define the convex set:

$$A_2 = \{(0, 0, s) \in \mathbb{R}^k \times \mathbb{R}^{m-k} \times \mathbb{R} \mid s < p^*\}.$$

We get that A_1 and A_2 do not intersect: Assume by contradiction $(u, v, t) \in A_1 \cap A_2$. With $(u, v, t) \in A_2$ we get $u = 0, v = 0$, and $t < p^*$. Because $(u, v, t) \in A_1$, we get x with $g_i(x) \leq 0, i = 1, \dots, k, Ax - b = 0$, and $f(x) \leq t < p^*$, in contradiction to the optimality of p^* .

With the separating hyperplane theorem we get $(w_1, w_2, w_3) \neq 0$ and $\alpha \in \mathbb{R}$ with

$$w_1^T u + w_2^T v + w_3 t \geq \alpha \text{ for all } (u, v, t) \in A_1, \tag{8.8}$$

and

$$w_1^T u + w_2^T v + w_3 t \leq \alpha \text{ for all } (u, v, t) \in A_2. \tag{8.9}$$

From (8.8) follows $w_1 \geq 0$ and $w_3 \geq 0$, as otherwise $w_1^T u + w_3 t$ is unbounded from below

on A_1 , in contradiction to (8.8). Condition (8.9) implies $w_3 t \leq \alpha$ for all $t \leq p^*$ and hence $w_3 p^* \leq \alpha$. Together with (8.8) we get for all $x \in S$ (choose $g(x) = u$ and $Ax - b = v$) that

$$\sum_{i=1}^k (w_1)_i g_i(x) + w_2^T(Ax - b) + w_3 f(x) \geq \alpha \geq w_3 p^*. \quad (8.10)$$

We argue now by a case distinction: If $w_3 > 0$, we divide (8.10) by w_3 and obtain

$$L(x, w_1/w_3, w_2/w_3) \geq p^*,$$

for all $x \in S$. We get $\mu(\lambda, v) \geq p^*$, where $\lambda = w_1/w_3$ and $v = w_2/w_3$. (Here $\lambda \in \mathbb{R}^k$ and $v \in \mathbb{R}^{m-k}$ are the corresponding multipliers for $g(x) \leq 0$ and $Ax = b$). With weak duality $\mu(\lambda, v) \leq p^*$ we get equality.

Now, we consider the case $w_3 = 0$ and lead this to a contradiction. From (8.10), we get for all $x \in S$:

$$\sum_{i=1}^k (w_1)_i g_i(x) + w_2^T(Ax - b) \geq 0. \quad (8.11)$$

We apply this to the point $\bar{x}$ satisfying the Slater-conditions and get

$$\sum_{i=1}^k (w_1)_i g_i(\bar{x}) \geq 0.$$

As $g_i(\bar{x}) < 0$ and $w_1 \geq 0$ we get $w_1 = 0$.

From $(w_1, w_2, w_3) \neq 0$ and $w_1 = 0, w_3 = 0$, we get $w_2 \neq 0$. Hence, (8.11) implies $w_2^T(Ax - b) \geq 0$ for all $x \in S$. The point $\bar{x}$ satisfies $w_2^T(A\bar{x} - b) = 0$. With $\bar{x} \in \text{int}(S)$ we get that $\bar{x} \pm \epsilon \in \text{int}(S)$ for $\epsilon \in \mathbb{R}^n$ with $\|\epsilon\|$ small enough. We get $A^T w_2 = 0$, as otherwise there are points in $\text{int}(S)$ with $w_2^T(Ax - b) < 0$, contradiction. Conditions $A^T w_2 = 0$ and $w_2 \neq 0$ contradict the assumption of A having full rank. $\square$

Chapter 9

Numerical Methods

We first recap the definition of the **matrix spectral norm**:

$$\|A\| := \max \left\{ \|Ax\| : x \in \mathbb{R}^n, \|x\| = 1 \right\}$$

This norm can be characterized by

$$\|A\| = \sqrt{\lambda_{\max}(A^T A)},$$

where $\lambda_{\max}(A^T A)$ denotes the largest eigenvalue of the symmetric and positive semidefinite matrix $A^T A$. In particular, for a symmetric matrix $A \in \mathbb{R}^{n \times n}$ we obtain

$$\|A\| = \max_i |\lambda_i(A)|.$$

9.1 Unconstrained Optimization Problems

We first consider

$$\min\{f(x) \mid x \in \mathbb{R}^n\} \tag{9.1}$$

without constraints, where $f \in C^2$ on $\mathbb{R}^n$.

Definition 9.1. $v \in \mathbb{R}^n$ is called **descent direction** of f in $\bar{x} \in \mathbb{R}^n$, if there is $\bar{t} > 0$ with

$$f(\bar{x} + tv) < f(\bar{x}) \text{ for all } t \in (0, \bar{t}).$$

We obtain the following Lemma.

Lemma 9.2. If $f'(\bar{x})v < 0$, then v is a descent direction of f in $\bar{x}$.

Proof. Define $\Phi(t) := f(\bar{x} + tv)$. We get $\Phi'(0) = f'(\bar{x})v < 0$ and the claim follows. $\square$

Remark 9.3.

- The condition $f'(\bar{x})v < 0$ means that the angle ϕ between v and $-f'(\bar{x})$ in $\bar{x}$ is less

than $\pi/2$ (or 90°). Consider:

$$0 > f'(\bar{x})v \Rightarrow 0 < -f'(\bar{x})v = \cos(\phi) \| -f'(\bar{x}) \| \|v\|.$$

We get $\cos(\phi) > 0$ and hence $\phi \in [0, \pi/2)$.

- The criterion $f'(\bar{x})v < 0$ is not necessary. If $\bar{x}$ is a strict local maximum, then all $v \in \mathbb{R}^n \setminus \{0\}$ are directions of descent for f in $\bar{x}$, but $f'(\bar{x})v < 0$ need not be satisfied.

Exercise 9.4. Let $B \in \mathbb{R}^{n \times n}$ be symmetric and positive definite. Then, $v = -B f'(\bar{x})^\top$ is a descent direction of f in $\bar{x}$, if $f'(\bar{x}) \neq 0$.

We describe a **descent method** to compute $\bar{x}$ with $f'(\bar{x}) = 0$.

1. Choose $x^0 \in \mathbb{R}^n$, $k = 0$ and fix $\epsilon_0 > 0$;
2. If $\|f'(x^k)\| \leq \epsilon_0$:STOP (Termination);
3. Compute a descent direction v^k with $f'(x^k)v^k < 0$;
4. Compute a **step-length** t_k with $f(x^k + t_k v^k) < f(x^k)$;
5. Set $x^{k+1} \leftarrow x^k + t_k v^k$; $k \leftarrow k + 1$ and go to step 2.

Figure 9.1: Generic Method of Descent.

Definition 9.5 (Gradient-Method, Newton-Method, Quasi-Newton-Method).

1. For

$$v^k := -f'(x^k)^\top$$

we obtain the **Gradient-Method**.

2. For the **Newton-Method**, we choose:

$$v^k := -f''(x^k)^{-1} f'(x^k)^\top.$$

In every point x^k in which the Hessian matrix $f''(x^k)$ is positive definite, the vector v^k is a descent direction (assuming $f'(x^k) \neq 0$).

3. The **Quasi-Newton-Method** chooses

$$v^k := -H_k^{-1} f'(x^k)^\top,$$

for a suitable positive definite matrix H_k .

Theorem 9.6. Let $H \in \mathbb{R}^{n \times n}$ be symmetric positive definite and $f'(x) \neq 0$. The direction

$$v_* = -\frac{H^{-1}f'(x)^\top}{\sqrt{f'(x)H^{-1}f'(x)^\top}}$$

minimizes $f'(x)v$ over $\|v\|_H = 1$, where $\|v\|_H = \sqrt{v^\top H v}$.

Proof. The Cauchy–Schwarz inequality gives

$$f'(x)v = (H^{-1/2}f'(x)^\top)^\top H^{1/2}v \geq -\sqrt{f'(x)H^{-1}f'(x)^\top} \|v\|_H.$$

Equality holds for $v = v_*$. □

We go back to the angle-condition discussed before.

