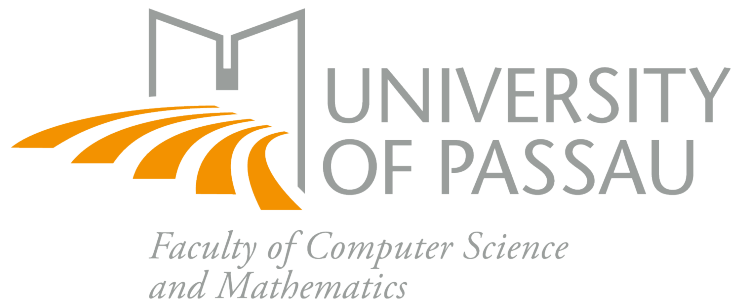




Lecture

Linear and Discrete Optimization

given in WS 2022/23, 2023/2024

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Preface

This scriptum has been developed during the optimization lectures given at Augsburg University 2015-2022. The contents of Chapters 2,3,5-12 are based on hand-written notes of myself taken as a student at a lecture given by Helmut Maurer (University of Münster) in 2003. Chapter 4 is based on lecture notes of Stefan Ulbrich (TU Darmstadt, as of WS 2013), which in turn are based on lecture notes of Martin Grötschel (ZIB and TU Berlin). Special thanks go to Lukas Graf and Georg Kraus who suggested several corrections to the scriptum during the course held in SS 2020.

Passau, October 2023
Prof. Dr. Tobias Harks

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Chapter 1

Introduction

1.1 Motivation and Terminology

We consider a normed vector space $(V, \|\cdot\|)$ for $K \subset V$ nonempty. We are given a function

$$f : K \rightarrow \mathbb{R}.$$

Our goal is to solve $\min\{f(x) \mid x \in K\}$. We sometimes write:

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{s.t.: } x \in K. \end{aligned} \tag{1.1}$$

An optimal solution of (1.1) is called global minimum.

Definition 1.1. $x^* \in K$ is a global minimum of f over K , if

$$f(x) \geq f(x^*) \text{ for all } x \in K.$$

If

$$f(x) > f(x^*) \text{ for all } x \in K, x \neq x^*,$$

we speak of a strict global minimum. Global maxima are defined analogously.

- infinite dimensional optimization (e.g., V is a function space, L_1 or L_p space.),
- finite dimensional optimization ($V = \mathbb{R}^n$),
- continuous optimization ($\text{int}(K) \neq \emptyset$),
- discrete optimization ($K \subseteq \mathbb{Z}^n$).

1.2 Examples and Applications

Example 1.2 (Optimal Supply). The goal is to buy an amount M of a certain commodity. We have the offers of n suppliers, where every supplier $i \in M$ has a maximum supply of

M_i units of the commodity. The prices of the i -th supplier are given by a function $f_i(x_i)$.

$$\begin{aligned} \min \quad & \sum_{i=1}^n f_i(x_i) \\ \text{s.t.} \quad & \sum_{i=1}^n x_i = M \\ & 0 \leq x_i \leq M_i, i = 1, \dots, n. \end{aligned}$$

Example 1.3 (Regression). An experiment shows the following data $(t_i, y_i), i = 1, \dots, m$. The hypothesis class under which this data is generated is given by a parameterized function $f(t, p)$. The goal is to choose parameters p in order to minimize the resulting error measured as:

$$\sum_{i=1}^m (y_i - f(t_i, p))^2$$

A more general **least-squares** problem is given as:

$$\begin{aligned} \min \quad & \sum_{i=1}^q \Phi_i(x)^2 \\ \text{s.t.} \quad & g_j(x) \leq 0, j = 1, \dots, r \\ & h_j(x) = 0, j = 1, \dots, s. \end{aligned}$$

Example 1.4 (Optimal control). We search for a control function that steers a car with minimal “effort” in a given time frame $[0, t_f]$ from A to B . We use Newton’s laws describing where $s(t)$ denotes the location at time t , $v(t)$ the speed at time t and $a(t)$ denotes the acceleration:

$$\dot{s}(t) = v(t), \dot{v}(t) = a(t).$$

Suppose we are in dimension 1 and there is a straight line between A nach B of length d . The coordinate of A is normalized to $s = 0$ and $s = d$ for B . We need to satisfy $s(0) = 0, v(0) = 0, s(t_f) = d, v(t_f) = 0$.

The control function corresponds to $a(t)$, where a positive sign is acceleration and negative sign is slow down. The effort accumulates quadratically in a :

$$\int_0^{t_f} a(t)^2 dt$$

We obtain the following optimal control problem:

$$\begin{aligned} \min \quad & \int_0^{t_f} a(t)^2 dt \\ & \dot{s}(t) = v(t) \\ & \dot{v}(t) = a(t) \\ & s(0) = 0, s(t_f) = d \end{aligned}$$

$$v(0) = 0, v(t_f) = 0.$$

1.3 Finite Dimensional Optimization

In this lecture, we consider only finite dimensional problems, that is, $V := \mathbb{R}^n$.

The set of points in K that attain the minimum is denoted by $\arg \min(f, K)$. We get

$$\alpha = \min\{f(x) : x \in K\} \Leftrightarrow \arg \min(f, K) = \{x \in K \mid f(x) = \alpha\}.$$

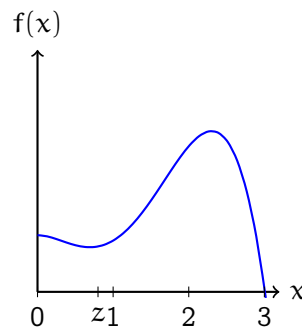


Figure 1.1: f attains on $K = [0, 3]$ its **global** minimum at $x^* = 3$. There is a further **local** minimum at z .

Definition 1.5. A point $x^* \in K$ is a local Minimum of f over K , if there is $\rho > 0$ such that

$$f(x) \geq f(x^*) \text{ for all } x \in K \cap B_\rho(x^*),$$

where

$$B_\rho(x) = \{y \in \mathbb{R}^n \mid \|x - y\| < \rho\}$$

denotes the open ball around x with radius $\rho > 0$. If

$$f(x) > f(x^*) \text{ for all } x \in K \cap B_\rho(x^*), x \neq x^*,$$

we speak of a strict local minimum.

Usually K is represented via functional inequalities or equalities. In this case, we obtain:

$$K = \{x \in \mathbb{R}^n \mid h_i(x) = 0, \ i \in I_1 = \{1, \dots, m\}, \ g_j(x) \leq 0, \ j \in I_2 = \{1, \dots, p\}\}, \quad (1.2)$$

where all functions satisfy $f, h_i, g_j \in C^2$ for all $i \in I_1, j \in I_2$.

We obtain the following classes of optimization problems:

- unrestricted optimization: $m = p = 0$
- restricted optimization: $m > 0$ oder $p > 0$
- linear optimization: f linear, g_j, h_i affin linear

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- quadratic optimization: $f(x) = x^T A x + b^T x + c$, g_j, h_i affin linear
- convex optimization: f convex, g_j convex, h_i affin linear
- L_2 -problems: the function f has the form

$$f(x) = \sum_{i=1}^n w_i (f_i(x))^2,$$

with smooth functions f_i and weights $w_i > 0$.

- minimax problems: f has the form

$$f(x) = \max\{f_i(x), i = 1, \dots, m\}$$

with smooth functions f_i .

The following questions are the key drivers for the content of this lecture:

- When do optimal solutions exist?
- Are they unique?
- Can we derive useful necessary and sufficient optimality conditions?
- What about algorithms for solving such problems?
- What is the dependence of optimal solutions on problem parameters?

We recap a fundamental result due to Weierstrass.

Theorem 1.6 (Weierstrass). Let $K \subset \mathbb{R}^n$ be nonempty and compact and $f : K \rightarrow \mathbb{R}$ continuous. Then, there is $x^* \in K$ with

$$f(x^*) \leq f(x) \text{ for all } x \in K.$$

Proof. As f is continuous, the image $f(K)$ of the compactum K is bounded in $\mathbb{R}$ and the infimum

$$A := \inf\{f(x) | x \in K\} \in \mathbb{R}$$

exists. Hence, there is a sequence $x_n \in K, n \in \mathbb{N}$ with

$$\lim_{n \rightarrow \infty} f(x_n) = A.$$

As $x_n, n \in \mathbb{N}$ is bounded, we can use the theorem of Bolzano/Weierstrass giving a convergent subsequence $x_{n_k}, k \in \mathbb{N}$ with

$$\lim_{k \rightarrow \infty} x_{n_k} =: x^* \in K.$$

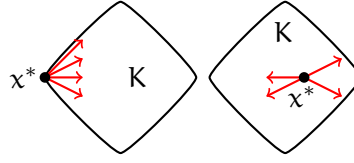


Figure 1.2: The red arcs represent feasible directions in $D_K(x^*)$.

With continuity of f we get

$$f(x^*) = \lim_{k \rightarrow \infty} f(x_{n_k}) = A.$$

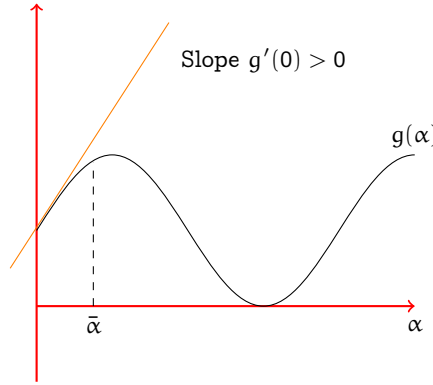
Thus, f attains at x^* its minimum over K . $\square$

1.4 Differentiable Classic Optimization

1.4.1 Variational Inequalities

Definition 1.7 (Feasible Directions). Let $x \in K \subseteq \mathbb{R}^n$ with $K \neq \emptyset$. The vector $d \in \mathbb{R}^n$ is a feasible direction at x , if there is $\bar{\alpha} > 0$ such that $x + \alpha d \in K$ for all $0 \leq \alpha \leq \bar{\alpha}$.

We denote by $D_K(x)$ the set of feasible directions at x . It is easy to see that $D_K(x)$ is a pointed cone containing 0 (cf. 2.1), hence, $D_K(x)$ is known as the cone of feasible directions.



For continuous optimization problems (1.1) we obtain the following necessary optimality conditions.

Theorem 1.8 (Variational Inequality). Let $K \subseteq \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuously differentiable. Let x^* be a local minimum of f over K and $d \in D_K(x^*)$. Then

$$\nabla f(x^*)^T d \geq 0.$$

Proof. Since $d \in D_K(x^*)$, there is $\bar{\alpha} > 0$ such that $x^*(\alpha) := x^* + \alpha d \in K$ for all $0 \leq \alpha \leq \bar{\alpha}$. We define a 1-dimensional function $g(\alpha) := f(x^*(\alpha))$. For a local minimum x^* (w.r.t. $B_\rho(x^*)$), we have

$$g(\alpha) \geq g(0) \text{ for all } \alpha \in [0, \min\{\bar{\rho}, \bar{\alpha}\}],$$

where

$$\bar{\rho} := \sup\{\alpha \geq 0 \mid x^* + \alpha d \in B_{\rho/2}(x^*)\}.$$

Thus,

$$\lim_{\alpha \rightarrow +0} \frac{g(\alpha) - g(0)}{\alpha} \geq 0.$$

With the differentiability of f we get

$$0 \leq \lim_{\alpha \rightarrow +0} \frac{g(\alpha) - g(0)}{\alpha} = g'(0) = \nabla f(x^*)^\top d.$$

□

Theorem 1.9. Let $K \subseteq \mathbb{R}^n$ and $f : K \rightarrow \mathbb{R}$ be continuously differentiable. If $x^* \in \text{int}(K)$ is a local minimum of f over K , then:

$$\nabla f(x^*) = 0. \quad (1.3)$$

In particular, we have (1.3) for every local minimum of an unconstrained optimization problem.

Proof. With Theorem 1.8 we get for every $d \in D_K(x^*)$: $\nabla f(x^*)^\top d \geq 0$. With $x^* \in \text{int}(K)$, we get $D_K(x^*) = \mathbb{R}^n$. □

Remark 1.10. Note that the concept of feasible directions of Definition 1.7 is not useful for sets given by algebraic manifolds. Here, we need **curved directions** leading to concepts of the **tangent cone** and **linearized cone** that we will see later.

1.4.2 Convex Optimization

We consider now a differentiable convex function f over a convex set $K \subset \mathbb{R}^n$.

Definition 1.11. A set $K \subset \mathbb{R}^n$ is convex, if for all $x, y \in K$ the segment between x and y lies in K , that is,

$$\lambda x + (1 - \lambda)y \in K \text{ for all } \lambda \in [0, 1].$$

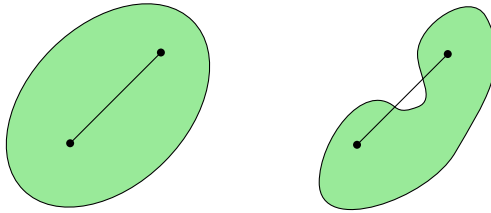


Figure 1.3: Left: convex set. Right: non-convex set.

Definition 1.12. Let $K \subset \mathbb{R}^n$ be convex. A function $f : K \rightarrow \mathbb{R}$ is convex, if for all $x, y \in K$ and $\lambda \in [0, 1]$ we have:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \quad (1.4)$$

f is strict convex, if for all $x \neq y$ and $\lambda \in (0, 1)$ the above inequality is strict. The function f is called (strictly) concave, if $-f$ is (strictly) convex.

Theorem 1.13. Let $K \subset \mathbb{R}^n$ be convex, and let $f_1, f_2 : K \rightarrow \mathbb{R}$ be convex functions and let $\alpha > 0$. Then, the functions $\alpha f_1, f_1 + f_2$ and $\max\{f_1, f_2\}$ are convex over K .

Proof. Exercise. □

Differences, products and minima of convex functions are not always convex!

Definition 1.14. Let $K \subset \mathbb{R}^n$ be convex, and $f : K \rightarrow \mathbb{R}$. The set

$$\text{Epi}(f) = \{(x, \alpha) \in K \times \mathbb{R} : f(x) \leq \alpha\}$$

is the epigraph of f . For $\beta \in \mathbb{R}$, we term the set

$$L(f, \beta) = \{x \in K : f(x) \leq \beta\}$$

lower level set of f with level β .

Theorem 1.15. Let $K \subseteq \mathbb{R}^n$ and $f : K \rightarrow \mathbb{R}$. Then:

1. f is convex $\Leftrightarrow \text{Epi}(f)$ is convex.
2. f is convex $\Rightarrow L(f, \beta)$ is convex for all $\beta \in \mathbb{R}$. The reverse need not be true.

Proof. Exercise. □

For convex differentiable functions we obtain the following characterization:

Theorem 1.16. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $f \in C^1$. Then:

1. f is convex over the convex set $K \subseteq \mathbb{R}^n$ iff for all $x, y \in K$:

$$f(y) \geq f(x) + \nabla f(x)^T(y - x). \quad (1.5)$$

2. f is strict convex $\Rightarrow (1.5)$ is strict for all $x \neq y \in K$.

Proof. We first show $\Leftarrow$ for the first statement. Assume (1.5) holds for all $x, y \in K$. Choose arbitrary $x, y \in K$ and $\lambda \in (0, 1)$. With convexity of K we get

$$z = \lambda x + (1 - \lambda)y \in K. \quad (1.6)$$

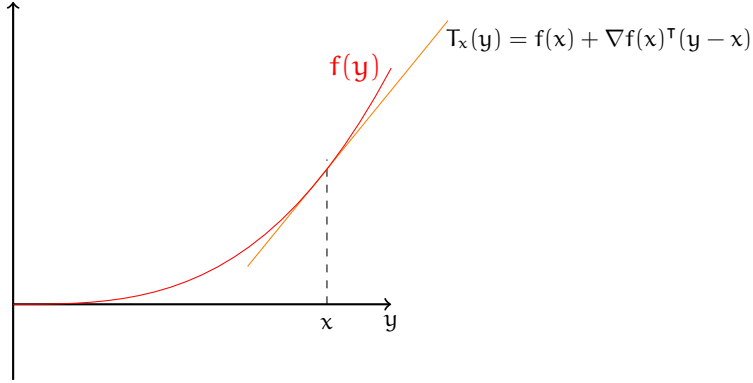


Figure 1.4: Illustration of inequality 1.5. $T_x(y)$ represents the tangent plane of f in x and we have $T_x(y) \leq f(y)$ for all $y \in K$.

With (1.5), we get for $x, y, z \in K$:

$$f(x) \geq f(z) + (x - z)^T \nabla f(z) \quad (1.7)$$

$$f(y) \geq f(z) + (y - z)^T \nabla f(z). \quad (1.8)$$

Multiply (1.7) with λ and (1.8) with $(1 - \lambda)$, add both inequalities and obtain:

$$\begin{aligned} \lambda f(x) + (1 - \lambda)f(y) &\geq f(z) + \left((\lambda(x - z) + (1 - \lambda)(y - z)) \right)^T \nabla f(z) \\ &= f(z) + \left(\lambda x + (1 - \lambda)y - z \right)^T \nabla f(z) \\ &= f(z). \end{aligned}$$

With (1.6), the second expression of the second equation is 0. Thus, f is convex.