Definition 9.7 (Angle-Condition). A generic descent method of the form (9.1) satisfies the [angle-condition](#), if:

$$\text{there is } c > 0, \text{ such that for all } k \in \mathbb{N} \text{ we have: } c_k := -\frac{f'(x^k)v^k}{\|f'(x^k)\| \|v^k\|} \geq c. \quad (9.2)$$

The weaker condition $\sum_{k=0}^{\infty} c_k^2 = \infty$, together with efficient step lengths and objective values bounded below, implies only $\liminf_k \|f'(x^k)\| = 0$. The proof below establishes the associated Zoutendijk summability estimate $\sum_k c_k^2 \|f'(x^k)\|^2 < \infty$.

Choice of the Step-Length

We discuss the degree of freedom regarding the step-length choice in 9.1. Let x^0 be the initial point and assume that $N(f, f(x^0)) = \{x \in \mathbb{R}^n | f(x) \leq f(x^0)\}$ is compact. With the continuity of $f''(x)$ on $N(f, f(x^0))$, there is $C > 0$ with

$$\|f''(x)\| \leq C \text{ for all } x \in N(f, f(x^0)).$$

We apply Taylor expansion of f in x in direction of $tv, v \in \mathbb{R}^n$:

$$\begin{aligned} f(x + tv) &= f(x) + tf'(x)v + \frac{t^2}{2}v^\top f''(z)v \\ &\leq f(x) + tf'(x)v + \frac{t^2}{2}C\|v\|^2. \end{aligned} \quad (9.3)$$

Here $z = x + \xi tv$ is an intermediate point with $0 < \xi < 1$. The bound (9.3) is valid for all $t > 0$ with $x + [0, t]v \subset N(f, f(x^0))$.

The last term of (9.3) is a polynomial $p(t)$ of degree two in t and attains at

$$t^* = -\frac{f'(x)v}{C\|v\|^2} > 0$$

its strict global minimum. Let $\bar{t}$ be the unique maximal step-length with

$$x + tv \in N(f, f(x^0)) \text{ for all } t \in [0, \bar{t}].$$

We get

$$p(\bar{t}) \geq f(x + \bar{t}v) \geq f(x) = p(0).$$

Using $p'(0) = f'(x)v < 0$ we get that t^* lies in $(0, \bar{t})$ and thus $x + t^*v \in N(f, f(x^0))$.

With (9.3) we get

$$f(x + t^*v) \leq p(t^*) = f(x) - \frac{1}{2C} \left(\frac{f'(x)v}{\|v\|} \right)^2.$$

This bounds the minimum descent. Since C is not known a priori, we define the following.

Definition 9.8. A step-length strategy $t(x, v)$ is called **efficient**, if for every $x^0 \in \mathbb{R}^n$, there is $\xi > 0$ with

$$f(x + t(x, v)v) \leq f(x) - \xi \left(\frac{f'(x)v}{\|v\|} \right)^2 \text{ for all } x \in N(f, f(x^0)),$$

and v is a descent direction of f in x with $f'(x)v < 0$.

Under the assumptions we get that $t \in \arg \min\{f(x + tv) | t > 0\}$ is efficient.

We obtain the following theorem on the general descent method (9.1) with efficient step-length strategies.

Theorem 9.9. Let $f \in C^2$ and let (9.1) with $\epsilon_0 = 0$ satisfy the condition (9.2). Suppose we choose an efficient step-length strategy. Then, one of the following statements is true:

1. After finitely many iterations we have $f'(x^k) = 0$.
2. $\lim_{k \rightarrow \infty} f(x^k) = -\infty$
3. $\lim_{k \rightarrow \infty} f'(x^k) = 0$, i.e., every accumulation point of x^k , $k \in \mathbb{N}$ is a zero of $f'(x)$.

Proof. If 9.1 terminates after finitely many iterations, we get using $\epsilon_0 = 0$ the condition $f'(x^k) = 0$.

So suppose that this does not hold. Using the angle- and efficiency condition we get for iteration k :

$$f(x^{k+1}) - f(x^k) \leq -\xi \left(\frac{f'(x^k)v^k}{\|v^k\|} \right)^2 = -\xi c_k^2 \|f'(x^k)\|^2.$$

After $N \in \mathbb{N}$ iterations we get

$$f(x^N) - f(x^0) = \sum_{k=0}^{N-1} (f(x^{k+1}) - f(x^k)) \leq -\xi \sum_{k=0}^{N-1} c_k^2 \|f'(x^k)\|^2.$$

We divide by $-\xi < 0$ and get

$$-\frac{f(x^N) - f(x^0)}{\xi} \geq \sum_{k=0}^{N-1} c_k^2 \|f'(x^k)\|^2.$$

If the decreasing sequence $f(x^k)$ does not tend to $-\infty$, it has a finite lower bound and we get

$$\lim_{N \rightarrow \infty} f(x^N) > -\infty$$

and therefore

$$\lim_{N \rightarrow \infty} -\frac{f(x^N) - f(x^0)}{\xi} < \infty.$$

Hence we get

$$\sum_{k=0}^{\infty} c_k^2 \|f'(x^k)\|^2 < \infty.$$

Using the angle- and efficiency condition, we get $\|f'(x^k)\| \rightarrow 0$.

□

We provide two additional step-length methods, the [Armijo-Rule](#) and the [Goldstein-Rule](#).

Definition 9.10 (Armijo-Rule). For $\sigma \in (0, 1)$, $\alpha \in (0, 1)$ we choose $t := \alpha^\ell$ with

$$\ell := \min\{j \in \mathbb{N}_0 \mid f(x + \alpha^j v) \leq f(x) + \sigma \alpha^j (f'(x)v)\}.$$

As for the interpretation of the Armijo-Rule, we define for

$$\Phi(t) = f(x + tv), t \geq 0$$

the auxiliary function

$$\Psi(t) = \Phi(t) - (f(x) + \sigma t (f'(x)v)).$$

Per construction, we get $\Psi(0) = 0$ and

$$\begin{aligned} \Psi'(0) &= \Phi'(0) - \sigma f'(x)v \\ &= f'(x)v - \sigma f'(x)v = (1 - \sigma)f'(x)v < 0. \end{aligned}$$

Since $\Psi'(0) < 0$, the Armijo condition holds for all sufficiently small positive t . Thus the minimum index ℓ exists, without any compactness assumption. Test $\ell = 0, 1, 2, \dots$ in this order.

Remark 9.11. A fixed initial trial step (even a large constant) does not ensure efficiency for arbitrarily scaled directions. A direction-dependent initial trial step is

$$s(x, v) = \eta \frac{-f'(x)v}{\|v\|^2}, \quad \eta > 0,$$

followed by $t = s(x, v)\alpha^\ell$ and the usual Armijo test. If $C > 0$ bounds the Hessian norm on all trial segments, Taylor's formula shows that this strategy is efficient with

$$\xi = \sigma \min\{\eta, 2\alpha(1 - \sigma)/C\} > 0.$$

Indeed, either the first trial is accepted, or the preceding rejected trial implies $t > 2\alpha(1 - \sigma)(-f'(x)v)/(C\|v\|^2)$. For a compact initial level set such a uniform Hessian bound exists: $s(x, v)\|v\| \leq \eta\|f'(x)\|$ bounds every trial segment in a compact set.

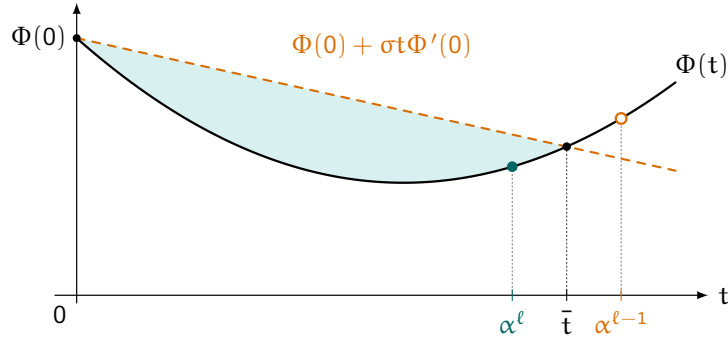


Figure 9.2: Armijo backtracking: the intersection projects vertically to $\bar{t}$. Here $\alpha^\ell < \bar{t} < \alpha^{\ell-1}$: the teal trial point is accepted, the open orange one rejected.

Definition 9.12 (Goldstein-Rule). In order to avoid small step-length, we bound the feasible space from below. The step-length $t > 0$ satisfies the Goldstein-Condition, if for fixed $\sigma \in (0, 0.5)$, we have

$$\Phi_u(t) \leq \Phi(t) \leq \Phi_o(t), \quad (9.4)$$

where $\Phi_u(t)$ and $\Phi_o(t)$ are defined as follows:

$$\Phi_o(t) := \Phi(0) + \sigma t \Phi'(0), \Phi_u(t) := \Phi(0) + (1 - \sigma)t \Phi'(0). \quad (9.5)$$

Any step-length method satisfying the Goldstein-Rule is efficient. We provide a concrete implementation.

Theorem 9.13. Let $f \in C^2(\mathbb{R}^n)$ and let $N_0 := \{y : f(y) \leq f(x^0)\}$ be compact. Choose $C > 0$ with $\|f''(y)\| \leq C$ for all $y \in N_0$. For $x \in N_0$, $f'(x)v < 0$, and any Goldstein step t with $0 < \sigma < 1/2$,

$$f(x + tv) \leq f(x) - \frac{\sigma}{C} \left(\frac{f'(x)v}{\|v\|} \right)^2.$$

Proof. Put $\alpha = -\Phi'(0) > 0$, where $\Phi(t) = f(x + tv)$, and let $t_* > 0$ be the first zero of Φ' . Such a zero exists by compactness of N_0 . The function Φ decreases on $[0, t_*]$, so this entire segment stays in N_0 and $|\Phi''(s)| \leq C\|v\|^2$ there.

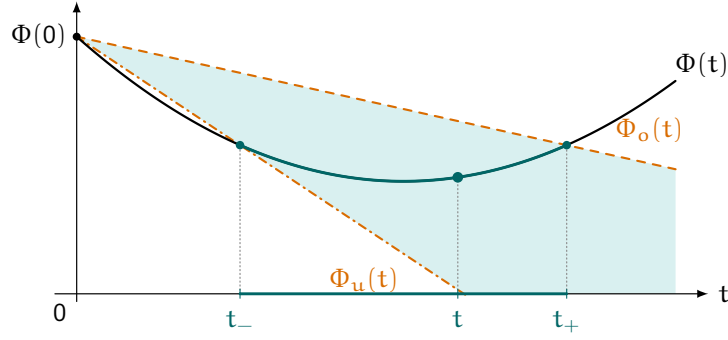


Figure 9.3: Goldstein step lengths: positive steps satisfy both bounds precisely for $t \in [t_-, t_+]$, highlighted in teal.

1. Set $t_u := 0$, $t_o > 0$ (arbitrary);
2. If $\Phi(t_o) < \Phi_u(t_o)$, set $t_u := t_o$, $t_o := 2t_o$.
Repeat until $\Phi(t_o) \geq \Phi_u(t_o)$;
3. If $\Phi(t_o) \leq \Phi_o(t_o)$, set $t := t_o$, Stop.;
4. Repeat: set $t := (t_u + t_o)/2$
If $\Phi(t) < \Phi_u(t)$, set $t_u := t$;
If $\Phi(t) > \Phi_o(t)$, set $t_o := t$;
Until: $\Phi_u(t) \leq \Phi(t) \leq \Phi_o(t)$, Stop.

Figure 9.4: Step-length choice after Goldstein.

If $t \leq t_*$, the lower Goldstein inequality and Taylor's formula give

$$-(1 - \sigma)at \leq \Phi(t) - \Phi(0) \leq -at + \frac{1}{2}Ct^2\|v\|^2.$$

Thus $t \geq \hat{t} := 2\sigma a / (C\|v\|^2)$. Monotonicity and Taylor's formula at $\hat{t}$ imply

$$\Phi(t) \leq \Phi(\hat{t}) \leq \Phi(0) - \frac{2\sigma(1 - \sigma)a^2}{C\|v\|^2} \leq \Phi(0) - \frac{\sigma a^2}{C\|v\|^2}.$$

If $t > t_*$, the mean value theorem gives $a = \Phi'(t_*) - \Phi'(0) \leq Ct_*\|v\|^2$. The upper Goldstein inequality then yields

$$\Phi(t) \leq \Phi(0) - \sigma at \leq \Phi(0) - \sigma at_* \leq \Phi(0) - \frac{\sigma a^2}{C\|v\|^2}.$$

□

9.2 Penalty Methods for Constrained Problems

We consider first the equality constrained problem:

$$\min\{f(x) \mid h(x) = 0, x \in \mathbb{R}^n\} \quad (9.6)$$

where $f \in C^1$ on $\mathbb{R}^n$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $h \in C^1$. The key idea of penalty approaches is to reduce (9.6) to an unconstrained problem with a changed objective that penalizes the violation of the constraint $h(x) = 0$. An example is the following penalty function

$$P(x; \alpha) := f(x) + \frac{\alpha}{2} \|h(x)\|^2.$$

1. Choose $\alpha_0 > 0$ and $k := 0$;

2. Compute optimal solution x^k to

$$\min P(x; \alpha_k) \text{ s.t. } x \in \mathbb{R}^n.$$

3. If $\|h(x^k)\| = 0$:STOP (Termination);

4. Choose $\alpha_{k+1} = \rho \alpha_k$ with a fixed $\rho > 1$; $k \leftarrow k + 1$ and go to step 2.

Figure 9.5: Penalty Method.

This is an idealized method: the subproblems are assumed to have global minimizers, and these minimizers must be computed.

Theorem 9.14. Let f, h be continuous functions, let $\{\alpha_k\}$ be a strictly increasing sequence with $\alpha_k \rightarrow \infty$, let the feasible set

$$S := \{x \in \mathbb{R}^n \mid h(x) = 0\}$$

be nonempty, and let $\{x^k\}$ be a sequence generated by Algorithm 9.5 (in particular, this sequence is assumed to exist). Then the following statements hold:

1. The sequence $\{P(x^k; \alpha_k)\}$ is monotonically increasing.
2. The sequence $\{\|h(x^k)\|\}$ is monotonically decreasing.
3. The sequence $\{f(x^k)\}$ is monotonically increasing.
4. $\lim_{k \rightarrow \infty} h(x^k) = 0$.
5. Every accumulation point of the sequence $\{x^k\}$ is a solution of problem (9.6).

Proof. 1: Since $\alpha_k < \alpha_{k+1}$, we have

$$P(x^k; \alpha_k) \leq P(x^{k+1}; \alpha_k) \leq P(x^{k+1}; \alpha_{k+1}),$$

which already proves statement 1:.

2: From

$$P(x^k; \alpha_k) \leq P(x^{k+1}; \alpha_k) \quad \text{and} \quad P(x^{k+1}; \alpha_{k+1}) \leq P(x^k; \alpha_{k+1}),$$

addition yields

$$P(x^k; \alpha_k) + P(x^{k+1}; \alpha_{k+1}) \leq P(x^{k+1}; \alpha_k) + P(x^k; \alpha_{k+1}).$$

This implies

$$\alpha_k \|h(x^k)\|^2 + \alpha_{k+1} \|h(x^{k+1})\|^2 \leq \alpha_k \|h(x^{k+1})\|^2 + \alpha_{k+1} \|h(x^k)\|^2,$$

and hence

$$(\alpha_k - \alpha_{k+1}) \left(\|h(x^k)\|^2 - \|h(x^{k+1})\|^2 \right) \leq 0.$$

Since $\alpha_k < \alpha_{k+1}$, it follows that

$$\|h(x^k)\| \geq \|h(x^{k+1})\| \quad \text{for all } k \in \mathbb{N}.$$

3.: From $P(x^k; \alpha_k) \leq P(x^{k+1}; \alpha_k)$ and part 2., it follows that

$$f(x^k) \leq f(x^{k+1}),$$

which proves statement 3.

4.: Since $S \neq \emptyset$, we obtain

$$P(x^k; \alpha_k) \leq \inf_{x \in S} P(x; \alpha_k) = \inf_{x \in S} f(x) < +\infty. \quad (9.7)$$

Because $\alpha_k \rightarrow \infty$ and $f(x^k) \geq f(x^0)$ by 3., it already follows that

$$\lim_{k \rightarrow \infty} \|h(x^k)\| = 0,$$

using that $\lim_{k \rightarrow \infty} P(x^k; \alpha_k) < +\infty$.