$\Rightarrow$: Let f be convex. We choose $x, y \in K$ and define $\psi : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\psi(\lambda) = (1 - \lambda)f(x) + \lambda f(y) - f((1 - \lambda)x + \lambda y).$$

With convexity of f we get for all $\lambda \in [0, 1]$ that $\psi(\lambda) \geq 0$. Moreover $\psi(0) = 0$. We compute the derivative of ψ at 0 and get

$$0 \leq \lim_{t \rightarrow 0+} \frac{\psi(t) - \psi(0)}{t} = \dot{\psi}(0) = -f(x) + f(y) - \nabla f(x)^T(y - x).$$

The second statement is easy. □

We obtain a sufficient optimality criterion for convex optimization problems.

Theorem 1.17. Let $K \subset \mathbb{R}^n$ be convex and let $f : K \rightarrow \mathbb{R}$ be a differentiable convex function. Then, every local minimum of f over K is also a global Minimum.

Proof. Let x^* be a local minimum. With Theorem 1.8, we get for every $d \in D_K(x^*)$ the condition $\nabla f(x^*)^T d \geq 0$. Because K is convex, for any $y \in K$, we get $x^* + \lambda(y - x^*) =$

$\lambda y + (1 - \lambda)x^* \in K$ for all $\lambda \in [0, 1]$. Hence, $y - x^* \in D_K(x^*)$. We get

$$f(y) \geq f(x^*) + \nabla f(x^*)^T(y - x^*) \geq f(x^*),$$

where the first inequality follows from Theorem 1.16 and the second one from the variational inequality. $\square$

For unrestricted convex problems, we get the following implication.

Corollary 1.18. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable convex function. Then, every $x^* \in \mathbb{R}^n$ with $\nabla f(x^*) = 0$ is a global minimum of the associated unrestricted optimization problem.

Chapter 2

Convexity and Separating Hyperplanes

2.1 Convex Sets and Cones

Definition 2.1. 1. For $M \subset \mathbb{R}^n$ we define

$$\text{co}(M) := \cap \{K \supset M \mid K \text{ convex}\}$$

as the convex hull of M . For $x^0, \dots, x^k \in \mathbb{R}^n$ we define

$$\text{co}(x^0, \dots, x^k) = \text{co}(\{x^0, \dots, x^k\}).$$

This set is known as the simplex spanned by the points $x^0, \dots, x^k$. If $x^1 - x^0, \dots, x^k - x^0$ are linearly independent, then the simplex is non-degenerate.

2. A subset $K \subset \mathbb{R}^n$ is a cone (pointed at 0), if for all $x \in K$ the half-ray through x lies in K , i.e.

$$\alpha x \in K \text{ for all } \alpha \geq 0.$$

3. Let $K \subset \mathbb{R}^n$ and $x \in K$. The cone

$$K(x) := \{\alpha(y - x) \mid y \in K, \alpha > 0\} = \bigcup_{\alpha > 0} \alpha(K - x)$$

is termed the conic hull of K w.r.t. x .

See Fig. 2.1 for an illustration.

Definition 2.2. Let $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$.

1. The set $H = \{x \in \mathbb{R}^n : a^T x = b\}$ is called hyperplane.
2. The sets $H^- = \{x \in \mathbb{R}^n : a^T x \leq b\}$ and $H^+ = \{x \in \mathbb{R}^n : a^T x \geq b\}$ are Halfspaces.
3. Let A be a real-valued $m \times n$ matrix and $b \in \mathbb{R}^m$.
 $K = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is a polyhedron and $K = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ is a polyhedron in standard form.

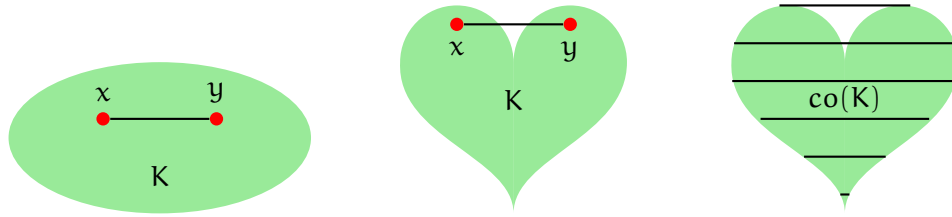


Figure 2.1: The first set is convex. The second heart is non-convex and the dashed set represents the convex hull.

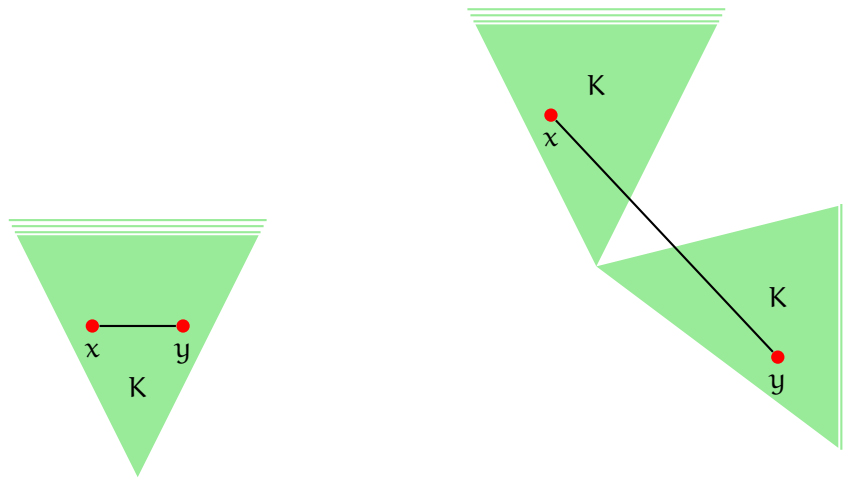


Figure 2.2: Left: convex cone. Right: non-convex cone.

2.2 Convex Combinations

Definition 2.3. Let $x^1, \dots, x^k \in \mathbb{R}^n$ und $\lambda_1, \dots, \lambda_k \in \mathbb{R}_{\geq 0}$ with $\lambda_1 + \dots + \lambda_k = 1$.

- The vector $\sum_{i=1}^k \lambda_i x^i$ is called convex combination of $x^1, \dots, x^k$.

Theorem 2.4.

1. The intersection of convex sets is convex.
2. Every polyhedron is convex.
3. A convex combination of a finite points of a convex set lies in the respective set.
4. The convex hull of a set $K \subset \mathbb{R}^n$ is the set of all finitely generated convex combinations of points in K . The set of convex combinations of a finite point set is convex.

Proof. (1): Let $X_i, i \in I$ be convex sets and define $X := \bigcap_{i \in I} X_i$. For $x, y \in X$ we have $x, y \in X_i$ for all $i \in I$, hence $\lambda x + (1 - \lambda)y \in X_i$ for all $i \in I$ and, hence, $\lambda x + (1 - \lambda)y \in X$.

(2): A polyhedron is the intersection of finitely many convex halfspaces.

(3): We prove via induction over k , that every convex combination of k points in X lies in X . For $k = 1$ the statement is trivial and for $k = 2$ the statement follows from the convexity of X . For the step $k - 1 \rightarrow k$ consider a convex combination $\mu_1 x^1 + \dots + \mu_k x^k$. If $\mu_i = 0$ for some $i \in \{1, \dots, k\}$ we can use the induction hypothesis, hence, we can assume w.l.o.g. that $\mu_k \in (0, 1)$. Define

$$v_1 = \frac{\mu_1}{1 - \mu_k}, \dots, v_{k-1} = \frac{\mu_{k-1}}{1 - \mu_k} \geq 0, \sum_{l=1}^{k-1} v_l = 1.$$

Set

$$y := \sum_{l=1}^{k-1} v_l x^l$$

and observe that $y \in X$ follows by the induction hypothesis. We get

$$x = (1 - \mu_k)y + \mu_k x^k \in X,$$

because the case $k = 2$ was shown already.

(4): Let L be the set of convex combinations of points in K .

$L \subseteq \text{co}(K)$: With (1) the set $\text{co}(K)$ is convex, hence, all convex combinations of points in $\text{co}(K)$ lie again in $\text{co}(K)$. With the definition of the convex hull, we get $K \subseteq \text{co}(K)$, thus, $L \subseteq \text{co}(K)$.

$\text{co}(K) \subseteq L$: Let $x, y \in L$ with

$$x = \sum_{i=1}^k \alpha_i x^i \text{ and } y = \sum_{j=1}^l \beta_j y^j, \text{ where } \alpha_i, \beta_j \geq 0, \sum_{i=1}^k \alpha_i = 1, \sum_{j=1}^l \beta_j = 1.$$

For $\lambda \in [0, 1]$ we get

$$z = \lambda x + (1 - \lambda)y = \sum_{i=1}^k \lambda \alpha_i x^i + \sum_{j=1}^l (1 - \lambda) \beta_j y^j,$$

and thus $z \in L$ because

$$0 \leq \lambda \alpha_i \leq 1 \forall i, \quad 0 \leq (1 - \lambda) \beta_j \leq 1 \forall j \text{ and}$$

$$\sum_{i=1}^k \lambda \alpha_i + \sum_{j=1}^l (1 - \lambda) \beta_j = \lambda \sum_{i=1}^k \alpha_i + (1 - \lambda) \sum_{j=1}^l \beta_j = \lambda + 1 - \lambda = 1,$$

which shows that z is a convex combination of points in K . Thus L is convex. Obviously $K \subseteq L$, since every $x^p \in K$ can be written as

$$x^p = \sum_{j \in J} \lambda_j x^j \text{ with } \lambda_p = 1 \text{ and } \lambda_j = 0 \text{ for } i \neq p.$$

Per definition we get $\text{co}(K) \subseteq L$, because L is a convex set containing K . $\square$

Corollary 2.5. The set of convex combinations of $x^1, \dots, x^k \in \mathbb{R}^n$ is the smallest (w.r.t. inclusion) convex subset of $\mathbb{R}^n$, which contains $x^1, \dots, x^k$.

Proof. Let X be the set of convex combinations of $x^1, \dots, x^k \in \mathbb{R}^n$. Define

$$Y := \text{co}(x^1, \dots, x^k) = \bigcap_{\substack{C \subseteq \mathbb{R}^n \\ C \text{ convex} \\ \{x^1, \dots, x^k\} \subset C}} C. \quad (2.1)$$

Y is well-defined because $\mathbb{R}^n$ is one candidate C . With Theorem 2.4(1.) Y is convex as intersection of convex sets. Y is also the smallest convex set containing $x^1, \dots, x^k$. Since X is convex (see Theorem 2.4(4.)) we get $Y \subseteq X$. Let $x \in X$. With the definition of X we get $x = \sum_{i=1}^k \lambda_i x^i$. As by assumption $x^1, \dots, x^k \in Y$ we get with Theorem 2.4(3.), that $x \in Y$. $\square$

Theorem 2.6 (Carathéodory). For $K \subset \mathbb{R}^n$, $\text{co}(K)$ is equal to the set of all convex combinations which require at most $(n + 1)$ points of K .

Proof. Let $x \in \text{co}(K)$. With Theorem 2.4 (4) there are $x^1, \dots, x^k \in K$ with

$$x = \sum_{i=1}^k \lambda_i x^i \text{ mit } \lambda_i \geq 0 \text{ for all } i = 1, \dots, k \text{ and } \sum_{i=1}^k \lambda_i = 1.$$

If $k \leq n + 1$ we are done. If $k > n + 1$, then we show that for the representation of x we can ignore one of the k points: Define the $(k - 1)$ vectors $y^i = x^i - x^k, i = 1, \dots, k - 1$. For $k > n + 1$, the points y^i are linearly dependent, i.e., there are $\alpha_1, \dots, \alpha_{k-1}$ with $\alpha_j \neq 0$ for at least one $j \in \{1, \dots, k - 1\}$ with

$$\begin{aligned} \sum_{i=1}^{k-1} \alpha_i y^i &= 0 \\ \Leftrightarrow \sum_{i=1}^{k-1} \alpha_i (x^i - x^k) &= 0 \\ \Leftrightarrow \sum_{i=1}^{k-1} \alpha_i x^i + \left(-\sum_{i=1}^{k-1} \alpha_i\right) x^k &= 0. \end{aligned}$$

With $\alpha_k = -\sum_{i=1}^{k-1} \alpha_i$ we get

$$\sum_{i=1}^k \alpha_i x^i = 0 \text{ and } \sum_{i=1}^k \alpha_i = 0.$$

Because $\alpha_j \neq 0$ for at least one $j \in \{1, \dots, k - 1\}$, the following value is well-defined:

$$i_0 = \arg \min_{i \in \{1, \dots, k\}} \left\{ \frac{\lambda_i}{\alpha_i} \mid \alpha_i > 0 \right\} = \frac{\lambda_{i_0}}{\alpha_{i_0}}.$$

We get

$$\lambda_i - \frac{\lambda_{i_0}}{\alpha_{i_0}} \alpha_i \geq 0 \quad \forall i \quad \text{and} \quad \sum_{i=1}^k \lambda_i - \frac{\lambda_{i_0}}{\alpha_{i_0}} \alpha_i = 1.$$

Moreover

$$x = \sum_{i=1}^k \lambda_i x^i = \sum_{i=1}^k \left(\lambda_i x^i - \frac{\lambda_{i_0}}{\alpha_{i_0}} \alpha_i x^i \right) = \sum_{i=1}^k \left(\lambda_i - \alpha_i \frac{\lambda_{i_0}}{\alpha_{i_0}} \right) x^i.$$

Here, we have $\lambda_{i_0} - \alpha_{i_0} \frac{\lambda_{i_0}}{\alpha_{i_0}} = 0$, and, hence, x can be represented as a convex combination of at most $k - 1$ points. $\square$

2.3 Separating Hyperplanes

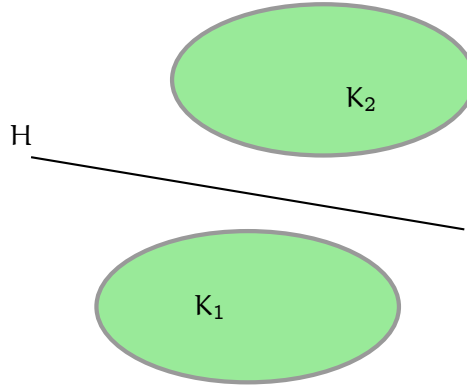


Figure 2.3: Illustration of the (strict) separation of two disjoint convex sets.

Definition 2.7. Two sets $K_1, K_2 \subset \mathbb{R}^n$ are called separable, if there are $c \in \mathbb{R}$ and row vector $\lambda \in \mathbb{R}^n, \lambda \neq 0$ with

$$\lambda x \leq c \leq \lambda y \quad \text{for all } x \in K_1, y \in K_2.$$

The hyperplane $H = \{x \in \mathbb{R}^n \mid \lambda x = c\}$ is called separating hyperplane (cf. Fig. 2.3); The sets K_1, K_2 are strictly separable via H , if $K_1 \cup K_2$ is not contained in H . The hyperplane H defines two halfspaces

$$H^+ = \{x \in \mathbb{R}^n \mid \lambda x \geq c\}, \quad H^- = \{x \in \mathbb{R}^n \mid \lambda x \leq c\}.$$

The sets K_1, K_2 are separable, if either $K_1 \subseteq H^+, K_2 \subseteq H^-$ or $K_1 \subseteq H^-, K_2 \subseteq H^+$.

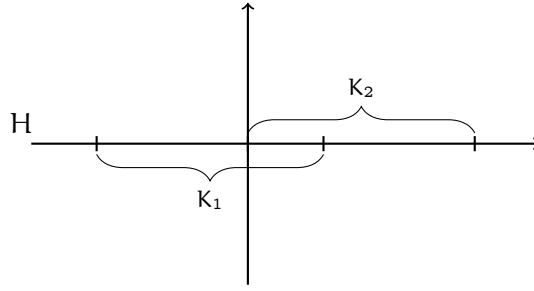


Figure 2.4: The two sets $K_1, K_2 \subset \mathbb{R}^2$ are separable but not strictly. The separating hyperplane is given as $H = \{x \in \mathbb{R}^2 | x_2 = 0\}$.

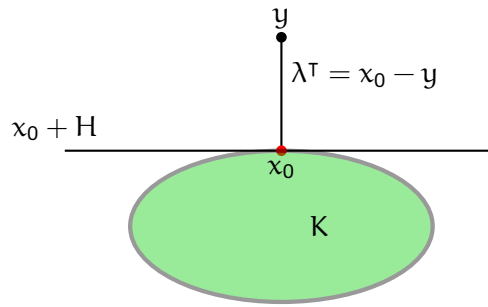


Figure 2.5: Illustration of the proof of Theorem 2.8. If $x_0 = y$ then $H + y$ is a separating supporting hyperplane.

Theorem 2.8. Let $K \subset \mathbb{R}^n$ be a non-empty, convex set and let $y \notin \overset{\circ}{K} := \text{int}(K)$. Then, $\{y\}$ and K are separable, i.e., there is a row vector $\lambda \in \mathbb{R}^n \setminus \{0\}$ with

$$\lambda y \leq \lambda x \text{ for all } x \in K.$$

If $\overset{\circ}{K} \neq \emptyset$ then $\{y\}$ and $\overset{\circ}{K}$ are strictly separable and we get

$$\lambda y < \lambda x \text{ for all } x \in \overset{\circ}{K}.$$

Proof. 1. Case: $y \notin \bar{K}$, where $\bar{K}$ denotes the topological closure of K . With $\|x\|$ we denote as usual the Euklidian norm. Set

$$d := \inf_{x \in \bar{K}} \|x - y\| > 0.$$

The function $f(x) := \|y - x\|$ is continuous and attains on $\bar{K} \cap \{x \in \mathbb{R}^n | \|x - y\| \leq 2d\}$ its minimum (Theorem of Weierstrass). As $\bar{K}$ is closed, there is $x_0 \in \bar{K}$ with $d = \|y - x_0\|$. With convexity of $\bar{K}$ one can further show that the point x_0 is unique (cf. Fig. 2.5).

We show $\lambda := (x_0 - y)^T \neq 0$ satisfies the conditions of the theorem. Let $x \in K$. With convexity of $\bar{K}$ we get

$$x_0 + \alpha(x - x_0) \in \bar{K} \text{ für } 0 \leq \alpha \leq 1.$$

Hence,

$$\|x_0 + \alpha(x - x_0) - y\|^2 \geq \|x_0 - y\|^2$$

and therefore

$$2\alpha(x_0 - y)^T(x - x_0) + \alpha^2 \|x - x_0\|^2 \geq 0.$$

Division by $\alpha > 0$ yields for $\alpha \downarrow 0$

$$(x_0 - y)^T(x - x_0) \geq 0$$

and therefore we get using $\lambda^T = (x_0 - y)$ and $d = \|\lambda\|$

$$\lambda x \geq \lambda x_0 = \lambda y + d^2 > \lambda y.$$

Thus, $H := \{x \in \mathbb{R}^n | \lambda x = \lambda x_0\}$ is a hyperplane separating $\{y\}$ and K .

2. Case: $y \in \partial K = \bar{K} - \overset{\circ}{K}$.

For $y \in \partial K$ there is a sequence $\{y_k\}$, $y_k \notin \bar{K}$, with $y = \lim_{k \rightarrow \infty} y_k$. For y_k we can choose according to Case 1. a row vector $\lambda_k \neq 0$ with

$$\lambda_k y_k \leq \lambda_k x \text{ for all } x \in K.$$

W.l.o.g. we can set $\|\lambda_k\| = 1$ and hence we can assume that the bounded sequence $\{\lambda_k\}$ converges with $\lambda = \lim_{k \rightarrow \infty} \lambda_k$, $\|\lambda\| = 1$. Taking the limit on both sides yields

$$\lambda y \leq \lambda x \text{ for all } x \in K.$$

The statement

$$\lambda y < \lambda x \text{ for all } x \in \overset{\circ}{K}$$

follows immediately. □

In case $y \in \partial K$ we call the hyperplane supporting. From the separating hyperplane theorem 2.8 we get:

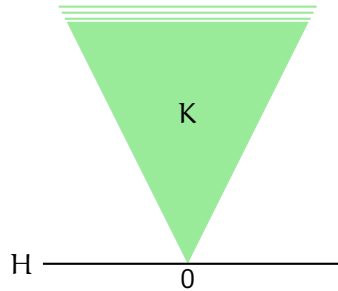


Figure 2.6: Separation of a convex cone via a hyperplane through 0.

Theorem 2.9. Let $K \subset \mathbb{R}^n$ be a non-empty convex and closed cone and suppose $y \notin K$. Then, there is $\lambda \in \mathbb{R}^n \setminus \{0\}$ with

$$\lambda y < 0 \leq \lambda x \text{ for all } x \in K.$$

Proof. With $\bar{K} = K$ there is – using the first statement of Theorem 2.8 – a row vector $\lambda \in \mathbb{R}^n \setminus \{0\}$ such that for all $x \in K$ we have $\lambda y < \lambda x$. From $0 \in K$ we get $\lambda y < \lambda 0 = 0$. Suppose there is $x \in K$ with $\lambda x < 0$. Then,

$$\lambda(\alpha x) = \alpha(\lambda x) \rightarrow_{\alpha \rightarrow \infty} -\infty,$$

contradicting boundedness of λK via λy from below. Thus, we get

$$\forall x \in K : \lambda y < 0 \leq \lambda x.$$

□

We get as an implication the following theorem of the alternatives:

Theorem 2.10 (Lemma of Farkas). Let B a $k \times n$ matrix and $d \in \mathbb{R}^k$. Then, exactly one of the following statements is true

1. $Bx = d, x \geq 0$ admits a solution $x \in \mathbb{R}^n$.
2. $\lambda B \geq 0, \lambda d < 0$ admits a solution $\lambda \in \mathbb{R}^k$.

Proof. The cone

$$K := \{Bx | x \geq 0\} \subset \mathbb{R}^k$$

non-empty convex and closed. Exactly one of the statements is true

(a) $d \in K$.

(b) $d \notin K$.

Statement (a) is statement (1) of the theorem. In case (b) we get with Theorem 2.8 the existence of some $\lambda \in \mathbb{R}^k$ with

$$\lambda d < 0 \leq \lambda z \text{ for all } z \in K.$$

Also $\lambda Bx \geq 0$ for all $x \geq 0$, i.e., $\lambda B \geq 0$. This is Statement (2) of the theorem. Note that (a) and (1) (and (b) and (2)) are equivalent, and, thus, (1) and (2) cannot be true simultaneously. □

Chapter 3

Introduction to Linear Optimization

3.1 Examples

3.1.1 Production Models

A company produces n products $P_1, \dots, P_n$, and for the production process, m activities $A_1, \dots, A_m$ (workers, materials, etc.) are needed. Product P_j requires a_{ij} units of resource A_i and yields a net gain of c_j euros per unit. For activity A_i there is an upper bound of b_i . The production amount x_j of product P_j should be determined in order to maximize net gain:

$$z(x) = \sum_{j=1}^n c_j x_j$$

subject to:

$$\sum_{j=1}^n a_{ij} x_j \leq b_i, \text{ for all } i = 1, \dots, m$$

$$x_j \geq 0, \text{ for all } j = 1, \dots, n.$$

Example 3.1. A shoe factory produces two types of products. There are 40 employees and 10 machines. The working time budget and material budget are shown in Table 3.1: With the decision variables

Table 3.1: Parameters.

	Type 1	Type 2	available
Production time [h]	20	10	8000
Machine hours [h]	4	5	2000
Material supply [dm ²]	6	15	4500
Net gain [EUR]	16	32	—

x_1 : amount type 1 x_2 : amount type 2

we get:

$$\text{maximize } z(x_1, x_2) = 16x_1 + 32x_2$$

s.t.:

$$20x_1 + 10x_2 \leq 8000$$

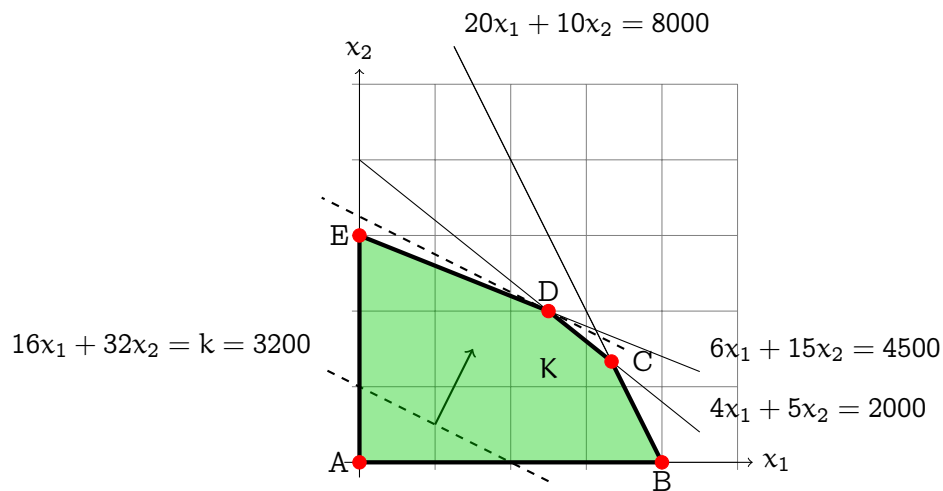
$$4x_1 + 5x_2 \leq 2000$$

$$6x_1 + 15x_2 \leq 4500$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

The level plane with level k is given by $z(x_1, x_2) = 16x_1 + 32x_2 = k$. One way of solving the problem is to determine the maximal k such that at least one point on $z(x_1, x_2) = k$ is feasible.



The function z attains its maximal value at vertex D of the feasible set K. The vertices of K are A, B, C, D, E:

Here, $y_1 = 8000 - 20x_1 - 10x_2$, $y_2 = 2000 - 4x_1 - 5x_2$, and $y_3 = 4500 - 6x_1 - 15x_2$ denote unused resources. One grid unit in the figure represents 100 units of production.

In this example, exactly two of the five variables are zero at each vertex (a non-degenerate basic solution). If we walk along vertices $B \rightarrow C \rightarrow D$ the value of z increases. The optimal vertex is D with objective value:

$$x_1 = 250, x_2 = 200, z(x_1, x_2) = 10400.$$

Table 3.2: Vertices of the feasible region of the production problem

Vertex	z	x_1	x_2	y_1	y_2	y_3
A	0	0	0	8000	2000	4500
B	6400	400	0	0	400	2100
C	9600	1000/3	400/3	0	0	500
D	10400	250	200	1000	0	0
E	9600	0	300	5000	500	0

3.2 Mathematical Formulation of Linear Optimization Problems

The linear optimization (LP) in **standard form** is given by

$$\begin{aligned}
 &\text{minimize } z(x_1, \dots, x_n) = \sum_{j=1}^n c_j x_j \\
 &\text{s.t.:} \\
 &\sum_{j=1}^n a_{ij} x_j = b_i, 1 \leq i \leq m \\
 &x_j \geq 0, j = 1, \dots, n
 \end{aligned}$$

In vector notation

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, c = (c_1, \dots, c_n) \in \mathbb{R}^n, b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m, A = (a_{ik}) \text{ } m \times n \text{ matrix}$$

we get

$$\begin{aligned}
 &\text{minimize } z(x) = cx \\
 &\text{s.t.} \\
 &Ax = b \\
 &x \geq 0.
 \end{aligned} \tag{3.1}$$

3.3 Reduction to Standard Form

Suppose we have an LP that contains inequalities or free variables.

3.3.1 Inequalities

Let A be an $m \times n$ matrix (not necessarily $m < n$).

$$\begin{aligned} &\text{minimize } z(x) = cx \\ &\text{s.t.:} \\ &\quad Ax \leq b \\ &\quad x \geq 0. \end{aligned} \tag{3.2}$$

Define slack-variables

$$y := b - Ax \in \mathbb{R}^m.$$

Then, $Ax \leq b$ is equivalent to

$$Ax + y = b, \quad y \geq 0.$$

With $\tilde{c} := (c, 0) \in \mathbb{R}^{n+m}$, $\tilde{x} := \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+m}$, I being the $m \times m$ identity matrix and the $m \times (n + m)$ -matrix $\tilde{A} := (A|I)$ we get that (3.2) is equivalent to the standard form formulation

$$\begin{aligned} &\text{minimize } \tilde{z}(\tilde{x}) = \tilde{c}\tilde{x} \\ &\text{s.t.:} \\ &\quad \tilde{A}\tilde{x} = b \\ &\quad \tilde{x} \geq 0. \end{aligned} \tag{3.3}$$

3.3.2 Free Variable

In standard form (3.1) let some x_i have no sign constraint, e.g. x_1 .

$$\begin{aligned} &\text{minimize } z(x) = cx \\ &\text{s.t.:} \\ &\quad Ax = b \\ &\quad x_2 \geq 0, \dots, x_n \geq 0. \end{aligned} \tag{3.4}$$

1. Method: Elimination of x_1

If the first column of A is nonzero, choose an index i with $a_{i1} \neq 0$ and eliminate from

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

the variable x_1 as linear combination of $x_2, \dots, x_n$. Substitute this expression into the objective as well; retain the resulting constant when recovering the original optimal value. If the first column is zero, discard x_1 when $c_1 = 0$. When $c_1 \neq 0$, the problem is unbounded whenever the remaining constraints are feasible. The second method works in all cases. When elimination is possible, it yields the reduced system

$$\tilde{A} \begin{pmatrix} x_2 \\ \vdots \\ x_n \end{pmatrix} = \tilde{b} \in \mathbb{R}^{m-1}.$$

2. Method:

Set $x_1 = u_1 - v_1$, $u_1 := \max\{0, x_1\} \geq 0$, $v_1 := \max\{0, -x_1\} \geq 0$. This yields an LP with $n + 1$ variables $u_1, v_1, x_2, \dots, x_n$.

Chapter 4

Theory of Polyhedra

We will study fundamental elements of the theory of polyhedra including vertices, faces and valid inequalities.

The set

$$K := \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$$

is a polyhedron in standard form. The description of K is given via matrices A, b , hence, we write $P^=(A, b) := K$. Similarly, for

$$K := \{x \in \mathbb{R}^n | Ax \leq b\}$$

we write $P(A, b) := K$.

| **Definition 4.1.** A bounded polyhedron is a polytope.

4.1 Faces of Polyhedra

We consider in the following polyhedra of the form $P(A, b)$, where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$.

| **Definition 4.2.** Let $K \subseteq \mathbb{R}^n, a \in \mathbb{R}^n, \alpha \in \mathbb{R}$.

1. The inequality $a^T x \leq \alpha$ is called valid w.r.t. K , if

$$K \subseteq \{x \in \mathbb{R}^n | a^T x \leq \alpha\}.$$

2. The hyperplane $H = \{x \in \mathbb{R}^n | a^T x = \alpha\}, a \neq 0$, is called supporting hyperplane of K , if $a^T x \leq \alpha$ is valid w.r.t. K and $K \cap H \neq \emptyset$.

Note that $a = 0$ in the first statement of Definition 4.2 is feasible.

| **Definition 4.3.** Let $K \subset \mathbb{R}^n$. A set $F \subseteq K$ is called a face of K , if there is a valid inequality w.r.t. K of the form $d^T x \leq \delta$ such that

$$F = K \cap \{x \in \mathbb{R}^n | d^T x = \delta\}.$$

A face is proper, if $F \neq K$. F is called non-trivial, if $\emptyset \neq F \neq K$.

| Is $d^T x \leq \delta$ valid w.r.t. K , then $K \cap \{x \in \mathbb{R}^n | d^T x = \delta\}$ is called **induced face** by $d^T x \leq \delta$.

Note again that $d = 0$ in Definition 4.3 is allowed.

Example 4.4. We give an example.

$$\begin{aligned} 2x_1 + x_2 &\leq 8 \\ 4x_1 + 5x_2 &\leq 20 \\ 2x_1 + 5x_2 &\leq 15 \\ x_1, x_2 &\geq 0 \end{aligned}$$

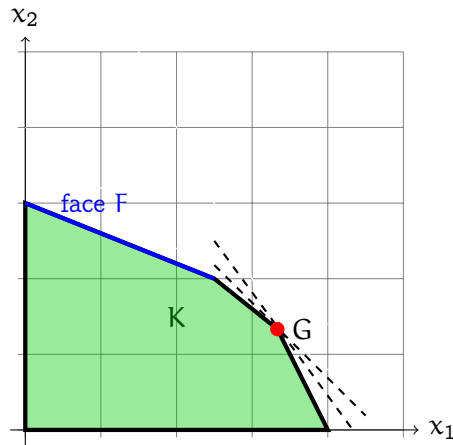


Figure 4.1: Example of a face.

The face F is induced by

$$2x_1 + 5x_2 \leq 15,$$

because

$$F = K \cap \{x \in \mathbb{R}_+^2 | 2x_1 + 5x_2 = 15\}.$$

Moreover, $G = \{(\frac{10}{3}, \frac{4}{3})\}$ is a face obtained by imposing equality in both of the following valid inequalities simultaneously (see Proposition 4.8):

$$\begin{aligned} 2x_1 + x_2 &\leq 8 \\ 4x_1 + 5x_2 &\leq 20. \end{aligned}$$

Note that $G = \{(\frac{10}{3}, \frac{4}{3})\}$ as a face of K can also be induced via (see Fig. 4.1)

$$\begin{aligned} 4x_1 + 3x_2 &\leq 52/3 \\ x_1 + x_2 &\leq 14/3 \end{aligned}$$

Proposition 4.5. Let $K = P(A, b) \subset \mathbb{R}^n$ be a polyhedron. Then, the following statements hold:

1. K is a face of itself.
2. $\emptyset$ is a face of K .
3. If $F = \{x | d^T x = \delta\} \cap K$ is a non-trivial face of K , then $d \neq 0$.