5.: Let x^* be an accumulation point of the sequence $\{x^k\}$ and let $\{x^k\}_{k \in L}$ be a subsequence converging to x^* . By 4. we have $h(x^*) = 0$, so x^* is at least a feasible point of (9.6). From (9.7) we further obtain

$$f(x^*) = \lim_{k \in L} f(x^k) \leq \lim_{k \in L} P(x^k; \alpha_k) \leq \inf_{x \in S} f(x).$$

Thus statement 5. is also proven. $\square$

We note that Theorem 9.14 does not require any differentiability assumptions on the

functions f and h , and the feasible set S does not need to satisfy any regularity condition. Only the continuity of f and h is required. Therefore, Theorem 9.14 also remains valid for the more general optimization problem:

$$\min f(x) \quad \text{s.t.} \quad h(x) = 0, g(x) \leq 0$$

since this can obviously be reformulated in the form

$$\min f(x) \quad \text{s.t.} \quad h(x) = 0, \max\{0, g(x)\} = 0$$

with

$$\max\{0, g(x)\} := (\max\{0, g_1(x)\}, \dots, \max\{0, g_k(x)\})^T \in \mathbb{R}^k.$$

The corresponding penalty function is then

$$P(x; \alpha) := f(x) + \frac{\alpha}{2} \|h(x)\|^2 + \frac{\alpha}{2} \sum_{i=1}^k \left(\max\{0, g_i(x)\} \right)^2.$$

In the following, we return to the equality-constrained problem (9.6) and investigate how we can construct an associated sequence of Lagrange multipliers $\{\mu^k\}$ from the sequence $\{x^k\}$ generated by the penalty method, so that the pair $\{(x^k, \mu^k)\}$ converges to a KKT point (x^*, μ^*) of (9.6). For this purpose, we assume f and h to be continuously differentiable. Since x^k is by construction a global minimum of $P(\cdot; \alpha_k)$, we have in particular

$$0 = P_x(x^k; \alpha_k) = f'(x^k) + \alpha_k \sum_{j=1}^m h_j(x^k) h_j'(x^k). \quad (9.8)$$

On the other hand, at a KKT point we have

$$0 = f'(x^*) + \sum_{j=1}^m \mu_j^* h_j'(x^*).$$

This suggests interpreting the values

$$\mu_j^k := \alpha_k h_j(x^k) \quad (j = 1, \dots, m) \quad (9.9)$$

as approximations for the Lagrange multipliers μ_j^* ($j = 1, \dots, m$). Indeed, the following result holds true.

Theorem 9.15. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be C^1 , $\{x^k\}$ a sequence generated by the penalty method with $\lim_{k \rightarrow \infty} x^k = x^*$, the gradients $h_1'(x^*), \dots, h_m'(x^*)$ linearly independent, and $\{\mu^k\}$ the sequence defined according to (9.9). Then:

1. The sequence $\{\mu^k\}$ converges to a vector $\mu^* \in \mathbb{R}^m$.
2. The pair (x^*, μ^*) with μ^* from 1. is a KKT point of (9.6), i.e., μ^* is the uniquely determined Lagrange multiplier corresponding to the solution x^* of (9.6).

Proof. (1): Let $A_k := h'(x^k) \in \mathbb{R}^{m \times n}$ and $A_* := h'(x^*) \in \mathbb{R}^{m \times n}$ be the Jacobian matrices of h at x^k and x^* , respectively. By continuity, $A_k \rightarrow A_*$. By assumption, the matrix $A_* A_*^T \in \mathbb{R}^{m \times m}$ is non-singular. For sufficiently large k , $A_k A_k^T$ is therefore also non-singular, and we have

$$(A_k A_k^T)^{-1} \rightarrow (A_* A_*^T)^{-1}.$$

With (9.8) and (9.9), we have

$$f'(x^k) + \sum_{j=1}^m \mu_j^k h_j'(x^k) = 0,$$

thus

$$A_k^T \mu^k = -f'(x^k)^T.$$

Multiplying from the left by A_k and solving for μ^k yields

$$\mu^k = -(A_k A_k^T)^{-1} A_k f'(x^k)^T.$$

Based on our preliminary considerations, the right-hand side converges, i.e., the sequence $\{\mu^k\}$ converges to some μ^* with

$$\mu^* = -(A_* A_*^T)^{-1} A_* f'(x^*)^T.$$

(2): This assertion follows immediately from (9.8) and (9.9) by taking the limit $k \rightarrow \infty$, since from part (1) we already know that the sequence $\{\mu^k\}$ also converges. $\square$

9.3 Lagrange-Newton Method

We recap the Newton-Method. Given is a C^1 function

$$F : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

and we search for

$$F(x) = 0.$$

The linear approximation in x^k is defined as

$$F_k(x) := F(x^k) + F'(x^k)(x - x^k).$$

Hence,

$$x^{k+1} = x^k - F'(x^k)^{-1} F(x^k),$$

if the inverse F'^{-1} exists. In a concrete implementation, we don't compute the inverse but a direction vector $v = x - x^k$ solving the Newton-Equation:

$$F'(x^k)v = -F(x^k).$$

For a solution v^k we set

$$x^{k+1} := x^k + v^k.$$

1. Choose $x^0 \in \mathbb{R}^n$ and $k := 0$;
2. If $F(x^k) = 0$:STOP (Termination);
3. Compute v^k as a solution to the equation system

$$F'(x^k)v = -F(x^k). \quad (9.10)$$

4. set $x^{k+1} = x^k + v^k$, $k \leftarrow k + 1$ and go to step 2.

Figure 9.6: Newton's Method.

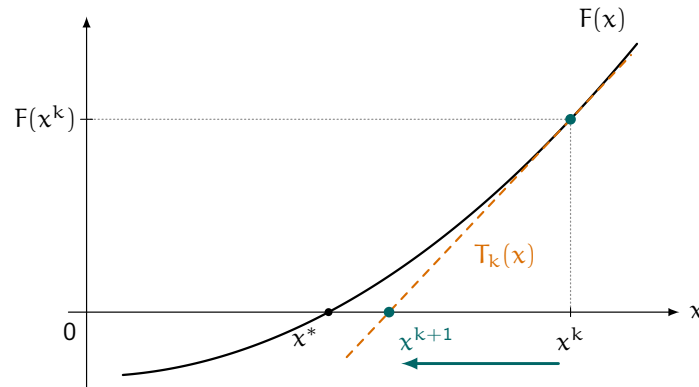


Figure 9.7: Newton step: the tangent $T_k(x) = F(x^k) + F'(x^k)(x - x^k)$ intersects the horizontal axis at x^{k+1} .

We now derive the so-called **perturbation lemma** which is heavily used in numerical linear algebra. We start with a preparatory lemma.

Lemma 9.16. Let $M \in M(n \times n, \mathbb{R})$ with $\|M\| < 1$. Then $I - M$ is regular, and the estimate

$$\|(I - M)^{-1}\| \leq \frac{1}{1 - \|M\|}$$

holds.

Proof. For every $x \in \mathbb{R}^n$ we have

$$\|(I - M)x\| = \|x - Mx\| \geq \|x\| - \|Mx\| \geq (1 - \|M\|) \|x\|. \quad (9.11)$$

Since $\|M\| < 1$, we have $1 - \|M\| > 0$, and thus from (9.11) it follows immediately that $(I - M)x \neq 0$ for $x \neq 0$, i.e., the matrix $I - M$ is non-singular. In particular, for $x := (I - M)^{-1}y$ it follows from (9.11) that

$$\|y\| \geq (1 - \|M\|) \|(I - M)^{-1}y\| \quad \text{for all } y \in \mathbb{R}^n.$$

From the definition of a matrix norm we therefore obtain

$$\|(I - M)^{-1}\| = \max_{y \neq 0} \frac{\|(I - M)^{-1}y\|}{\|y\|} \leq \frac{1}{1 - \|M\|},$$

which proves the claimed inequality. $\square$

We now turn to the announced perturbation lemma.

Lemma 9.17. Let $A, B \in \mathbb{R}^{n \times n}$ satisfy $\|I - BA\| < 1$. Then both A and B are non-singular, and the estimate

$$\|B^{-1}\| \leq \frac{\|A\|}{1 - \|I - BA\|}$$

holds. (A corresponding inequality also holds for A^{-1} .)

Proof. Let $M := I - BA$. By assumption, $\|M\| < 1$. By Lemma 9.16, it follows that

$$I - M = I - (I - BA) = BA$$

is non-singular. By the determinant multiplication theorem, both A and B are therefore non-singular. Moreover, Lemma 9.16 yields the estimate

$$\|(I - M)^{-1}\| \leq \frac{1}{1 - \|M\|} = \frac{1}{1 - \|I - BA\|}. \quad (9.12)$$

Since $I - M = BA$, we have $B^{-1} = A(I - M)^{-1}$, and hence, using (9.12),

$$\|B^{-1}\| \leq \|(I - M)^{-1}\| \|A\| \leq \frac{\|A\|}{1 - \|I - BA\|}.$$

$\square$

Lemma 9.18. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuously differentiable, $x^* \in \mathbb{R}^n$, and assume that $F'(x^*)$ is regular. Then there exists an $\varepsilon > 0$ such that $F'(x)$ is also regular for all $x \in B_\varepsilon(x^*)$. Moreover, there exists a constant $c > 0$ such that the estimate

$$\|F'(x)^{-1}\| \leq c$$

holds for all $x \in B_\varepsilon(x^*)$.

Proof. Since F' is continuous at x^* by assumption, there exists an $\varepsilon > 0$ such that

$$\|F'(x^*) - F'(x)\| \leq \frac{1}{2 \|F'(x^*)^{-1}\|}$$

for all $x \in B_\varepsilon(x^*)$. Consequently,

$$\begin{aligned} \|I - F'(x^*)^{-1}F'(x)\| &\leq \|F'(x^*)^{-1}\| \|F'(x^*) - F'(x)\| \\ &\leq \frac{1}{2} \end{aligned}$$

for all $x \in B_\varepsilon(x^*)$. By the perturbation lemma (Lemma 9.17) it follows that for all $x \in B_\varepsilon(x^*)$ the matrix $F'(x)$ is regular with

$$\|F'(x)^{-1}\| \leq \frac{\|F'(x^*)^{-1}\|}{1 - \|I - F'(x^*)^{-1}F'(x)\|} \leq 2 \|F'(x^*)^{-1}\| =: c.$$

This proves the lemma. $\square$

Definition 9.19 (Lipschitz-Continuity). A function $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **Lipschitz** continuous in $D \subset \mathbb{R}^n$ with Lipschitz constant $L > 0$, if the following holds true

$$\|F(x) - F(y)\| \leq L \|x - y\| \text{ for all } x, y \in D.$$

F is said to be **locally Lipschitz** at x , if there is $\varepsilon > 0$ so that F is Lipschitz on $B_\varepsilon(x)$.

Lemma 9.20. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\{x^k\} \subseteq \mathbb{R}^n$ be a sequence converging to some $x^* \in \mathbb{R}^n$. Then the following hold:

1. If F is continuously differentiable, then

$$\|F(x^k) - F(x^*) - F'(x^k)(x^k - x^*)\| = o(\|x^k - x^*\|).$$

2. If F is continuously differentiable and F' is locally Lipschitz continuous, then

$$\|F(x^k) - F(x^*) - F'(x^k)(x^k - x^*)\| = O(\|x^k - x^*\|^2).$$

Proof. 1.: Using the triangle inequality we obtain

$$\begin{aligned} &\|F(x^k) - F(x^*) - F'(x^k)(x^k - x^*)\| \\ &\leq \|F(x^k) - F(x^*) - F'(x^*)(x^k - x^*)\| + \|F'(x^*) - F'(x^k)\| \|x^k - x^*\|. \end{aligned}$$

Since F is differentiable at x^* by assumption, we have

$$\|F(x^k) - F(x^*) - F'(x^*)(x^k - x^*)\| = o(\|x^k - x^*\|).$$

Moreover, because F' is continuous at x^* , it follows that

$$\|F'(x^*) - F'(x^k)\| \rightarrow 0.$$

Combining the last three observations yields precisely

$$\left\| F(x^k) - F(x^*) - F'(x^k)(x^k - x^*) \right\| = o(\|x^k - x^*\|),$$

which proves claim 1.

2.: First, by the mean value theorem in integral form we have

$$\begin{aligned} F(x^k) - F(x^*) - F'(x^k)(x^k - x^*) &= \int_0^1 F'(x^* + \tau(x^k - x^*))(x^k - x^*) \, d\tau - F'(x^k)(x^k - x^*) \\ &= \int_0^1 [F'(x^* + \tau(x^k - x^*)) - F'(x^k)](x^k - x^*) \, d\tau. \end{aligned}$$

Denoting by $L > 0$ the local Lipschitz constant of F' in a neighborhood of x^* , we obtain for all sufficiently large $k \in \mathbb{N}$:

$$\begin{aligned} &\left\| F(x^k) - F(x^*) - F'(x^k)(x^k - x^*) \right\| \\ &\leq \int_0^1 \left\| F'(x^* + \tau(x^k - x^*)) - F'(x^k) \right\| \, d\tau \, \|x^k - x^*\| \\ &\leq L \|x^k - x^*\| \int_0^1 \|(\tau - 1)(x^k - x^*)\| \, d\tau \\ &= \frac{L}{2} \|x^k - x^*\|^2 \\ &= O(\|x^k - x^*\|^2). \end{aligned}$$

This is exactly claim 2. □

Definition 9.21 (Convergence). 1. A sequence $\{x^k\} \subset \mathbb{R}^n$ converges **superlinearly** towards x^* , if $x^k \rightarrow x^*$ and there is a sequence $\{\epsilon_k\} \subset \mathbb{R}_+$ with $\epsilon_k \rightarrow 0$ such that

$$\|x^{k+1} - x^*\| \leq \epsilon_k \|x^k - x^*\| \text{ for all } k \in \mathbb{N}.$$

2. A sequence $\{x^k\} \subset \mathbb{R}^n$ converges **quadratically** towards x^* , if $\{x^k\}$ converges to x^* and there is a constant C with

$$\|x^{k+1} - x^*\| \leq C \|x^k - x^*\|^2 \text{ for all } k \in \mathbb{N}.$$

Theorem 9.22 (Local Convergence of Newton's Method). Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuously differentiable, $x^* \in \mathbb{R}^n$ a zero of F , and suppose the Jacobian matrix $F'(x^*)$ is regular. Then there exists an $\varepsilon > 0$ such that for every $x^0 \in B_\varepsilon(x^*)$ the following hold:

1. The Newton method (Algorithm 9.6) is well-defined and generates a sequence $\{x^k\}$ converging to x^* .

2. The rate of convergence is superlinear.
3. If, in addition, F' is locally Lipschitz continuous, then the rate of convergence is even quadratic.

Proof. By Lemma 9.18 there exists an $\varepsilon_1 > 0$ such that $F'(x)$ is regular for all $x \in B_{\varepsilon_1}(x^*)$ with

$$\|F'(x)^{-1}\| \leq c$$

for some constant $c > 0$. Furthermore, by Lemma 9.20(a) there exists an $\varepsilon_2 > 0$ such that

$$\|F(x) - F(x^*) - F'(x)(x - x^*)\| \leq \frac{1}{2c} \|x - x^*\|$$

for all $x \in B_{\varepsilon_2}(x^*)$. Set $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ and choose $x^0 \in B_{\varepsilon}(x^*)$. Then x^1 is well-defined, and we have

$$\begin{aligned} \|x^1 - x^*\| &= \|x^0 - x^* - F'(x^0)^{-1}F(x^0)\| \\ &\leq \|F'(x^0)^{-1}\| \|F(x^0) - F(x^*) - F'(x^0)(x^0 - x^*)\| \\ &\leq c \frac{1}{2c} \|x^0 - x^*\| \\ &= \frac{1}{2} \|x^0 - x^*\|. \end{aligned}$$

Hence, $x^1 \in B_{\varepsilon}(x^*)$ as well, and by induction we obtain

$$\|x^k - x^*\| \leq \left(\frac{1}{2}\right)^k \|x^0 - x^*\| \quad \text{for all } k \in \mathbb{N}.$$

Thus, the sequence $\{x^k\}$ is well-defined and converges to x^* , which proves part 1.