Proof. (1): $K = K \cap \{x \in \mathbb{R}^n | 0^T x = 0\}$.

(2): Let $\delta > 0$ be arbitrary. We have $0^T x \leq \delta$ is a valid inequality for K and we get $\emptyset = K \cap \{x \in \mathbb{R}^n | 0^T x = \delta\}$.

(3): For $d = 0$ the case $\delta \geq 0$ yields a valid inequality of the form $d^T x \leq \delta$ and, hence, one of the first two cases applies. $\square$

Theorem 4.6. Let $K = P(A, b)$ be a non-empty polyhedron and $c^T \in \mathbb{R}^n$. Consider the LP

$$\min\{cx | x \in K\}.$$

Let F^* be the solution set and in case $F^* \neq \emptyset$ let $z^* = \min\{cx | x \in K\}$. Then:

1. If $F^* \neq \emptyset$, then $F^* = \{x \in K | cx = z^*\}$ is a non-empty face of K and if $c \neq 0$, the set $\{x \in \mathbb{R}^n | cx = z^*\}$ is a supporting hyperplane of K .
2. The set of optimal solutions of $\min\{cx | x \in K\}$ is a face of K .

Proof. For (1):

Let $F^* \neq \emptyset$. With the definition of z^* we get $cx \geq z^*$ for all $x \in K$, thus the inequality $-cx \leq -z^*$ is valid for K . Moreover, $F^* = \{x \in \mathbb{R}^n | cx = z^*\} \cap K$, implying that F^* is a non-empty face of K . We get immediately that $\{x \in \mathbb{R}^n | cx = z^*\}$ is a supporting hyperplane in case $c \neq 0$.

For (2): For $F^* = \emptyset$ the statement is clear, otherwise we get that (2) is just a reformulation of (1). $\square$

An illustration of the above theorem appears in Fig. 4.2.

For $A \in \mathbb{R}^{m \times n}$ with row index set $M := \{1, \dots, m\}$ and column index set $N = \{1, \dots, n\}$ we denote by $a_i, i \in M$ the rows and by $a^j, j \in N$ the columns of A . For $I \subseteq M$ we denote by A_I the submatrix consisting of the rows $a_i, i \in I$.

Definition 4.7. Let $K = P(A, b) \subset \mathbb{R}^n$ and M the row index set of A . For $F \subseteq K$ let

$$eq(F) := \{i \in M | a_i x = b_i \forall x \in F\},$$

i.e. $eq(F)$ consists of the inequalities that are tight at every point of F . For $I \subseteq M$ denote by

$$fa(I) := \{x \in K | A_I x = b_I\}$$

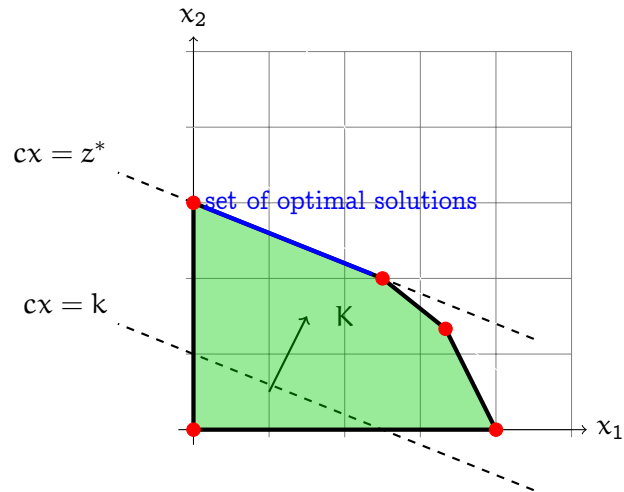


Figure 4.2: For the objective $\min(-2x_1 - 5x_2)$, the optimal face F illustrates Theorem 4.6.

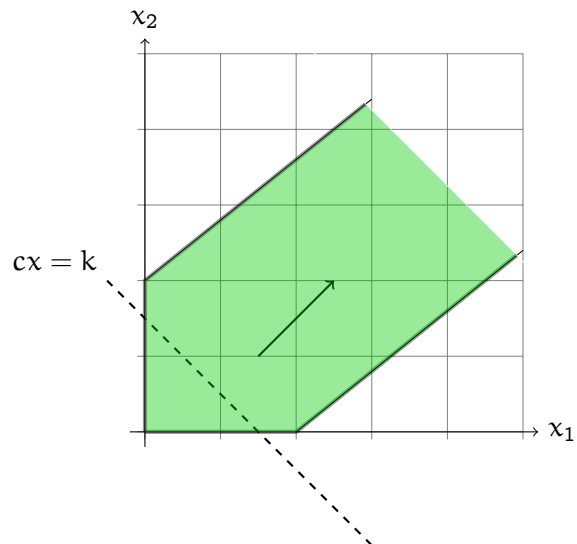


Figure 4.3: Minimizing $-x_1 - x_2$ over this set is unbounded below, so there is no optimal solution.



Figure 4.4: The black points of the polytopes are vertices. For a closed disk, every point on the boundary is a vertex.

| the induced face by valid inequalities corresponding to index set I .

We verify that indeed $\text{fa}(I)$ is a face of K .

| **Proposition 4.8.** The set $F := \text{fa}(I)$ defined in Definition 4.7 is a face of K .

Proof. If $I = \emptyset$, then $F = K$ is a trivial face of K . Let $|I| \geq 1$. Define

$$a^T := \sum_{i \in I} a_i \text{ and } \gamma := \sum_{i \in I} b_i.$$

We have that $a^T x \leq \gamma$ is a valid inequality and for $x \in K \setminus F$, at least one of the summed inequalities is strict, hence,

$$a^T x = \gamma \quad \text{for } x \in F, \quad a^T x < \gamma \quad \text{for } x \in K \setminus F.$$

We get $F = \{x \in K \mid a^T x = \gamma\} = K \cap \{x \in \mathbb{R}^n \mid a^T x = \gamma\}$. □

We consider again the Example 4.4. Here, we have $M = \{1, 2, 3, 4, 5\}$, $\text{fa}(\{1, 2\}) = G$ and $\text{eq}(G) = \{1, 2\}$.

4.2 Vertices and Extreme Points

Definition 4.9. Let $K \subset \mathbb{R}^n$.

1. $x \in K$ is called extreme point of K , if there are no two distinct points $y, z \in K$ with

$$x = \alpha y + (1 - \alpha)z \text{ for some } \alpha \in (0, 1).$$

2. $x \in K$ is called vertex of K , if $\{x\}$ is a 0-dimensional face of K .

Definition 4.9 works for any set $K \subset \mathbb{R}^n$.

| **Theorem 4.10.** Let $K = P(A, b) \subset \mathbb{R}^n$ be a polyhedron and $x \in K$. Then, the following is equivalent:

1. x is a vertex of K .

2. $\{x\}$ is a 0-dimensional face of K .
3. x is an extreme point of K .
4. $\text{rank}(A_{\text{eq}(\{x\})}) = n$.
5. There is $c^T \in \mathbb{R}^n \setminus \{0\}$, such that x is the unique solution to the LP $\min\{cy : y \in K\}$.

Proof. The statements (1) and (2) just correspond to the Definition 4.9 of a vertex. The proof works as follows: (2) $\Rightarrow$ (5), (5) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2).

(2) $\Rightarrow$ (5):

Per definition, x is a face, hence there is a valid inequality w.r.t. K $d^T y \leq \gamma$, such that $\{y \in K | d^T y = \gamma\} = \{x\}$.

Thus, x is the unique optimal solution to $\min\{cy : y \in K\}$ for $c := -d^T$.

If $K \neq \{x\}$, then $c \neq 0$ because of Proposition 4.5(3), otherwise we can choose $c \neq 0$ arbitrarily.

(5) $\Rightarrow$ (3):

Let x be the unique optimal solution to $\min\{cy | y \in K\}$ with value γ . If $x = \lambda w + (1 - \lambda)z$ for $w, z \in K, w \neq z, 0 < \lambda < 1$, then

$$\begin{aligned} \gamma &= cx = c(\lambda w + (1 - \lambda)z) \\ &= \lambda cw + (1 - \lambda)cz \\ &> \lambda\gamma + (1 - \lambda)\gamma = \gamma, \end{aligned}$$

contradiction.

(3) $\Rightarrow$ (4):

Suppose (4) does not hold. Then, there is $d \neq 0$ with $A_{\text{eq}(\{x\})}d = 0$. For small $\epsilon > 0$ we get

$$A(x \pm \epsilon d) \leq b.$$

With $y = x - \epsilon d$, and $z = x + \epsilon d$ we get $y, z \in K$ and $x = 1/2y + 1/2z$. Hence, we also get that (3) is not valid.

(4) $\Rightarrow$ (2):

We know that $\text{fa}(\text{eq}(\{x\}))$ is a face and with $\text{rank}(A_{\text{eq}(\{x\})}) = n$ we get

$$\text{fa}(\text{eq}(\{x\})) = \{y \in K | A_{\text{eq}(\{x\})}y = b_{\text{eq}(\{x\})}\} = \{x\}.$$

Thus, $\{x\}$ is a face and its dimension is 0. □

We get a similar result for polyhedra in standard form.

Theorem 4.11. Let $K = P^=(A, b)$, i.e.

$$K := \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}.$$

$x \in K$ is an extreme point or equivalently a vertex, if and only if the column vectors a^k of A that correspond to indices k with $x_k > 0$ are linearly independent.

Proof. $\Rightarrow$: Let $x \in K$ be a vertex. W.l.o.g.

$$x = (x_1, \dots, x_r, 0, \dots, 0)^T, x_i > 0, i = 1, \dots, r.$$

If $r = 0$, the column set is empty, and the set is linearly independent. For $r > 0$ we have

$$\sum_{i=1}^r a^i x_i = b.$$

Contra-position: suppose $a^1, \dots, a^r$ are linearly dependent. Then, there are scalars $d_1, \dots, d_r, d_i \neq 0$ for at least one i , with

$$\sum_{i=1}^r a^i d_i = 0.$$

If $x_i > 0$, then for $\epsilon > 0$ small enough, we get

$$x_i \pm \epsilon d_i > 0, \text{ for } i = 1, \dots, r.$$

We set

$$d := (d_1, \dots, d_r, 0, \dots, 0)^T, y := x + \epsilon d, z := x - \epsilon d.$$

Then, $y, z \geq 0$ and with

$$\sum_{i=1}^r a^i (x_i \pm \epsilon d_i) = \sum_{i=1}^r a^i x_i \pm \epsilon \underbrace{\sum_{i=1}^r a^i d_i}_{=0} = b,$$

we have $y, z \in K$. With $x \neq y, z$, and $x = \frac{y+z}{2}$ we get a contradiction that x is an extreme point of K . Hence, $a^1, \dots, a^r$ are linearly independent.

$\Leftarrow$: W.l.o.g. assume that the first r components of x are positive, and assume that $a^1, \dots, a^r$ are linearly independent.

1. Case: $r = 0 \Rightarrow x = 0$. If $x = 0$ is no extreme point, there are $y, z \in K, y \neq z$ and $0 < \alpha < 1$ with

$$0 = x = \alpha y + (1 - \alpha)z.$$

With $y, z \geq 0$ and $\alpha \neq 0$, we get $y = 0, z = 0$, contradiction.

2. Case: $r > 0$: Per definition we have $\sum_{i=1}^r a^i x_i = b$.

Contra-position: x is no extreme point of K . Then, there are $y, z \in K, y \neq z$, and $0 < \alpha < 1$ with

$$x = \alpha y + (1 - \alpha)z.$$

As in Case 1., we get

$$y_{r+1} = \cdots = y_n = 0, z_{r+1} = \cdots = z_n = 0.$$

Moreover,

$$Ay = Az = b \text{ hence } A(y - z) = 0 \Rightarrow \sum_{i=1}^r a^i(y_i - z_i) = 0$$

As $a^1, \dots, a^r$ are linearly independent, we get

$$y_i = z_i \text{ for } i = 1, \dots, r \Rightarrow y = z \Rightarrow x = y = z, \text{ contradiction.} \quad \square$$

Definition 4.12. A polyhedron is called pointed, if it contains a vertex.

We define terms like edge and line of a polyhedron.

Definition 4.13. A polyhedron $K \subset \mathbb{R}^n$ contains a line, if $x \in K$ and there is $d \in \mathbb{R}^n \setminus \{0\}$, such that

$$x + \lambda d \in K \text{ for all } \lambda \in \mathbb{R}.$$

An edge of K is a face of dimension 1. In a pointed polyhedron, an edge is either a segment joining two vertices or a ray starting at a vertex.

Theorem 4.14. Let $K = P(A, b) \subset \mathbb{R}^n$ be non-empty. The following statements are equivalent:

1. K is pointed.
2. $\text{rank}(A) = n$.
3. Every non-empty face of K is pointed.

Proof. (1) $\Rightarrow$ (2): Is x a vertex of K , then with Theorem 4.10 we get

$$n = \text{rank}(A_{\text{eq}(\{x\})}) \leq \text{rank}(A) \leq n.$$

Hence, $\text{rank}(A) = n$.

(2) $\Rightarrow$ (1) : We choose $x \in K$ such that the set $I = \text{eq}(\{x\})$ is inclusion maximal. Let

$$F = \{x \in K | A_I x = b_I\}.$$

If $\text{rank}(A_I) = n$, then we get with Thm. 4.10 that x is a vertex, hence, assume $\text{rank}(A_I) < n$. Then, the kernel of A_I contains $d \neq 0$ with

$$x \pm \epsilon d \in K, \text{ for } \epsilon \text{ small enough.}$$

The line $\{x + \lambda d | \lambda \in \mathbb{R}\}$ hits at least one of the hyperplanes $H_j = \{x \in \mathbb{R}^n | a_j x = b_j\}$ for some $j \notin I$. (Suppose not, then the entire line lies in K . Then,

$$a_i(x + \lambda d) \leq b_i, \text{ for all row indices } i, \text{ and all } \lambda \in \mathbb{R}.$$

This implies $Ad = 0$ and with $\text{rank}(A) = n$ we get $d = 0$, contradiction.) Hence, there is $\delta \in \mathbb{R}$ such that $x + \delta d \in K$ and $\text{eq}(\{x + \delta d\}) \supset I$, in contradiction to the maximality of I .

(3) $\Rightarrow$ (1): Is trivial as K is a face of itself.

(2) $\Rightarrow$ (3): Let $F = \{x \in K | dx = \delta\}$ be a nonempty face induced by a valid inequality $dx \leq \delta$. Adding $dx \leq \delta$ and $-dx \leq -\delta$ to $Ax \leq b$ gives an inequality representation of F . The augmented matrix still has rank n , so the equivalence already proved shows that F is pointed. $\square$

We get an important corollary for polyhedra in standard form.

Corollary 4.15. Let $K = P^=(A, b)$, then

$$K \neq \emptyset \Leftrightarrow K \text{ is pointed.}$$

Proof. We obtain a representation of K via

$$K = P^=(A, b) = P(D, d) \text{ with } D = \begin{pmatrix} A \\ -A \\ -I \end{pmatrix}, d = \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}.$$

D has rank n . From Thm. 4.14 the statement follows. $\square$

For polytopes we get a similar result.

Corollary 4.16. Let $K = P(A, b)$ be a polytope. Then,

$$K \neq \emptyset \Leftrightarrow K \text{ is pointed.}$$

Proof. As K is bounded, there is u with $K \subseteq \{x | x \leq u\}$. Thus we get a representation of K via

$$K = P(A, b) = P(D, d) \text{ with } D = \begin{pmatrix} A \\ I \end{pmatrix}, d = \begin{pmatrix} b \\ u \end{pmatrix}.$$

D has rank n . From Thm. 4.14 the statement follows. $\square$

Corollary 4.17. Let $K = P(A, b)$ be a pointed polyhedron and suppose the LP

$$\min cx \text{ s.t. } x \in K$$

| has a finite optimal solution. Then, the LP has an optimal solution that is a vertex.

Proof. $F = \{x \in K | cx = \min\{cy | y \in K\}\}$ is a non-empty face of K and contains using Thm. 4.14 a vertex. A vertex of F is also a vertex of K : in any nontrivial convex decomposition into points of K , optimality forces both points to belong to F . $\square$

Corollary 4.18. If K is a non-empty polytope, then, every LP of the form

$$\min cx \text{ s.t. } x \in K$$

has an optimal vertex solution.

We collect these results in the main theorem of linear programming.