To show statements 2 and 3, we proceed similarly to the above estimate and obtain

$$\begin{aligned} \|x^{k+1} - x^*\| &= \|x^k - x^* - F'(x^k)^{-1}F(x^k)\| \\ &\leq \|F'(x^k)^{-1}\| \|F(x^k) - F(x^*) - F'(x^k)(x^k - x^*)\| \\ &\leq c \|F(x^k) - F(x^*) - F'(x^k)(x^k - x^*)\|. \end{aligned}$$

Applying Lemma 9.20 then yields claims 2 and 3. □

We consider now a constrained optimization problem:

$$\min \{f(x) \mid g(x) = 0\}, \text{ where } g : \mathbb{R}^n \rightarrow \mathbb{R}^m. \quad (9.13)$$

Using the Lagrange-Function

$$L(x, \lambda) = f(x) + \lambda^T g(x) = f(x) + \sum_{i=1}^m \lambda_i g_i(x)$$

the KKT-conditions read as

$$F(x, \lambda) := \begin{pmatrix} L_x(x, \lambda)^T \\ g(x) \end{pmatrix} = \begin{pmatrix} f'(x)^T + g'(x)^T \lambda \\ g(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (9.14)$$

The equation (9.14) is a nonlinear system of $n + m$ equations in x, λ . The corresponding Newton-Update reads as

$$\begin{pmatrix} x^{k+1} \\ \lambda^{k+1} \end{pmatrix} = \begin{pmatrix} x^k \\ \lambda^k \end{pmatrix} - F'(x^k, \lambda^k)^{-1} F(x^k, \lambda^k). \quad (9.15)$$

Formally, we get:

1. Choose $x^0 \in \mathbb{R}^n, \lambda^0 \in \mathbb{R}^m, k = 0$ and fix $\epsilon_0 > 0$;
2. If $\|F(x^k, \lambda^k)\| \leq \epsilon_0$:STOP (Termination);
3. Compute $(\Delta x^k, \Delta \lambda^k)$ as solution of

$$F'(x^k, \lambda^k) \begin{pmatrix} \Delta x \\ \Delta \lambda \end{pmatrix} = -F(x^k, \lambda^k). \quad (9.16)$$

4. Set $x^{k+1} \leftarrow x^k + \Delta x^k; \lambda^{k+1} \leftarrow \lambda^k + \Delta \lambda^k; k \leftarrow k + 1$ and go to step 2.

Figure 9.8: Lagrange-Newton Method.

We obtain the following result regarding the Lagrange-Newton method.

Theorem 9.23. Let f, g be C^2 near a normal KKT point $(\bar{x}, \bar{\lambda})$ of (9.13), and assume the second-order sufficient conditions. With $\epsilon_0 = 0$ and (x^0, λ^0) sufficiently close to $(\bar{x}, \bar{\lambda})$, the Lagrange-Newton method converges to this point superlinearly. If the second derivatives of f, g are locally Lipschitz, the convergence is quadratic. A terminated sequence is continued constantly at its solution.

Proof. The KKT Jacobian is nonsingular by the same argument used for A_0 in Theorem 7.1. The local Newton theorem applies; its quadratic estimate additionally requires the stated Lipschitz assumption. $\square$

Inequality Constraints

We now consider optimization problems with inequality constraints of the following form:

$$\min \{f(x) \mid g_i(x) \leq 0, i = 1, \dots, k, h_i(x) = 0, i = 1, \dots, m - k\}. \quad (9.17)$$

and we also write

$$\min \{f(x) \mid g(x) \leq 0, h(x) = 0\}.$$

With the Lagrangian

$$L(x, \lambda, \mu) = f(x) + \lambda^T h(x) + \mu^T g(x),$$

the KKT conditions are

$$\begin{aligned} L_x(x, \lambda, \mu) &= f'(x) + \lambda^T h'(x) + \mu^T g'(x) = 0 \\ g(x) &\leq 0 \\ h(x) &= 0 \\ \mu_i g_i(x) &= 0, i = 1, \dots, k \\ \mu_i &\geq 0, i = 1, \dots, k. \end{aligned} \quad (9.18)$$

We now introduce a so-called NCP function.

Definition 9.24. A function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is called an **NCP function** if

$$\phi(a, b) = 0 \iff a \geq 0, b \geq 0, ab = 0.$$

Example 9.25. The following mappings are NCP functions:

1. Minimum: $\phi(a, b) = 2 \min\{a, b\}$.
2. Fischer–Burmeister: $\phi(a, b) = a + b - \sqrt{a^2 + b^2}$.

We obtain the following **equivalent** representation of the KKT conditions:

$$\begin{aligned} L_x(x, \lambda, \mu) &= f'(x) + \mu^T g'(x) + \lambda^T h'(x) = 0 \\ h(x) &= 0 \\ \phi(-g_i(x), \mu_i) &= 0, i = 1, \dots, k. \end{aligned}$$

This can again be formulated as a nonlinear system of equations

$$F(x, \lambda, \mu) = 0 \text{ with}$$

$$F(x, \lambda, \mu) := \begin{pmatrix} L_x(x, \lambda, \mu)^\top \\ h(x) \\ \phi(-g(x), \mu) \end{pmatrix},$$

where

$$\phi(-g(x), \mu) := (\phi(-g_1(x), \mu_1), \dots, \phi(-g_k(x), \mu_k))^\top \in \mathbb{R}^k.$$

Remark 9.26. In principle, one can apply a Newton method to the above system of equations. However, the function ϕ is not differentiable everywhere; therefore, the Newton method may need to be modified (using subdifferentials). A more detailed treatment of this topic can be found in Geiger and Kanzow (2011).

9.4 SQP Methods

For this section, write the equality constraints as $h(x) = 0$ and use column multipliers μ for equalities. For the general problem below, use column multipliers λ for inequalities and define

$$L(x, \lambda, \mu) = f(x) + \lambda^\top g(x) + \mu^\top h(x).$$

For the equality-only problem set $L(x, \mu) = f(x) + \mu^\top h(x)$ and $F(x, \mu) = \begin{pmatrix} L_x(x, \mu)^\top \\ h(x) \end{pmatrix}$. In the Lagrange-Newton method, we set

$$x^{k+1} := x^k + \Delta x^k, \quad \mu^{k+1} := \mu^k + \Delta \mu^k, \quad (9.19)$$

where $(\Delta x^k, \Delta \mu^k)$ denotes a solution of the linear system

$$F'(x^k, \mu^k) \begin{pmatrix} \Delta x \\ \Delta \mu \end{pmatrix} = -F(x^k, \mu^k). \quad (9.20)$$

Taking into account the special structure of $F'(x^k, \mu^k)$, this system can be written equivalently as

$$\begin{aligned} H_k \Delta x + h'(x^k)^\top \Delta \mu &= -L_x(x^k, \mu^k)^\top, \\ h'_j(x^k) \Delta x &= -h_j(x^k), \quad \forall j = 1, \dots, m, \end{aligned} \quad (9.21)$$

with $H_k = L_{xx}(x^k, \mu^k)$. In the following, we also allow symmetric approximations $H_k \approx L_{xx}(x^k, \mu^k)$. Defining $\mu^+ := \mu^k + \Delta \mu$, system (9.21) is equivalent to

$$\begin{aligned} H_k \Delta x + h'(x^k)^\top \mu^+ &= -f'(x^k)^\top, \\ h'_j(x^k) \Delta x &= -h_j(x^k), \quad \forall j = 1, \dots, m. \end{aligned} \quad (9.22)$$

System (9.22) admits another interpretation: it corresponds exactly to the Karush–Kuhn–Tucker (KKT) conditions of the quadratic optimization problem

$$\begin{aligned} \min_{\Delta x} \quad & f'(x^k) \Delta x + \frac{1}{2} \Delta x^T H_k \Delta x \\ \text{subject to} \quad & h_j(x^k) + h'_j(x^k) \Delta x = 0, \quad (j = 1, \dots, m). \end{aligned} \quad (9.23)$$

This observation motivates, in particular, the use of the quadratic subproblem

$$\begin{aligned} \min_{\Delta x} \quad & f'(x^k) \Delta x + \frac{1}{2} \Delta x^T H_k \Delta x \\ \text{subject to} \quad & g_i(x^k) + g'_i(x^k) \Delta x \leq 0, \quad (i = 1, \dots, \ell), \\ & h_j(x^k) + h'_j(x^k) \Delta x = 0, \quad (j = 1, \dots, m - \ell). \end{aligned} \quad (9.24)$$

for the general problem

$$\min \{f(x) \mid g(x) \leq 0, h(x) = 0\}. \quad (9.25)$$

with

$$g : \mathbb{R}^n \rightarrow \mathbb{R}^\ell \text{ and } h : \mathbb{R}^n \rightarrow \mathbb{R}^{m-\ell}.$$

(SQP method with $H_k = L_{xx}(x^k, \lambda^k, \mu^k)$)

1. Choose $(x^0, \lambda^0, \mu^0) \in \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ and set $k := 0$.
2. If (x^k, λ^k, μ^k) is a KKT point of (9.25): **STOP**.
3. Compute a KKT point $(x^{k+1}, \lambda^{k+1}, \mu^{k+1}) \in \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ of the quadratic subproblem

$$\begin{aligned} \min_x \quad & f'(x^k)(x - x^k) + \frac{1}{2}(x - x^k)^T L_{xx}(x^k, \lambda^k, \mu^k)(x - x^k) \\ \text{subject to} \quad & g_i(x^k) + g'_i(x^k)(x - x^k) \leq 0, \quad i = 1, \dots, \ell, \\ & h_j(x^k) + h'_j(x^k)(x - x^k) = 0, \quad j = 1, \dots, m - \ell. \end{aligned} \quad (9.26)$$

If this quadratic subproblem has several KKT points, choose the KKT point $(x^{k+1}, \lambda^{k+1}, \mu^{k+1})$ such that

$$\|(x^{k+1}, \lambda^{k+1}, \mu^{k+1}) - (x^k, \lambda^k, \mu^k)\|$$

is minimal.

4. Set $k \leftarrow k + 1$ and return to (2).

Figure 9.9: SQP Method.

The quadratic subproblem arising in step (3) of Algorithm 9.9 may admit multiple KKT points. The restriction imposed in step (3) on the choice of $(x^{k+1}, \lambda^{k+1}, \mu^{k+1})$ is hardly

realizable in practice; however, it allows us to prove the following local convergence result. After finite termination, extend the sequence constantly at the solution.

Theorem 9.27 (Local Convergence of SQP). Assume f, g, h are C^2 near x^* .

Let $(x^*, \lambda^*, \mu^*) \in \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ be a KKT point of (9.25) satisfying the following assumptions:

- (i) $g_i(x^*) + \lambda_i^* \neq 0$ for all $i = 1, \dots, \ell$ (strict complementarity).
- (ii) The gradients $h_j'(x^*)$ ($j = 1, \dots, m - \ell$) and $g_i'(x^*)$ for $i \in I(x^*) := \{i \mid g_i(x^*) = 0\}$ are linearly independent (LICQ).
- (iii)

$$d^T L_{xx}(x^*, \lambda^*, \mu^*) d > 0$$

for all $d \neq 0$ such that

$$h_j'(x^*)d = 0 \quad (j = 1, \dots, m - \ell), \quad g_i'(x^*)d = 0 \quad (i \in I(x^*)),$$

(second-order sufficient condition).

Then there exists $\varepsilon > 0$ such that for every initial vector $(x^0, \lambda^0, \mu^0) \in B_\varepsilon(x^*, \lambda^*, \mu^*)$ and every sequence $\{(x^k, \lambda^k, \mu^k)\}$ generated by Algorithm 9.9, the following statements hold:

- (a) Algorithm 9.9 is well defined, and $(x^k, \lambda^k, \mu^k) \rightarrow (x^*, \lambda^*, \mu^*)$.
- (b) The rate of convergence is superlinear.
- (c) If f'' , g_i'' ($i = 1, \dots, \ell$), and h_j'' ($j = 1, \dots, m - \ell$) are locally Lipschitz continuous, then the convergence rate is quadratic.

Proof. We give the local argument. Define

$$F(x, \lambda, \mu) = \begin{pmatrix} L_x(x, \lambda, \mu)^T \\ h(x) \\ \min\{-g(x), \lambda\} \end{pmatrix},$$

where the last block has components $\min\{-g_i(x), \lambda_i\}$, $i = 1, \dots, \ell$. Its zeros are precisely the KKT points. Strict complementarity fixes the branch of every minimum near the solution: it equals $-g_i(x)$ for active constraints and λ_i for inactive ones. Hence F is C^1 there. LICQ and the second-order condition together make its Jacobian nonsingular, by the bordered-matrix argument from Theorem 7.1.

Newton's method is therefore locally superlinearly convergent, and quadratically convergent under the stated Lipschitz assumption. Its nearby iterates satisfy the KKT conditions of the SQP subproblem with the same active set: active multipliers remain positive and

inactive linearized constraints remain strictly slack. Locally this KKT solution is unique. For sufficiently close starting points, it is the nearest KKT solution selected by the algorithm; all other solutions lie outside a fixed neighbourhood. Thus the SQP and Newton iterates agree locally. $\square$

9.5 The Frank-Wolfe Algorithm

In this chapter, we consider the Frank-Wolfe (FW) algorithm for a problem of type

$$\min f(x), \text{ s.t. } x \in K, \quad (9.27)$$

for a nonempty compact convex set $K \subset \mathbb{R}^m$, with f continuously differentiable on a neighbourhood of K . FW relies on the following basic idea: we compute the gradient $f'(x_k)$ of f for $f \in C^1$ at some iterate $x_k \in K$. Then, we solve the simpler linear-objective problem

$$\min_{v \in K} f'(x_k)v, \quad (9.28)$$

and for an optimal solution v_k , we choose the new iterate as

$$x_{k+1} = (1 - \gamma_k)x_k + \gamma_k v_k. \quad (\text{FW-Alg.})$$

We require $0 \leq \gamma_k \leq 1$ and start at $x_0 \in K$. A common choice of γ_k is $\gamma_k = \frac{2}{k+2}$. In the literature, the linear optimization problem is assumed to be easily solvable and we can abstract away the resulting running time by having access to a Linear Minimization Oracle (LMO).

9.5.1 Convex Objectives

In order to analyze the convergence of FW, we need some more concepts.

Definition 9.28 (Smoothness). A continuously differentiable function $f : K \rightarrow \mathbb{R}$ satisfies the L -smoothness upper bound, with $L > 0$, if

$$f(y) - f(x) \leq f'(x)(y - x) + \frac{L}{2} \|y - x\|^2 \text{ for all } x, y \in K. \quad (9.29)$$

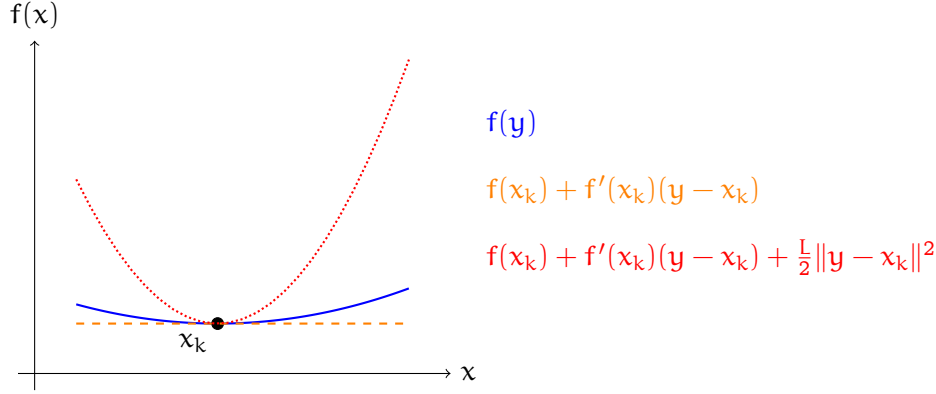


Figure 9.10: Smoothness property.

Lemma 9.29. Let $f : K \rightarrow \mathbb{R}$ with $f \in C^1$ be convex and let K be convex and compact. Then, the following statements hold.

1. x^* is optimal for (9.27) if and only if $f'(x^*)(x^* - v^*) \leq 0$, where $v^* \in K$ is optimal for (9.28) with $x_k = x^*$.
2. For all $x \in K$ it holds:

$$\underbrace{f(x) - f(x^*)}_{\text{primal gap}} \leq \underbrace{f'(x)(x - x^*)}_{\text{linearization bound}} \leq \max_{v \in K} f'(x)(x - v) = f'(x)(x - v^*), \quad (\text{FW-gap})$$

where $v^* \in K$ is optimal for (9.28) with $x_k = x$.

The proof follows by convexity of f and K and the property of v^* . Now we measure the progress during the execution of (FW-Alg.).

Lemma 9.30. (Primal progress)

Consider (FW-Alg.) and fix $k \in \mathbb{N}$. Let $f : K \rightarrow \mathbb{R}$ with $f \in C^1$ and L -smooth. Let K be convex and compact. Then, we have

$$f(x_k) - f(x_{k+1}) \geq \gamma_k f'(x_k)(x_k - v_k) - (\gamma_k)^2 \frac{L}{2} \|x_k - v_k\|^2. \quad (9.30)$$

Proof. We just insert the definition of x_{k+1} according to (FW-Alg.) and x_k into (9.29). $\square$

The two basic lemmas 9.29 and 9.30 lead to the following convergence result.

Theorem 9.31. Let f be an L -smooth convex function and let K be a compact convex set of diameter $\text{diam}(K) = D$, where $\text{diam}(K) := \max_{x, y \in K} \|y - x\|$. Consider the iterates of (FW-Alg.) with $\gamma_k = \frac{2}{k+2}$. Then the following holds for every integer $k \geq 1$:

$$f(x_k) - f(x^*) \leq \frac{2LD^2}{k+2} \quad (9.31)$$

and thus, for a given accuracy $\epsilon > 0$ we have $f(x_k) - f(x^*) \leq \epsilon$ for all integers $k \geq 1$ with $k \geq \frac{2LD^2}{\epsilon}$.