Theorem 4.19. If the LP $\min\{cx \mid Ax = b, x \geq 0\}$ is feasible and its objective is bounded below, then it has an optimal vertex.

Proof. Let x be feasible. If the columns corresponding to $x_i > 0$ are dependent, choose $d \neq 0$ with $Ad = 0$ and $d_i = 0$ whenever $x_i = 0$. Orient d so that some $d_i > 0$ and $cd \geq 0$. If d has both signs, choose its sign according to cd . Otherwise, take $d \geq 0$; then $cd < 0$ would contradict boundedness along $x + td$, $t \geq 0$. Set

$$\mu = \min_{d_i > 0} \frac{x_i}{d_i}, \quad x' = x - \mu d.$$

Then x' is feasible, $cx' \leq cx$, and x' has fewer positive components. Repetition yields a vertex with no larger objective value by Theorem 4.11. There are only finitely many vertices, since each independent column set determines at most one solution on its support. The best of these vertices is therefore optimal. $\square$

Thm. 4.19 can be used to solve an LP in standard form by trying out all vertices.

Example 4.20.

$$\begin{aligned} \min & x_1 + 2x_2 + 3x_3 \\ \text{s.t.:} & \\ & 2x_1 + x_2 + 5x_3 = 5 \\ & x_1 + 2x_2 + x_3 = 4 \\ & x_i \geq 0, i = 1, 2, 3 \end{aligned}$$

Solution: For a vertex, one component is 0.

1. Possibility: $x_1 = 0$. Solve

$$\begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \Rightarrow x_2 = 5/3, x_3 = 2/3$$

Vertex: $(0, 5/3, 2/3)^T$, objective $16/3$.

2. Possibility: $x_2 = 0$. Solve

$$\begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \Rightarrow x_1 = 5, x_3 = -1$$

Infeasible.

3. Possibility: $x_3 = 0$. Solve

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \Rightarrow x_1 = 2, x_2 = 1$$

Vertex: $(2, 1, 0)^T$, objective 4.

The set K is bounded, hence the objective function attains at $(2, 1, 0)^T$ its minimum.

In contrast to exhaustive search (cf. Chapter 5), the simplex algorithm yields an algorithm that is often efficient in practice for finding an optimal vertex.

4.3 Basic Solutions

If $m = n$ and $\text{rank}(A) = n$, there is only one candidate $x = A^{-1}b$; it remains to test $x \geq 0$. The nontrivial simplex case is $m < n$.

We consider an LP in standard form:

$$\begin{aligned} \min z(x) &= cx \\ \text{s.t.:} \\ Ax &= b \\ x &\geq 0 \end{aligned}$$

Notation: $K := P^-(A, b) = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$.

We assume $\text{rank}(A) = m \leq n$. (Later we will see that this holds w.l.o.g.). We consider the linear equation system $Ax = b$, whose solution space is an $(n - m)$ -dimensional affine subspace. The columns of A are denoted by $a^j, j = 1, \dots, n$.

Definition 4.21. An index vector $B = (i_1, \dots, i_m)$ with m distinct indices $i_j \in \{1, \dots, n\}$ is called basis, if the corresponding column vectors are linearly independent. The complement vector to B is denoted by $N = (j_1, \dots, j_{n-m})$, $j_k \in \{1, \dots, n\}$ and is called non-basis. We have $B \oplus N = \{1, \dots, n\}$. With A_B and A_N we denote the submatrices, defined via column vectors corresponding to B and N :

$$\begin{aligned} A_B &: m \times m \text{ matrix with column vectors } a^i, i \in B \\ A_N &: m \times (n - m) \text{ matrix with column vectors } a^j, j \in N \end{aligned}$$

For such a subdivision of the set $\{1, \dots, n\}$ we write the set K as

$$(A_B, A_N) \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b, x_B, x_N \geq 0.$$

For B , we denote A_B as basis matrix and A_N as non-basis matrix. The variables $x_i, i \in B$ are called basic variables and the variables $x_j, j \in N$ are called non-basic variables. $x = (x_B, x_N)$ with

$$x_B := A_B^{-1}b \text{ and } x_N := 0$$

is termed basic solution w.r.t. basis B . A basic solution is feasible, if $x_B \geq 0$. A feasible basic solution is non-degenerate, if $x_B > 0$; if more than $n - m$ components of x are equal 0, we speak of a degenerate basic solution.

We obtain a characterization of basic solutions via Thm. 4.11:

Theorem 4.22. Let $K = P^=(A, b)$. Then, the following is equivalent:

1. $x \in K$ is an extreme point.
2. $x \in K$ is a vertex.
3. $x \in K$ is a feasible basic solution w.r.t. a basis B .

Proof. (1) $\Leftrightarrow$ (2) follows from Thm. 4.10.

(1) $\Rightarrow$ (3) : Let $x \in K$ be an extreme point. With Thm. 4.11 we obtain that all column vectors $a^j, j \in J$ with $J := \{i \in \{1, \dots, n\} : x_i > 0\}$ are linearly independent and they can be extended to a basis B . Per definition of J we get $x_N = 0$, where N denotes the non-basis w.r.t. B . Hence,

$$b = Ax = A_B x_B,$$

and using that A_B is invertible

$$x_B = A_B^{-1}b.$$

Thus, x is a feasible basic solution w.r.t. basis B .

(3) $\Rightarrow$ (1) : Let x be a feasible basic solution to basis B , i.e., $x = (x_B, x_N)$ with $x_N = 0$ and $x_B := A_B^{-1}b \geq 0$. The set of indices with positive entries of x is a subset of B . As the vectors $a^i, i \in B$ are linearly independent, the statement follows by Thm. 4.11 .

□

$$\text{vertex} \overset{\text{unique}}{\Leftrightarrow} \text{feasible basic solution} \overset{\text{non-unique}}{\Leftrightarrow} \text{feasible basis}.$$

We obtain a corollary on the number of vertices of a polyhedron K with $\text{rank}(A) = m \leq n$.

Corollary 4.23. There are at most $\binom{n}{m}$ distinct vertices of K .

Proof. Every vertex leads to a feasible basic solution with a basic matrix. Every basic matrix has m linearly independent columns of A and there are at most $\binom{n}{m}$ possibilities to choose m linearly independent columns from A . □

We close this chapter with showing that $\text{rank}(A) = m$ is w.l.o.g.

Theorem 4.24. Let $K = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ be a non-empty polyhedron in standard form with matrix $A \in \mathbb{R}^{m \times n}$. Let $\text{rank}(A) = k < m$ and suppose that the row vectors $a_{i_1}, \dots, a_{i_k}$ are linearly independent. Consider

$$P := \{x \in \mathbb{R}^n | a_{i_1}x = b_{i_1}, \dots, a_{i_k}x = b_{i_k}, x \geq 0\}.$$

Then, $K = P$.

Proof. W.l.o.g. $i_1 = 1, \dots, i_k = k$. Trivially $K \subseteq P$.

We show only $P \subseteq K$. With $\text{rank}(A) = k$ we that the row space of A has dimension k and the vectors $a_1, \dots, a_k$ form a basis of that space. Hence, every row a_i of A can be represented as $a_i = \sum_{j=1}^k \lambda_{ij} a_j$ for scalars λ_{ij} . Let $x \in K$. We have

$$b_i = a_i x = \sum_{j=1}^k \lambda_{ij} a_j x = \sum_{j=1}^k \lambda_{ij} b_j, \quad i = 1, \dots, m. \quad (4.1)$$

Let $y \in P$. For all $i = 1, \dots, m$ we get with (4.1)

$$a_i y = \sum_{j=1}^k \lambda_{ij} a_j y = \sum_{j=1}^k \lambda_{ij} b_j = b_i,$$

hence $y \in K$. □

4.4 Degeneracy of Basic Solutions

For a feasible basic solution, the m equations hold and at least $n - m$ nonnegativity constraints are active, since the nonbasic variables are zero. If more than $n - m$ variables are equal 0, we speak of a degenerate basic solution, see Definition 4.21.

We give an example.

Example 4.25. 1. Redundant variables:

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_3 &= 0 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

2. Redundant inequalities:

$$\begin{aligned} 2x_1 + x_2 + y_1 &= 3 \\ x_1 + 2x_2 + y_2 &= 3 \\ x_1 + x_2 + y_3 &= 2 \\ x_1, x_2, y_1, y_2, y_3 &\geq 0. \end{aligned}$$

In $x = (1, 1, 0, 0, 0)^T$ the ineq. $y_3 \geq 0$ is redundant.

3. geometric reasons (see octahedron in Fig. 4.5).

Remark 4.26. For fixed A of full row rank, perturbing b according to a distribution with a density makes every basic solution non-degenerate with probability one. Such a perturbation may change feasibility and the optimal solution of the original problem.

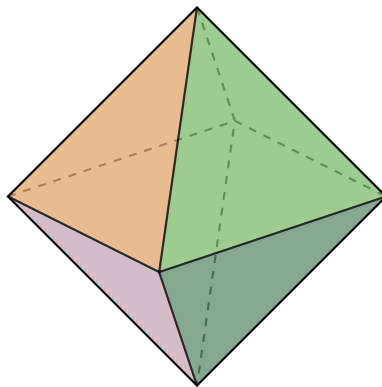


Figure 4.5: The vertices of the octahedron are degenerate.

Chapter 5

The Simplex Method

A fundamental method for solving an LP in standard form

$$\begin{aligned} &\text{minimize } z(x) = cx \\ &\text{s.t.} \\ &Ax = b \\ &x \geq 0. \end{aligned} \tag{5.1}$$

is the Simplex-Method. We assume $\text{rank}(A) = m < n$. With Thm. (4.19) we know that an optimal solution (if it exists) is a vertex of

$$K = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}.$$

The simplex-method consists of executing the following steps:

Description in terms of bases:

1. Find a feasible basis (Phase I), or establish infeasibility.
2. If all reduced costs satisfy $r_j \leq 0$, the current basic solution is optimal. Otherwise, choose $r_s > 0$.
3. If the ratio test has no leaving variable, the LP is unbounded. Otherwise, perform a feasible pivot and return to Step 2.

A non-degenerate step decreases the objective and moves to an adjacent vertex. A degenerate step may leave the vertex unchanged, so we keep track of bases as well as vertices.

5.1 Parametrization of the Solution Space

For basis B we get that the linear equation system $Ax = b$ can be represented as

$$Ax = A_B x_B + A_N x_N = b.$$

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This way, we obtain a parametrization of the $n - m$ dimensional solution space of $Ax = b$ via

$$x_B = A_B^{-1}b - A_B^{-1}A_N x_N, \quad x_N \in \mathbb{R}^{n-m}, \quad (5.2)$$

where x_B is the dependent variable and x_N denotes the independent variable. Subdivide c in $c_B \in \mathbb{R}^m$ and $c_N \in \mathbb{R}^{n-m}$. Inserting (5.2) into the objective leads to

$$\begin{aligned} z(x) &= cx = c_B x_B + c_N x_N \\ &= c_B A_B^{-1}b - (c_B A_B^{-1}A_N - c_N)x_N \\ &=: z_0 - r_N x_N, \end{aligned} \quad (5.3)$$

where

$$\begin{aligned} z_0 &:= c_B A_B^{-1}b \\ r_N &:= c_B A_B^{-1}A_N - c_N = (r_j)_{j \in N} \in \mathbb{R}^{n-m} \text{ (vector of reduced costs)} \end{aligned}$$

With Thm. 4.22 the feasible basic solutions describe the vertices of K . The representation (5.3) of the objective yields the following optimality criterion:

Theorem 5.1. Let B be a basis with

1. the corresponding solution x is feasible, i.e. $x_B \geq 0$,
2. $r_N = c_B A_B^{-1}A_N - c_N \leq 0$.

Then, x is optimal for the LP (5.1) and the optimal value is $z_0 = c_B A_B^{-1}b$.

Proof. For every feasible $\tilde{x}$ we get $\tilde{x}_N \geq 0$ and with (5.3) and using $r_N \leq 0$, we get

$$z(\tilde{x}) = c\tilde{x} = z_0 - r_N \tilde{x}_N \geq z_0 = z(x).$$

□

For a special case of linear inequality restrictions of the form

$$\min cx \text{ s.t. } Ax \leq b, x \geq 0, \text{ where } b \geq 0,$$

we get an equivalent reformulation

$$\min cx + 0y \text{ s.t. } Ax + y = \tilde{A} \begin{pmatrix} x \\ y \end{pmatrix} = b, x \geq 0, y \geq 0,$$

where $\tilde{A} = (A, I)$. Choose basis $B = \{n+1, \dots, n+m\}$, $N = \{1, \dots, n\}$ and we get a basic

solution $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} \in \mathbb{R}^{n+m}$. The reduced cost are $\tilde{A}_B = I, \tilde{c}_B = 0, \tilde{c}_N = c$ with

$$r_N = \tilde{c}_B \tilde{A}_B^{-1} \tilde{A}_N - \tilde{c}_N = -c \in \mathbb{R}^n$$

$$z_0 = 0.$$

5.2 Basis Exchange

The Simplex-Method is based on the sufficient optimality conditions of Thm. 5.1. We search for a basis B with

1. $x_B \geq 0$
2. $r_N \leq 0$.

We start with B satisfying 1. If 2. is violated, we go to an adjacent basis B' via a basis exchange step so that the objective value does not increase.

Definition 5.2. Two bases B, B' are adjacent if $|B \cap B'| = m - 1$, so they differ by one exchange of a basic and a nonbasic variable. Under degeneracy, their feasible basic solutions may describe the same vertex.

For B , we use (5.2)

$$x_B = A_B^{-1}b - A_B^{-1}A_N x_N, \quad x_N \in \mathbb{R}^{n-m}.$$

Computing A_B^{-1} can be done with elementary row operations (Gauss-Jordan-Elimination). The following example demonstrates one exchange-step (pivot step):

Example 5.3.

$$\begin{aligned} \text{minimize } z(x_1, x_2, x_3) &= -x_1 - 2x_2 - 3x_3 \\ \text{s.t.} \\ 2x_1 + x_2 + 5x_3 &= 5 \\ x_1 + 2x_2 + x_3 &= 4 \\ x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0. \end{aligned}$$

We choose basis $B = (1, 2)$ and we use a tableau form $b|A$:

b	x_1	x_2	x_3	
5	2	1	5	·1/2
4	1	2	1	

b	x_1	x_2	x_3	
5/2	1	1/2	5/2	
4	1	2	1	−1. row

b	x_1	x_2	x_3	
5/2	1	1/2	5/2	
3/2	0	3/2	−3/2	·2/3

b	x_1	x_2	x_3	
5/2	1	1/2	5/2	−1/2 · 2. row
1	0	1	−1	

b	x_1	x_2	x_3
2	1	0	3
1	0	1	−1

We get:

$$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} x_3 = A_B^{-1}b - A_B^{-1}A_N x_N.$$

The basic solution

$$x_1 = 2, x_2 = 1, x_3 = 0$$

is feasible but not optimal, as we have

$$r_N = r_3 = c_B A_B^{-1} A_N - c_N = (-1, -2) \begin{pmatrix} 3 \\ -1 \end{pmatrix} - (-3) = 2 > 0.$$

For $0 \leq x_3 \leq 2/3$ we get that $x_B \geq 0$ is feasible: geometrically, we follow an edge of K . For $x_3 = 2/3$ we get a new basic solution $x = (0, 5/3, 2/3)^T$. The transition $x_3 = 0 \rightarrow x_3 = 2/3$ corresponds to a basic exchange step

$$B = (1, 2) \rightarrow B' = (3, 2), N = (3) \rightarrow N' = (1).$$

The non-basic variable x_3 is exchanged with the basic variable x_1 : $x_3 = 2/3 - 1/3x_1$. For

the new basis B' we get

$$x_{B'} = \begin{pmatrix} x_3 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 5/3 \end{pmatrix} - \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix} x_1$$

with

$$r_{N'} = r'_1 = (-3, -2) \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix} - (-1) = -2/3 < 0.$$

Hence, the found solution is optimal.

We will formalize this idea now. Suppose

$$A_B = I, B = (1, \dots, m) \text{ and } N = (m+1, \dots, n)$$

and $x_B = b \geq 0$, i.e. the starting x is feasible. If $r_N \leq 0$, x is optimal with Thm. 5.1. Let $r_s > 0$ for some $s \in N$. In

$$x_B = A_B^{-1}b - A_B^{-1}A_N x_N, z(x) = z_0 - r_N x_N,$$

we insert

$$A_B = I, x_j = 0 \text{ for } j \in N \setminus \{s\}$$

and get

$$\begin{aligned} x_B &= b - a^s x_s \\ z(x) &= z_0 - r_s x_s. \end{aligned} \tag{5.4}$$

1. Case: $a^s \leq 0$.

In this case, we get

$$x_B = b - a^s x_s \geq 0 \text{ for } x_s \rightarrow \infty.$$

We get $z(x) = z_0 - r_s x_s \rightarrow -\infty$ for $x_s \rightarrow \infty$ and hence the problem has no finite solution; the polyhedron K is unbounded in a descent direction of the objective.