Proof. Using Lemma 9.30, we get

$$\begin{aligned} f(x_k) - f(x_{k+1}) &\geq \gamma_k f'(x_k)(x_k - v_k) - (\gamma_k)^2 \frac{L}{2} \|x_k - v_k\|^2 \\ &\geq \gamma_k (f(x_k) - f(x^*)) - (\gamma_k)^2 \frac{L}{2} \|x_k - v_k\|^2. \end{aligned} \quad (\text{By (FW-gap)})$$

Multiplying with -1 and using $\|x_k - v_k\|^2 \leq D^2$, we obtain:

$$f(x_{k+1}) \leq (1 - \gamma_k)f(x_k) + \gamma_k f(x^*) + (\gamma_k)^2 \frac{LD^2}{2}.$$

Subtracting $f(x^*)$ on both sides yields:

$$f(x_{k+1}) - f(x^*) \leq (1 - \gamma_k)(f(x_k) - f(x^*)) + (\gamma_k)^2 \frac{LD^2}{2}.$$

Now, we proceed via induction over k . The first update ($k = 0$) establishes the base case $k = 1$: $\gamma_0 = 1$ and thus

$$f(x_1) - f(x^*) \leq \frac{LD^2}{2} \leq \frac{2LD^2}{3}.$$

For the induction step $k \rightarrow k + 1$, we get:

$$\begin{aligned} f(x_{k+1}) - f(x^*) &\leq (1 - \gamma_k)(f(x_k) - f(x^*)) + (\gamma_k)^2 \frac{LD^2}{2} \\ &\leq \frac{k}{k+2}(f(x_k) - f(x^*)) + \frac{4}{(k+2)^2} \frac{LD^2}{2} \\ &\leq \left(\frac{k}{k+2}\right) \left(\frac{2LD^2}{k+2}\right) + \frac{4}{(k+2)^2} \frac{LD^2}{2} \quad (\text{induction hypothesis}) \\ &= 2LD^2 \left(\frac{(1+k)}{(k+2)^2}\right) = \left(\frac{2LD^2}{k+3}\right) \left(\frac{(k+3)(1+k)}{(k+2)^2}\right) \\ &\leq \left(\frac{2LD^2}{k+3}\right), \end{aligned}$$

where the last inequality follows from $\left(\frac{(k+3)(1+k)}{(k+2)^2}\right) \leq 1$. $\square$

Example 9.32 (Primal Gap Lower Bound). We provide an example where $\Omega(LD^2/\epsilon)$ linear minimizations are needed to achieve a primal gap additive error at most ϵ for an L -smooth convex function over a feasible region with diameter D , for $L, D > 0$, sufficiently small $\epsilon > 0$, and sufficiently large dimension.

We consider the problem

$$\min_{x \in \text{co}(e_1, \dots, e_n)} \|x\|_2^2,$$

that is, minimizing the quadratic objective function $f(x) = \|x\|_2^2$ over the probability

simplex $P = \text{co}(e_1, \dots, e_n) \subset \mathbb{R}^n$, where the e_i are the coordinate vectors, i.e., the vectors in the standard basis of $\mathbb{R}^n$. The unique optimal solution to the problem is $x_i^* = (\frac{1}{n})$, $i = 1, \dots, n$, the point whose coordinates are all equal to $1/n$.

If we now run the Frank-Wolfe algorithm from any extreme point x_0 of P , and require the LMO to return an extreme-point minimizer, then after $k < n$ iterations, we have made k LMO calls, and hence have picked up at most $k + 1$ of the n standard basis vectors. Each Frank-Wolfe update is a convex combination of the current iterate and the new oracle vertex. Hence the iterate is supported on at most $k + 1$ coordinate vectors. The extreme-point oracle assumption is essential: an oracle returning arbitrary minimizers may return a dense point at a tie. Thus it holds

$$f(x_k) \geq \min_{\substack{x \in \text{co}(S) \\ S \subseteq \{e_1, \dots, e_n\} \\ |S| \leq k+1}} f(x) = \frac{1}{k+1}.$$

Therefore the primal gap after k iterations satisfies

$$f(x_k) - f(x^*) \geq \frac{1}{k+1} - \frac{1}{n},$$

and for $n \gg 1/\epsilon$, we need $\Omega(1/\epsilon)$ LMO calls to guarantee a primal gap of at most ϵ . For arbitrary $L, D > 0$, use the scaled simplex $K = (D/\sqrt{2})\text{co}(e_1, \dots, e_n)$ and $f(x) = (L/2)\|x\|^2$. Then

$$f(x_k) - f(x^*) \geq \frac{LD^2}{4} \left(\frac{1}{k+1} - \frac{1}{n} \right).$$

Thus the worst-case bound is $\Omega(LD^2/\epsilon)$ when the dimension is sufficiently large and ϵ is small relative to LD^2 . More generally, every feasible point with at most s nonzero coordinates in the unscaled example has primal gap at least $1/s - 1/n$.

9.5.2 Non-Convex Objectives

Now we turn to general non-convex functions $f : K \rightarrow \mathbb{R}$ with $f \in C^1$ and K nonempty, compact, and convex.

Definition 9.33 (Frank-Wolfe Gap). The Frank-Wolfe gap of a function $f : K \rightarrow \mathbb{R}$ with $f \in C^1$ at $x \in K$ is defined as

$$g(x) := \max_{v \in K} f'(x)(x - v). \quad (9.32)$$

With $x \in K$, we get $g(x) \geq 0$.

Note that for convex functions f , the condition $g(x) = 0$ is by convexity of K equivalent to x being a global minimum. For non-convex functions f , the condition $g(x) = 0$ is by convexity of K **only necessary** but not sufficient. Yet, due to the general difficulty of non-convex problems, we search for points x^* with $g(x^*) = 0$ and the quantity $g(x)$ for the

current iterate x will serve as a measure for termination, namely, for a prescribed precision $\epsilon > 0$, we will ask for convergence towards some point $x \in K$ with $g(x) \leq \epsilon$.

Theorem 9.34. Let f be an L -smooth function and let K be a compact convex set of diameter $\text{diam}(K) = D$. Consider the iterates of (FW-Alg.) with $\gamma_k = \gamma = \frac{1}{\sqrt{T+1}}$ for a fixed integer $T \geq 0$. After $T + 1$ oracle calls and updates of (FW-Alg.) the following holds:

$$G_T := \min_{0 \leq k \leq T} g(x_k) \leq \frac{\max\{2h_0, LD^2\}}{\sqrt{T+1}}, \quad (9.33)$$

where x^* is a global minimizer on K and $h_0 := f(x_0) - f(x^*)$ denotes the primal gap at $x_0 \in K$.

This bounds the smallest gap among $x_0, \dots, x_T$, not necessarily the gap of the last iterate.

Proof. At iterate x_k we can bound the progress via Lemma 9.30 as

$$f(x_k) - f(x_{k+1}) \geq \gamma f'(x_k)(x_k - v_k) - \frac{\gamma^2 L}{2} \|x_k - v_k\|^2.$$

Summing up the above for $k = 0, \dots, T$ and rearranging gives

$$\begin{aligned} \gamma \sum_{k=0}^T f'(x_k)(x_k - v_k) &\leq f(x_0) - f(x_{T+1}) + \gamma^2 \sum_{k=0}^T \frac{L}{2} \|x_k - v_k\|^2 \\ &\leq f(x_0) - f(x^*) + \gamma^2 \sum_{k=0}^T \frac{LD^2}{2} = h_0 + \gamma^2(T+1) \frac{LD^2}{2}. \end{aligned}$$

We divide by $\gamma(T+1)$ on both sides to obtain

$$G_T \leq \frac{1}{T+1} \sum_{k=0}^T f'(x_k)(x_k - v_k) \leq \frac{h_0}{\gamma(T+1)} + \gamma \frac{LD^2}{2},$$

and for $\gamma = \frac{1}{\sqrt{T+1}}$ we get

$$G_T \leq \frac{1}{T+1} \sum_{k=0}^T f'(x_k)(x_k - v_k) \leq \frac{2h_0 + LD^2}{2\sqrt{T+1}} \leq \frac{\max\{2h_0, LD^2\}}{\sqrt{T+1}},$$

completing the proof. □