2. Case: $a_{is} > 0$ for some $i \in \{1, \dots, m\}$. Define $p \in B$ via

$$\frac{b_p}{a_{ps}} = \min \left\{ \frac{b_i}{a_{is}} \mid a_{is} > 0, i = 1, \dots, m \right\}.$$

In order to ensure $x_B \geq 0$, the value of x_s can be at most

$$x_s = \frac{b_p}{a_{ps}} \Rightarrow x_p = 0.$$

This corresponds to an exchange of the non-basic variable x_s with the basic variable x_p (recall $B = (1, \dots, m)$). The value $a_{ps} > 0$ is called pivot element. we get

$$z(x) = z_0 - r_s \frac{b_p}{a_{ps}}. \tag{5.5}$$

We can differentiate the following cases:

- (a) $b_p = 0$: basic solution is degenerate, $z(x) = z_0$
- (b) $b_p > 0$: We get a strict improvement $z(x) < z_0$

We get the following necessary optimality condition:

Corollary 5.4. Let the basic solution x with $x_B \geq 0$ and $x_N = 0$ optimal. If $x_B > 0$, then $r_N \leq 0$.

The index $s \in N$ (pivot column) can be computed as follows:

1. (Rule of Dantzig): Choose smallest $s \in N$ with:

$$r_s = \max_{j \in N} r_j. \quad (5.6)$$

Choose smallest $p \in B$ with

$$\frac{b_p}{a_{ps}} = \min \left\{ \frac{b_i}{a_{is}} \mid a_{is} > 0, i = 1, \dots, m \right\}. \quad (5.7)$$

2. (Rule of Bland): Choose smallest $s \in N$ with $r_s > 0$. Choose smallest $p \in B$, among those satisfying $\frac{b_p}{a_{ps}} = \min \left\{ \frac{b_i}{a_{is}} \mid a_{is} > 0, i = 1, \dots, m \right\}$.

Let us describe a basis exchange after executing the Gauss-Jordan elimination. That is, we start with

$$A_B = I, B = (1, \dots, m), N = (m+1, \dots, n),$$

and thus

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j, \quad i \in B. \quad (5.8)$$

The p -th equation can be solved for x_s using $a_{ps} > 0$:

$$x_p = b_p - \sum_{j \in N} a_{pj} x_j = b_p - \sum_{j \in N, j \neq s} a_{pj} x_j - a_{ps} x_s$$

Solving for x_s yields:

$$x_s = \frac{b_p}{a_{ps}} - \sum_{j \in N, j \neq s} \frac{a_{pj}}{a_{ps}} x_j - \frac{1}{a_{ps}} x_p. \quad (5.9)$$

Insert x_s in the other equations (5.8) with $i \neq p$:

$$x_i = b_i - a_{is} \underbrace{\frac{b_p}{a_{ps}}}_{=b'_i} - \sum_{j \in N, j \neq s} \left(a_{ij} - a_{is} \frac{a_{pj}}{a_{ps}} \right) x_j + \frac{a_{is}}{a_{ps}} x_p. \quad (5.10)$$

Define new basis: Transition

$$B = (1, \dots, m) \rightarrow B' = (1, \dots, p-1, s, p+1, \dots, m).$$

New non-basis:

$$N = (m + 1, \dots, n) \rightarrow N' = (m + 1, \dots, s - 1, p, s + 1, \dots, n).$$

have the form

$$x_{B'} = \begin{pmatrix} x_1 \\ \vdots \\ x_{p-1} \\ x_s \\ x_{p+1} \\ \vdots \\ x_m \end{pmatrix} = b' - A'_{N'} x_{N'},$$

and hence have the form (5.2). The elements of the matrix

$$\begin{bmatrix} b' & A'_{N'} \end{bmatrix}$$

are given via

Pivot element: reciprocal value :

$$a'_{ps} = \frac{1}{a_{ps}}$$

Other row p: divide by pivot element:

$$a'_{pj} = \frac{a_{pj}}{a_{ps}}, j \neq s$$

$$b'_p = \frac{b_p}{a_{ps}}$$

Other column s: divide by negative pivot element:

$$a'_{is} = -\frac{a_{is}}{a_{ps}}, i \neq p$$

Other elements: subtract the a_{is} -multiple of the new row p from the i-th row:

$$a'_{ij} = a_{ij} - a_{is} \frac{a_{pj}}{a_{ps}} = a_{ij} - a_{is} a'_{pj}, i \neq p, j \neq s$$

$$b'_i = b_i - a_{is} \frac{b_p}{a_{ps}} = b_i - a_{is} b'_p, i \neq p$$

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We give an example.

	b	x_1	x_2	x_3	x_4	x_5	x_6
x_1	5	1	0	0	1	1	-1
x_2	3	0	1	0	2	-3	1
x_3	-1	0	0	1	-1	2	-1

Note that the current basic solution is infeasible. Change non-basic variable x_4 with basic variable x_1 :

	b'	x_1	x_2	x_3	x_4	x_5	x_6
x_4	5	1	0	0	1	1	-1
x_2	-7	-2	1	0	0	-5	3
x_3	4	1	0	1	0	3	-2

This example illustrates only the algebraic pivot operation. Its starting solution is already infeasible, so the primal simplex feasibility guarantee does not apply. After the pivot, $x_2 = -7$.

As the identity matrix $A_B = I$ contains no new information if we know B, N, A_N, b , we will consider in the following only the reduced tableau.

	b'	x_1	x_5	x_6
x_4	5	1	1	-1
x_2	-7	-2	-5	3
x_3	4	1	3	-2

We can verify:

$$b'_2 = 3 - 2 \cdot 5 = -7$$

$$b'_3 = -1 - (-1) \cdot 5 = 4$$

$$x_{B'} = \begin{pmatrix} x_4 \\ x_2 \\ x_3 \end{pmatrix} = b' = \begin{pmatrix} 5 \\ -7 \\ 4 \end{pmatrix}$$

$$B' = (4, 2, 3)$$

5.3 Objective Function and Reduced Costs

Via elimination of x_s in (5.9), we get

$$\begin{aligned}
 z(x) &= z_0 - \sum_{j \in N} r_j x_j \\
 &= z_0 - \sum_{j \in N, j \neq s} r_j x_j - r_s \left(\frac{b_p}{a_{ps}} - \sum_{j \in N, j \neq s} \frac{a_{pj}}{a_{ps}} x_j - \frac{1}{a_{ps}} x_p \right) \\
 &= z_0 - r_s \frac{b_p}{a_{ps}} - \sum_{j \in N, j \neq s} \left(r_j - r_s \frac{a_{pj}}{a_{ps}} \right) x_j + \frac{r_s}{a_{ps}} x_p \\
 &=: z'_0 - \sum_{j \in N'} r'_j x_j
 \end{aligned}$$

with

$$\begin{aligned}
 z'_0 &:= z_0 - r_s \frac{b_p}{a_{ps}} \\
 r'_p &:= -\frac{r_s}{a_{ps}} \\
 r'_j &:= r_j - r_s \frac{a_{pj}}{a_{ps}} = r_j - r_s a'_{pj}, \quad j \in N \setminus \{s\} = N' \setminus \{p\}.
 \end{aligned} \tag{5.11}$$

The row (z_0, r_N) will be handled as the rows in

b	A_N
---	-------

This leads to the extended tableau form:

$$\begin{array}{cc|cc}
 & & \multicolumn{2}{c}{x_j, j \in N} \\
 z(x) & \begin{array}{|c|c|} \hline z_0 & r_N \\ \hline \end{array} \\
 x_i, i \in B & \begin{array}{|c|c|} \hline b & A_N \\ \hline \end{array}
 \end{array} \tag{5.12}$$

Figure 5.1: Extended Simplex-Tableau.

We give an example.

Example 5.5.

$$\begin{aligned}
 \text{minimize } z(x_1, x_2, x_3) &= -x_1 - 2x_2 - 3x_3 \\
 \text{s.t.} \\
 2x_1 + x_2 + 5x_3 &= 5 \\
 x_1 + 2x_2 + x_3 &= 4
 \end{aligned}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

For $B = (1, 2), N = (3)$ we can transform the linear equation system to $A_B = I$:

$$\begin{bmatrix} \mathbf{b} & A_N \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}, \quad x_B = \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} > 0$$

$$\mathbf{c} = (-1, -2, -3)$$

$$z_0 = \mathbf{c}_B \mathbf{b} = (-1, -2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -4$$

$$r_3 = \mathbf{c}_B A_B^{-1} A_N - \mathbf{c}_3 = 2 > 0.$$

We get the Simplex-Tableau:

	x_N	
$z(x), r_N$	-4	2
x_1	2	3
x_2	1	-1

Exchange x_3 with x_1 : $r'_1 = -2/3 < 0$, hence the basic solution $x_1 = 0, x_2 = 5/3, x_3 = 2/3$

	x_N	
$z(x), r_N$	-16/3	-2/3
x_3	2/3	1/3
x_2	5/3	1/3

is optimal.

The algorithmic execution of the Simplex-Method is illustrated in Fig. 5.1 with a corresponding tableau-matrix.

$$T = (t_{ij}) = \begin{bmatrix} z_0 & r_N \\ \mathbf{b} & A_N \end{bmatrix} \quad \begin{array}{l} 0 \leq i \leq m \\ 0 \leq j \leq n - m \end{array} \quad (5.13)$$

Now, we formally describe the simplex-method (with the rule of Dantzig).

1. **Start:** Let B be a basis with $A_B = I$. Suppose the corresponding basic solution is feasible, i.e., $x_B = b \geq 0$. Compute with N the tableau-matrix $T = (t_{ij})$ in (5.13). Denote with $B(i)$ the basis index corresponding to the i -th row of T for $1 \leq i \leq m$. Analogously denote with $N(j)$ the non-basis index corresponding to the j -th column of T for $1 \leq j \leq n - m$.

2. If there is an s with $1 \leq s \leq n - m$ and $t_{0s} > 0$, go to 3. Otherwise, the current basis-solution is optimal. Set

$$\begin{aligned} x_{B(i)} &:= t_{i0}, 1 \leq i \leq m \\ x_{N(j)} &:= 0, 1 \leq j \leq n - m \\ z &:= t_{00} \end{aligned}$$

Stop.

3. **Compute the Exchange-Column:** Choose index $1 \leq s \leq n - m$ (with $N(s)$ smallest) with

$$t_{0s} = \max_{1 \leq j \leq n-m} t_{0j}.$$

4. If $t_{is} \leq 0$, for all $1 \leq i \leq m$, there is no finite solution. **Stop.**
If there is $t_{is} > 0$ for some $1 \leq i \leq m$, go to step 5.

5. **Compute the Exchange-Row:** Choose index $1 \leq p \leq m$ (with $B(p)$ smallest) with

$$\frac{t_{p0}}{t_{ps}} = \min \left\{ \frac{t_{i0}}{t_{is}} \mid t_{is} > 0, i = 1, \dots, m \right\}.$$

Go to step 6.

6. Exchange the s -th element of N with the p -th element of B :

$$(B(p), N(s)) \leftarrow (N(s), B(p)) \quad \text{simultaneously.}$$

Execute pivot operation with pivot element $t_{ps} > 0$:

- pivot element: $t'_{ps} := 1/t_{ps}$
- pivot row: $t'_{pj} := \frac{t_{pj}}{t_{ps}}, j = 0, 1, \dots, n - m, j \neq s$
- pivot column: $t'_{is} := -\frac{t_{is}}{t_{ps}}, i = 0, 1, \dots, m, i \neq p$
- other elements: $t'_{ij} := t_{ij} - t_{is} \frac{t_{pj}}{t_{ps}}, i \neq p, j \neq s$
- set $t_{ij} := t'_{ij}$ and go to step 2.

Figure 5.2: Formal Execution of the Simplex-Method.

5.4 Application to LPs with Inequalities

The LP

$$\min\{cx \mid Ax \leq b, x \geq 0\}, x \in \mathbb{R}^n, b \in \mathbb{R}^m, b \geq 0$$

is equivalent to

$$\min\{cx + 0 \cdot y \mid Ax + y = b, x \geq 0, y \geq 0\}, \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+m}, b \in \mathbb{R}^m$$

Start: $B = (n+1, \dots, n+m)$, $N = (1, \dots, n)$

Basic solution: $x = 0, y = b \geq 0, z_0 = 0, r_N = -c$. The corresponding $(m+1) \times (n+1)$ dimensional Simplex-Tableau:

z	0	$-c$
y	b	A

We solve the example from Subsec. 3.1.1. The following tableaux solve the production problem by minimizing negative profit, that is, $\min(-16x_1 - 32x_2)$. Their objective entries are therefore negative profits. To follow the illustrated path, we first choose the improving variable x_1 ; Dantzig's rule would choose x_2 .

$$\begin{aligned} &\text{maximize } z(x_1, x_2) = 16x_1 + 32x_2 \\ &\text{s.t.} \\ &20x_1 + 10x_2 \leq 8000 \\ &4x_1 + 5x_2 \leq 2000 \\ &6x_1 + 15x_2 \leq 4500 \\ &x_1 \geq 0 \\ &x_2 \geq 0 \end{aligned}$$

The geometric sequence $A \rightarrow B \rightarrow C \rightarrow D$ is executed algebraically via the Tableau-Form.

		x_1	x_2
z	0	16	32
y_1	8000	20	10
y_2	2000	4	5
y_3	4500	6	15

Let us choose 20 as pivot element and we exchange x_1 with y_1 : This corresponds to $A \rightarrow B$.

		y_1	x_2
z	-6400	$-4/5$	24
x_1	400	$1/20$	$1/2$
y_2	400	$-1/5$	(3)
y_3	2100	$-3/10$	12

Then, we exchange x_2 with y_2 : This leads to $B \rightarrow C$.

		y_1	y_2
z	-9600	$4/5$	-8
x_1	$1000/3$	$1/12$	$-1/6$
x_2	$400/3$	$-1/15$	$1/3$
y_3	500	($1/2$)	-4

Exchange y_1 with y_3 : This is $C \rightarrow D$.

		y_3	y_2
z	-10400	$-8/5$	$-8/5$
x_1	250	$-1/6$	$1/2$
x_2	200	$2/15$	$-1/5$
y_1	1000	2	-8

Here $r_N = (-8/5, -8/5) < 0$, hence $x_1 = 250, x_2 = 200, z_{\min} = -10400$, i.e. $z_{\max} = 10400$, is optimal.

We show next that for non-degenerate problems the simplex-method always terminates.

Theorem 5.6. If, during the execution of the simplex-method, all computed basic-solutions are non-degenerate, the method terminates.

Proof. For every pivot step j we get

$$z_0^j = z_0^{j-1} - r_s^{j-1} \frac{b_p^{j-1}}{a_{ps}^{j-1}}.$$

As for every j per assumption $b_p^j > 0$, we get a strictly monotone sequence

$$z_0^1 > z_0^2 > \dots$$

Hence, no basis is visited twice and since there are only finitely many bases, the algorithm terminates. $\square$

5.5 Cycling of the Simplex-Method and Lexicographical Pivoting

For a pivot operation with $b_p = 0$ the objective value does not change

$$\begin{aligned} z(x) &= z_0 - r_s \frac{b_p}{a_{ps}} = t_{00} - t_{0s} \frac{t_{p0}}{t_{ps}} \\ &= z_0 \end{aligned}$$

There are examples showing that the simplex-method (using a “wrong” pivot-rule) may cycle forever (see exercise).

In order to avoid cycling, we will now introduce a lexicographical variant.

Definition 5.7. Let $k \in \mathbb{N}$. Then, we can define a total order on $\mathbb{R}^k$ via

$$x \succeq y :\Leftrightarrow x = y \text{ or } x_i > y_i \text{ for } i = \min\{s | x_s \neq y_s\}$$

that is compatible with addition on $\mathbb{R}^k$. This order is called lexicographical order.

If $0 \in \mathbb{R}^k$ and $x \in \mathbb{R}^k$ with $x \succ 0$ (that means $x \succeq 0$ and $x \neq 0$), then x is called lexicographically positive.

We consider now the complete tableau in standard form:

$$T = (t_{ij})_{\substack{i=0,\dots,m \\ j=0,\dots,n}} = \begin{array}{|c|c|c|} \hline z_0 & 0 & r_N \\ \hline b & A_B = I & A_N \\ \hline \end{array}$$

The rows $t_i, i = 1, \dots, m$ are vectors in $\mathbb{R}^{n+1}$. For basis B , we denote by $B(i)$ the i -th entry of the current basis B . Initially $B = (1, \dots, m)$ and $N = (m+1, \dots, n)$.

Throughout the pivot operations, the columns of the complete tableau retain their initial order. Basis indices are updated, but columns are not reordered into basic and nonbasic blocks.

Definition 5.8 (Lexicographical Rule (LEX)). Let B be the current basis and denote the corresponding tableau by T .

- Choose arbitrary column index $s \in N$ with $r_s = t_{0s} > 0$.
- Consider $I = \{i \in \{1, \dots, m\} | t_{is} > 0\}$. If $I = \emptyset$, the LP is unbounded and the

algorithm stops. Otherwise, Choose $p \in I$ with

$$\frac{t_p}{t_{ps}} = \text{lexmin} \left\{ \frac{t_i}{t_{is}} \mid i \in I \right\}.$$

This lex. minimum satisfies

$$\frac{t_i}{t_{is}} \succ \frac{t_p}{t_{ps}} \text{ for all } i \in I \setminus \{p\}.$$

Then, $B(p)$ leaves the current basis B and s enters the new basis, that is, $B'(p) = s$.

By choice of p , we get

$$\frac{t_{p0}}{t_{ps}} = \min \left\{ \frac{t_{i0}}{t_{is}} \mid i \in I \right\}.$$

Hence, the pivot step leads to a feasible new basis.

Theorem 5.9. Suppose we start the simplex-method using LEX with a tableau that contains only lex. positive rows (except the reduced cost row). Then, the method terminates.

Proof. We first claim that the assumption of lex. positive rows of the theorem are easily satisfiable via applying the Gauss-Jordan transformation, i.e., $(b|I|A_N)$ (except first row). Because of feasibility $x_B \geq 0$, the first column contains only numbers greater equal than 0 (except first row). As the first sub-matrix is lex. positive, the claim is shown.

We show inductively over the execution of LEX, that this property is preserved. Consider a basis B with the required property. Let s and p be chosen according to LEX. Denote the new basis by B' . For the p -th row t'_p of the new tableau T' we get

$$t'_p = \frac{t_p}{t_{ps}} \succ 0,$$

since by assumption $t_p \succ 0$ and $t_{ps} > 0$. For $i \in \{1, \dots, m\}, i \neq p$, the i -th row t'_i of T' is given as:

$$t'_i = t_i - \frac{t_{is}}{t_{ps}} t_p.$$

For $t_{is} > 0$, we get according to LEX

$$t'_i = t_{is} \left(\frac{t_i}{t_{is}} - \frac{t_p}{t_{ps}} \right) \succ 0.$$

For $t_{is} \leq 0$, we get

$$t'_i = t_i + \frac{|t_{is}|}{t_{ps}} t_p \geq t_i \succ 0.$$

Overall, the rows remain lex. positive.

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Let us consider the first row t_0 and t'_0 . We have

$$t'_0 = t_0 - \frac{t_{0s}}{t_{ps}} t_p.$$

With $t_{0s} = r_s > 0$ we get

$$t_0 = t'_0 + \frac{t_{0s}}{t_{ps}} t_p \succ t'_0.$$

Thus, after executing a pivot step using LEX, the 0-th row decreases strictly lexicographically and hence no tableau can appear twice. $\square$

There are finitely many bases and hence finitely many associated tableaux. Since the first row strictly decreases in lexicographic order at each LEX step, no basis can recur; this proves finite termination.

5.6 Computing a Feasible Basic Solution - Two-Phase Method

Consider the LP in standard form:

$$\begin{aligned} &\text{minimize } z(x) = cx \\ &\text{s.t.} \\ &Ax = b \\ &x \geq 0. \end{aligned} \tag{5.14}$$

W.l.o.g. assume $b \geq 0$. If there is some row with negative b_i just multiply with -1 . We define the following auxiliary problem:

$$\begin{aligned} &\text{minimize } z(x, y) = \sum_{i=n+1}^{n+m} y_i \\ &\text{s.t.} \\ &Ax + y = b \\ &x, y \geq 0. \end{aligned} \tag{5.15}$$

We use here additional artificial variables y_i , where $i \in \{n+1, \dots, n+m\}$. One of the following statements holds true.

1. The auxiliary problem has an optimal solution with value 0. Then, the system $Ax = b, x \geq 0$ is feasible.
2. The auxiliary problem has an optimal solution with value > 0 . Then, $Ax = b, x \geq 0$ is infeasible.

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The auxiliary problem can be solved with the simplex-method, where we use as start basis $y_i, i \in \{n+1, \dots, n+m\}$ (recall $b \geq 0$). We write

$$B = (n+1, \dots, n+m), N = (1, \dots, n), c_B = (1, 1, \dots, 1) \in \mathbb{R}^m \text{ and } c_N = 0 \in \mathbb{R}^n.$$

This yields

$$\begin{aligned} z_0 &= c_B b = \sum_{i=1}^m b_i \geq 0, \\ r_N &= c_N A_N, \end{aligned} \tag{5.16}$$

where $r_j = \sum_{i=1}^m a_{ij}, j = 1, \dots, n$. The start tableau reads

$$\begin{array}{cc} & x \\ z(x, y), r_N & \begin{array}{|c|c|} \hline z_0 & r_N \\ \hline \end{array} \\ y & \begin{array}{|c|c|} \hline b & A_N \\ \hline \end{array} \end{array} \tag{5.17}$$

The two-phase method for LP (5.14) consists of the following steps:

Phase I: Solve the auxiliary problem (5.15), for example using LEX. If its optimal value is positive, the original problem is infeasible and the method stops. If its value is zero, all artificial variables are zero, but some may still be basic. Pivot each such variable out using a nonzero coefficient of an original nonbasic variable in its tableau row. Since the right-hand side is zero, this preserves the basic solution and feasibility. If there is no such coefficient, remove the redundant equation. This yields a feasible basis for the original problem.

Phase II: Compute starting tableau for LP (5.14) via

- Delete the columns belonging to $y_{n+1}, \dots, y_{n+m}$,
- Compute z_0 and r_N .

Example 5.10. We apply the two-phase method to the following problem:

$$\begin{aligned} \min \quad & 4x_1 + x_2 + x_3 \\ \text{s.t.} \quad & 2x_1 + x_2 + 2x_3 = 4 \\ & 3x_1 + 3x_2 + x_3 = 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The auxiliary problem reads as:

$$\begin{aligned}
 \min \quad & y_4 + y_5 \\
 \text{s.t.} \quad & 2x_1 + x_2 + 2x_3 + y_4 = 4 \\
 & 3x_1 + 3x_2 + x_3 + y_5 = 3 \\
 & x_1, x_2, x_3, y_4, y_5 \geq 0
 \end{aligned}$$

We start with basis $B = (4, 5)$ and $N = (1, 2, 3)$.

$$z_0 = b_1 + b_2 = 4 + 3 = 7 \quad \text{and} \quad r_N = \left(\sum_{i=1}^2 a_{ij} \right)_{j=1,2,3} = (5, 4, 3).$$

The start tableau reads as:

		x_1	x_2	x_3
$z(x, y), r_N$	7	5	4	3
y_4	4	2	1	2
y_5	3	3	3	1

With the rule of Dantzig we get:

		y_5	x_2	x_3
$z(x, y), r_N$	2	-5/3	-1	4/3
y_4	2	-2/3	-1	4/3
x_1	1	1/3	1	1/3

		y_5	x_2	y_4
$z(x, y), r_N$	0	-1	0	-1
x_3	3/2	-1/2	-3/4	3/4
x_1	1/2	1/2	5/4	-1/4

We see that $r_N \leq 0$ and $z(x, y) = 0$, where y_4 and y_5 non-basic. Hence, a feasible solution to $Ax = b, x \geq 0$ is given via basis $B = (3, 1)$ with

$$x_3 = 3/2, \quad x_1 = 1/2, \quad x_2 = 0.$$

Note that the basic columns in the transformed tableau already form the identity matrix and hence phase II can be easily started. We delete the column vectors corresponding

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to y_4 and y_5 . We get $c = (4, 1, 1)$, $B = (3, 1)$, $N = (2)$, $c_B = (1, 4)$ and $b = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$ (see tableau). For the new costs and the reduced costs we get:

$$z_0 = c_B b = (1, 4) \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} = 3/2 + 4/2 = 7/2 \text{ and}$$

$$r_N = r_2 = c_B a^2 - c_2 = (1, 4) \begin{pmatrix} -3/4 \\ 5/4 \end{pmatrix} - 1 = 13/4,$$

where a^2 needs to be taken from the tableau.

The start tableau for phase II reads as:

	x_2	
$z(x), r_N$	7/2	13/4
x_3	3/2	-3/4
x_1	1/2	5/4

After one pivot step we get

	x_1	
$z(x), r_N$	11/5	-13/5
x_3	9/5	3/5
x_2	2/5	4/5

As $r_1 = -13/5 < 0$, the solution is optimal

$$x_1 = 0, \quad x_2 = 2/5, \quad x_3 = 9/5 \quad \text{with objective } 11/5.$$

Let us turn to problems with inequality constraints.

$$\begin{aligned}
 &\text{minimize } z(x) = cx \\
 &\text{s.t.} \\
 &a_i x = b_i, \quad i = 1, \dots, k \\
 &a_i x \leq b_i, \quad i = k+1, \dots, m \\
 &x \geq 0.
 \end{aligned} \tag{5.18}$$

We introduce artificial variables $y_{n+1}, \dots, y_{n+k}$ for the first k equations and slack-variables $x_{n+k+1}, \dots, x_{n+m}$ for the inequalities. Let us assume $b \geq 0$. Then, we need to solve in phase I the following auxiliary problem (as slack-variables can be ≥ 0 , they don't appear in the objective):

$$\begin{aligned}
& \text{minimize} && \sum_{i=n+1}^{n+k} y_i \\
& \text{s.t.} && \\
& a_i x + y_{n+i} = b_i, && i = 1, \dots, k \\
& a_i x + x_{n+i} = b_i && i = k+1, \dots, m \\
& x, y \geq 0.
\end{aligned} \tag{5.19}$$

The start tableau reads as in 5.17, where

$$z_0 = \sum_{i=1}^k b_i \text{ and } r_j = \sum_{i=1}^k a_{ij}, j = 1, \dots, n.$$

Note that we assumed $b \geq 0$ which is this time not w.l.o.g. Let us consider the general case now:

$$\begin{aligned}
& \text{minimize } z(x) = cx \\
& \text{s.t.} \\
& a_i x \geq b_i, && i = 1, \dots, k \\
& a_i x \leq b_i, && i = k+1, \dots, m \\
& x \geq 0.
\end{aligned} \tag{5.20}$$

Multiply each row with a negative right-hand side by -1 , reversing its inequality direction. Then regroup the rows and redefine k accordingly, so that $b \geq 0$.

Here we can assume w.l.o.g. $b \geq 0$. We introduce slack-variables $x_{n+1}, \dots, x_{n+m}$ and obtain a problem in standard form:

$$\begin{aligned}
& \text{minimize } z(x) = cx \\
& \text{s.t.} \\
& a_i x - x_{n+i} = b_i, && i = 1, \dots, k \\
& a_i x + x_{n+i} = b_i, && i = k+1, \dots, m \\
& x_i \geq 0, && i = 1, \dots, n+m.
\end{aligned} \tag{5.21}$$

We use the artificial variables $y_{n+m+1}, \dots, y_{n+m+k}$ for the first k equations (since here the slack-variables have a negative sign).

The auxiliary problem reads as

$$\begin{aligned}
& \text{minimize} && \sum_{i=n+m+1}^{n+m+k} y_i \\
& \text{s.t.} && \\
& a_i x - x_{n+i} + y_{n+m+i} = b_i, && i = 1, \dots, k \\
& a_i x + x_{n+i} = b_i, && i = k+1, \dots, m \\
& x_i \geq 0, && i = 1, \dots, n+m \\
& y_i \geq 0, && i = n+m+1, \dots, n+m+k.
\end{aligned} \tag{5.22}$$

Let us sort the $n+m+k$ variables as

$$\underbrace{x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}}_N, \underbrace{y_{n+m+1}, \dots, y_{n+m+k}, x_{n+k+1}, \dots, x_{n+m}}_B.$$

Then, we get

$$\begin{aligned}
c_B &= (\underbrace{1, 1, \dots, 1}_k, \underbrace{0, 0, \dots, 0}_{m-k}) \in \mathbb{R}^m, \\
z_0 &= c_B b = \sum_{i=1}^k b_i, \\
r_j &= c_B a^j - c_j = \begin{cases} \sum_{i=1}^k a_{ij}, & j = 1, \dots, n \\ -1, & j = n+1, \dots, n+k \end{cases}
\end{aligned}$$

Finally, we can compute the start tableau for Phase I:

	$x_1 \cdots x_n$		$x_{n+1} \cdots x_{n+k}$
$z(x), r_N$	$\sum_{i=1}^k b_i$	$\sum_{i=1}^k a_{ij}$	$-1 \cdots -1$
y_{n+m+1}	b_1	a_1	$-I_k$
$\vdots$	$\vdots$	$\vdots$	
y_{n+m+k}	b_k	a_k	
x_{n+k+1}	b_{k+1}	a_{k+1}	0
$\vdots$	$\vdots$	$\vdots$	
x_{n+m}	b_m	a_m	

Remark 5.11. A zero artificial basic variable indicates degeneracy of the auxiliary problem. The basic solution may remain degenerate after all artificial variables have been removed. If A has full row rank, the cleanup pivots remove all artificial basic

| variables; otherwise, redundant equations are deleted as needed.

Chapter 6

LP Duality

We consider an LP in standard form

$$\begin{aligned} \text{minimize } z(x) &= cx \\ Ax &= b \\ x &\geq 0, \end{aligned} \tag{P}$$

for $A \in \mathbb{R}^{m \times n}$, $c^T \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Denote

$$K = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}.$$

Let $x \in K$ and consider $\lambda^T \in \mathbb{R}^m$. We obtain

$$b = Ax \Leftrightarrow \lambda b = \lambda Ax \quad \forall \lambda^T \in \mathbb{R}^m.$$

We get a lower bound on cx via:

$$\lambda b \leq cx \quad \forall \lambda^T \in \mathbb{R}^m \text{ with } \lambda A \leq c.$$

Let us compute the best lower bound on cx . This leads to the dual problem:

$$\begin{aligned} \text{maximize } z^*(\lambda) &= \lambda b \\ \lambda A &\leq c. \end{aligned} \tag{D}$$

We denote by K^* the polyhedron for the dual problem.

$$K^* := \{\lambda^T \in \mathbb{R}^m | \lambda A \leq c\}.$$

Note that λ^T has no sign constraints.

6.1 Dual Programs

Every LP has a corresponding dual LP. One can differentiate between a [symmetric and asymmetric forms](#).

Symmetric Form:

Primal problem (P-sym):

$$\begin{aligned} &\text{minimize } cx \\ &Ax \geq b \\ &x \geq 0, \end{aligned} \quad (\text{P-sym})$$

Dual problem (D-sym):

$$\begin{aligned} &\text{maximize } \lambda b \\ &\lambda A \leq c \\ &\lambda \geq 0. \end{aligned} \quad (\text{D-sym})$$

Here, A is an $m \times n$ matrix, $x \in \mathbb{R}^n$ is a column vector, and $\lambda \in \mathbb{R}^m$ a row vector.

Remark 6.1. The dual problem of the dual problem is the primal problem. Proof: Exercise.

We give an example.

Primal problem (P):

$$\begin{aligned} &\text{minimize } 3x_1 + 4x_2 + 5x_3 \\ &x_1 + 2x_2 + 3x_3 \geq 5 \\ &2x_1 + 2x_2 + x_3 \geq 6 \\ &x_i \geq 0, i = 1, 2, 3 \end{aligned}$$

Dual problem (D):

$$\begin{aligned} &\text{maximize } 5\lambda_1 + 6\lambda_2 \\ &\lambda_1 + 2\lambda_2 \leq 3 \\ &2\lambda_1 + 2\lambda_2 \leq 4 \\ &3\lambda_1 + \lambda_2 \leq 5 \\ &\lambda_i \geq 0, i = 1, 2. \end{aligned}$$

The LP in standard form can be reduced to the symmetric form. The equation system

$$Ax = b$$

corresponds to

$$\begin{aligned} &Ax \geq b \\ &-Ax \geq -b. \end{aligned}$$

The dual problem for $w \in \mathbb{R}^{2m}$ is then using (P-sym)

$$\begin{aligned} & \text{maximize } w \begin{pmatrix} b \\ -b \end{pmatrix} \\ & w \begin{pmatrix} A \\ -A \end{pmatrix} \leq c \\ & w \geq 0. \end{aligned}$$

Insert $w := (u, v)$, $u, v \in \mathbb{R}^m$, and we obtain

$$\begin{aligned} & \text{maximize } ub - vb \\ & uA - vA \leq c \\ & u \geq 0, v \geq 0. \end{aligned}$$

With $\lambda := u - v$ we obtain the dual problem (D). Note that $\lambda = u - v$ has no sign constraints.

Here are some computing rules for dualizing.

Table 6.1: Dualizing rules.

Primal LP	Dual LP
(P ₁) $\max cx, Ax \leq b, x \geq 0$	(D ₁) $\min \lambda b, \lambda A \geq c, \lambda \geq 0$
(P ₂) $\min cx, Ax \geq b, x \geq 0$	(D ₂) $\max \lambda b, \lambda A \leq c, \lambda \geq 0$
(P ₃) $\max cx, Ax = b, x \geq 0$	(D ₃) $\min \lambda b, \lambda A \geq c$
(P ₄) $\min cx, Ax = b, x \geq 0$	(D ₄) $\max \lambda b, \lambda A \leq c$
(P ₅) $\max cx, Ax \leq b$	(D ₅) $\min \lambda b, \lambda A = c, \lambda \geq 0$
(P ₆) $\min cx, Ax \geq b$	(D ₆) $\max \lambda b, \lambda A = c, \lambda \geq 0$

We give a general version now.

Lemma 6.2. Consider matrices A, B, C, D and vectors a, b, c, d . For the primal LP

$$\begin{aligned} & \text{maximize } cx + dy \\ & Ax + By \leq a \\ & Cx + Dy = b \\ & x \geq 0, \end{aligned} \tag{LP}$$

we get the dual LP

$$\begin{aligned}
& \text{minimize } u a + v b \\
& \quad u A + v C \geq c \\
& \quad u B + v D = d \\
& \quad u \geq 0.
\end{aligned} \tag{DP}$$

The variables y and multipliers v are unrestricted in sign.

Proof. Exercise. □

6.2 The Strong Duality Theorem

$$\begin{aligned}
& \text{minimize } z(x) = cx \\
& \quad Ax = b \\
& \quad x \geq 0,
\end{aligned} \tag{P}$$

$$\begin{aligned}
& \text{maximize } z^*(\lambda) = \lambda b \\
& \quad \lambda A \leq c.
\end{aligned} \tag{D}$$

The feasible sets of the LPs are denoted by K and K^* .

We get the following relationship of (P) and (D).

Theorem 6.3 (Weak-Duality). For $x \in K, \lambda^T \in K^*$ we have $\lambda b \leq cx$.

Proof. Let $x \in K$ and $\lambda^T \in K^*$. We get

$$b = Ax \Rightarrow \lambda b = \lambda Ax \leq_{\lambda^T \in K^*} cx.$$

□

Corollary 6.4. Let $x^* \in K$ and $(\lambda^*)^T \in K^*$ with $\lambda^* b = cx^*$. Then, x^* and $(\lambda^*)^T$ are optimal for the respective problems (P) and (D).

Proof. For arbitrary $x \in K$ we get

$$cx \geq \lambda^* b = cx^*.$$

For arbitrary $\lambda \in K^*$

$$\lambda b \leq cx^* = \lambda^* b.$$

□

Theorem 6.5 (Strong Duality for Linear Optimization). If one of the problems (P) or (D) admits a finite optimal solution, then also the other admits a finite optimal solution and their objective values are equal:

$$\min\{cx | x \in K\} = \max\{\lambda b | \lambda^T \in K^*\}.$$

Proof. W.l.o.g. $x^* \in K$ is optimal with $cx^* = z_0$ finite. Define

$$C := \{(r, w) = t(cx - z_0, b - Ax)^T | x \geq 0, t > 0\} \subset \mathbb{R}^{m+1}.$$

C is a convex cone containing 0. We get the following alternative representation of C :

$$C = \{(r, w) = (cx - tz_0, tb - Ax)^T | x \geq 0, t > 0\} \subset \mathbb{R}^{m+1}.$$

(Define $\tilde{x} = x/t$). Since $Ax^* = b$ and $cx^* = z_0$, replacing (x, t) by $(x + x^*, t + 1)$ leaves the represented point unchanged. Thus allowing $t = 0$ does not enlarge C . Consequently, C is the cone generated by the finitely many vectors $(c_j, -a^j)^T$ and $(-z_0, b)^T$, and is closed. To justify the last assertion, eliminate linear dependencies from any nonnegative representation until its support is independent. The cone is therefore a finite union of cones with independent generators. Each such cone is closed: a continuous left inverse determines its coefficients, which remain nonnegative under limits.

We claim that $(-1, 0) \notin C$. This follows since $t > 0$ and $z_0 \leq cx$ for all $x \in K$. Hence, we can use the separation theorem for convex and closed cones (cf. Thm. 2.9). There is $(\lambda_0, \lambda)^T \in \mathbb{R}^{m+1} \setminus \{0\}$ with $\lambda_0 \in \mathbb{R}, \lambda^T \in \mathbb{R}^m$ such that

$$(\lambda_0, \lambda)^T \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\lambda_0 < 0 \leq (\lambda_0, \lambda)^T \begin{pmatrix} cx - tz_0 \\ tb - Ax \end{pmatrix}$$

for all $t > 0, x \geq 0$. We get $\lambda_0 > 0$ and w.l.o.g. $\lambda_0 = 1$ (multiply with $1/\lambda_0$). Computing the scalar products, we get for $t > 0, x \geq 0$

$$\begin{aligned} 0 &\leq (cx - tz_0) + \lambda(tb - Ax) \\ \Leftrightarrow tz_0 - cx &\leq \lambda(tb - Ax). \end{aligned} \tag{6.1}$$

Take limit $t \downarrow 0$:

$$cx \geq \lambda Ax \text{ for all } x \geq 0$$

and hence

$$c \geq \lambda A$$

implying $\lambda \in K^*$. Insert in (6.1) the values $t = 1, x = 0$ to obtain

$$z_0 \leq \lambda b.$$

With weak duality (cf. Thm. 6.3) we get $z_0 = \lambda b$ and using Corollary 6.4 we get that $x^* \in K$ and $\lambda \in K^*$ are optimal. $\square$

| Remark 6.6. The proof does not use $\text{rank}(A) = m$.

With Theorems (6.3) and (6.5) we obtain the following existence- and optimality criteria.

Theorem 6.7 (Existence). The following statements are equivalent.

1. (P) and (D) admit feasible solutions.
2. (P) has a finite optimal solution.
3. (P) and (D) have finite optimal solutions with equal optimal value.
4. (D) admits a finite optimal solution.
5. (P) admits feasible solutions and the objective is bounded from below or (D) admits feasible solutions and the objective is bounded from above.

Proof. (1) $\Rightarrow$ (2) : With Thm. (6.3) we get that the objective of (P) is bounded from below. Hence, (since (P) admits feasible solutions by assumption) by Theorem 4.19 we get the existence of an optimal finite solution.

(2) $\Rightarrow$ (3) : This is the statement of Thm. (6.5).

(3) $\Rightarrow$ (4) $\Rightarrow$ (5) is immediate. Under (5), Theorem 4.19, after conversion to standard form if needed, gives an optimum for one problem. Strong duality then implies (1). $\square$

Theorem 6.8 (Complementarity). Let $x \in K$ and $\lambda \in K^*$ be feasible for (P) and (D). Then, the following statements are equivalent.

1. x and λ are optimal.
2. $(\lambda A - c)x = 0$
3. For all $i = 1, \dots, n$, we get

$$\begin{aligned} x_i > 0 &\Rightarrow \lambda a^i = c_i \\ \lambda a^i < c_i &\Rightarrow x_i = 0. \end{aligned}$$

Proof: Exercise (trivial).

6.3 The Dual Simplex-Method

Consider the primal-dual programmes

$$\min\{cx \mid Ax = b, x \geq 0\} \quad (P)$$

$$\max\{\lambda b \mid \lambda A \leq c\}. \quad (D)$$

We assume $\text{rank}(A) = m$. The basic solution w.r.t. a basis B of (P) is given as

$$x_B = A_B^{-1}b, x_N = 0.$$

The row vector

$$\lambda := c_B A_B^{-1} \in \mathbb{R}^m \quad (6.2)$$

is called Simplex- or Lagrange-Multiplier. Hence, the reduced cost read as

$$r_N = c_B A_B^{-1} A_N - c_N = \lambda A_N - c_N.$$

We obtain:

Theorem 6.9. 1. $\lambda = c_B A_B^{-1}$ is feasible for (D) $\Leftrightarrow r_N \leq 0$.

2. Let the basic solution x be optimal and non-degenerate, i.e.,

$$x_B = A_B^{-1} b > 0, x_N = 0.$$

Then, $\lambda = c_B A_B^{-1}$ is optimal for (D) and we get

$$\lambda b = c x.$$

Proof. For (1): With $\lambda = c_B A_B^{-1}$ we get

$$\begin{aligned} \lambda A &\leq c \Leftrightarrow (\lambda A_B, \lambda A_N) \leq (c_B, c_N) \\ &\Leftrightarrow \lambda A_N \leq c_N \\ &\Leftrightarrow r_N = \lambda A_N - c_N \leq 0. \end{aligned}$$

For (2): Is x optimal and non-degenerate, then $r_N \leq 0$ (see Corollary 5.4). Thus, $\lambda = c_B A_B^{-1}$ is feasible for (D) (using (1)) and we have

$$\lambda b = c_B A_B^{-1} b = c_B x_B = c x.$$

Because of Corollary (6.4) we get that $\lambda = c_B A_B^{-1}$ is optimal for D. □

The connection of primal optimality and dual feasibility is visible in the following definition.

Definition 6.10. A basic-solution x with

$$x_B = A_B^{-1} b, x_N = 0$$

is called dual feasible for (P), if and only if

$$\lambda = c_B A_B^{-1}$$

is feasible for (D), i.e., if $r_N \leq 0$.

Let us now explain the idea of the dual simplex-method. Suppose we have a basis B and a basic solution x that is

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- primal infeasible, i.e., $x_B = A_B^{-1}b \geq 0$ is not valid
- dual feasible, i.e., $r_N \leq 0$.

We consider the primal tableau (P) but solve it from the dual point of view:

1. maintain $r_N \leq 0$
2. until we have feasibility of x , i.e. $x_B = A_B^{-1}b \geq 0$
3. The objective value of the current basic solution does not decrease; a degenerate step may leave it unchanged.

Assuming $A_B = I$ the tableau for (P) is given via (5.13)

$$\begin{array}{c}
 \begin{array}{cc}
 & x_j, j \in N \\
 z(x) & \begin{array}{|c|c|} \hline z_0 & r_N \\ \hline \end{array} \\
 x_i, i \in B & \begin{array}{|c|c|} \hline b & A_N \\ \hline \end{array}
 \end{array}
 \end{array} \quad (6.3)$$

Figure 6.1: Reduced Simplex-Tableau.

with tableau-matrix:

$$T = (t_{ij}) = \begin{array}{|c|c|} \hline z_0 & r_N \\ \hline b & A_N \\ \hline \end{array} \quad \begin{array}{l} 0 \leq i \leq m \\ 0 \leq j \leq n - m \end{array} \quad (6.4)$$

The tableau (6.4) is

1. primal feasible, if $b \geq 0$,
2. dual feasible, if $r_N \leq 0$,
3. optimal, if $b \geq 0, r_N \leq 0$.

Let us write down the dual Simplex-Algorithms formally (see Fig. 6.2).

We give an application. The primal problem is given as:

$$\min\{cx \mid Ax \geq b, x \geq 0\}, c \geq 0 \quad (P)$$

$Ax \geq b$ is equivalent to

$$Ax - y = b, y \geq 0.$$

Because $c \geq 0$, the slack basis is dual feasible, so we can start the dual simplex method directly for (P). Consider

$$-Ax + y = -b, y \geq 0, x \geq 0.$$

1. **Start:** Let x be a dual feasible basis B . Compute $T = (t_{ij})$, as in (6.4). Denote by $B(i)$ the basis index for $1 \leq i \leq m$ that corresponds to the i -th row of T . Analogously we denote by $N(j)$ the index $1 \leq j \leq n - m$ that belongs to the j -th column of T .
2. If $t_{i0} \geq 0$ ($1 \leq i \leq m$), then the current solution is optimal.
Set

$$\begin{aligned} x_{B(i)} &:= t_{i0}, 1 \leq i \leq m \\ x_{N(j)} &:= 0, 1 \leq j \leq n - m \\ z &:= t_{00} \end{aligned}$$

Stop. Otherwise go to 3.

3. **Compute exchange row:** Choose $p \in [m]$ (with $B(p)$ smallest) such that

$$t_{p0} = \min_{1 \leq i \leq m} t_{i0} \quad (t_{i0} = b_i).$$

4. If $t_{pj} \geq 0, 1 \leq j \leq n - m$, then The primal problem (P) is infeasible and the dual problem (D) is unbounded. **Stop.**
If there is some $t_{pj} < 0$, go to Step 5.

5. **Compute exchange column:** Choose $s \in [n - m]$ (with $N(s)$ smallest) such that

$$\frac{t_{0s}}{t_{ps}} = \min \left\{ \frac{t_{0j}}{t_{pj}} \mid t_{pj} < 0, j = 1, \dots, n - m \right\} \quad (t_{0j} = r_{N(j)} \leq 0).$$

Go to Step 6.

6. Exchange the s -th element of N with the p -th element of B :

$$(B(p), N(s)) \leftarrow (N(s), B(p)) \quad \text{simultaneously.}$$

Execute in T the pivot step with pivot element $t_{ps} < 0$ as in Fig. 5.2, point (6):

- Pivot element: $t'_{ps} := 1/t_{ps}$
- Pivot column: $t'_{is} := -\frac{t_{is}}{t_{ps}}, i = 0, 1, \dots, m, i \neq p$
- Pivot row: $t'_{pj} := \frac{t_{pj}}{t_{ps}}, j = 0, 1, \dots, n - m, j \neq s$
- other elements: $t'_{ij} := t_{ij} - t_{is} \frac{t_{pj}}{t_{ps}}, i \neq p, j \neq s$
- Set $t_{ij} := t'_{ij}$ and go to Step 2.

Figure 6.2: Dual Simplex-Method

We obtain the start-tableau

$$\begin{aligned} B &= (n + 1, \dots, n + m), N = (1, \dots, n) \\ r_N &= -c \leq 0. \end{aligned}$$

Hence $x = 0, y = -b$ is dual feasible and we get the tableau:

$$\begin{array}{c|cc} & \mathbf{x} & \\ \hline z & z_0 & \mathbf{r_N} = -\mathbf{c} \leq 0 \\ \hline y & -\mathbf{b} & -\mathbf{A} \end{array}$$

(6.5)

We solve the example from the last section:

$$\begin{aligned} \text{minimize } & 3x_1 + 4x_2 + 5x_3 \\ & x_1 + 2x_2 + 3x_3 \geq 5 \\ & 2x_1 + 2x_2 + x_3 \geq 6 \\ & x_i \geq 0, i = 1, 2, 3 \end{aligned}$$

The start-tableau has the form:

		x_1	x_2	x_3
z	0	-3	-4	-5
y_1	-5	-1	-2	-3
y_2	-6	-2	-2	-1

We get

$$\min\{-5, -6\} = -6 \Rightarrow p = 2$$

$$\min\{3/2, 4/2, 5/1\} = 3/2 \Rightarrow s = 1.$$

Exchange y_2 and x_1 :

		y_2	x_2	x_3
z	9	-3/2	-1	-7/2
y_1	-2	-1/2	-1	-5/2
x_1	3	-1/2	1	1/2

Exchange y_1 and x_2 :

		y_2	y_1	x_3
z	11	-1	-1	-1
x_2	2	1/2	-1	5/2
x_1	1	-1	1	-2

The tableau is optimal; the optimal solution is

$$x_1 = 1, x_2 = 2, x_3 = 0.$$

The optimal dual solution is

$$\lambda = (1, 1).$$

6.4 Sensitivity and Shadow Prices

Suppose the LP

$$\min\{cx \mid Ax = b, x \geq 0\} \quad (6.6)$$

has an optimal basis B with non-degenerate basic solution $x_B = A_B^{-1}b > 0$, $x_N = 0$. The corresponding optimal dual solution is $\lambda = c_B A_B^{-1}$. For a sufficiently small change $b \mapsto b + \Delta b$, the basis B remains feasible. The new basic solution is

$$x'_B = A_B^{-1}(b + \Delta b) = x_B + \Delta x_B, \quad x'_N = 0, \quad \Delta x_B = A_B^{-1}\Delta b.$$

Since the reduced costs $r_N = c_B A_B^{-1} A_N - c_N \leq 0$ do not depend on b , this basic solution remains optimal as long as it is feasible. Writing $z(b)$ for the optimal value, we obtain

$$z(b + \Delta b) - z(b) = cx' - cx = c_B \Delta x_B = \lambda \Delta b.$$

Thus, locally around b , the shadow price formula holds:

$$\frac{\partial z}{\partial b_j}(b) = \lambda_j, \quad j = 1, \dots, m. \quad (6.7)$$

Here, λ_j is the marginal price of a change in b_j . Larger changes may require a different basis and hence a different shadow price.

Chapter 7

Postoptimality Analysis

In this section we consider how an optimal solution of an LP can be updated when the input data are perturbed, so that the new solution is optimal for the modified data. We want to compute these changes with as little effort as possible; of course, we could always solve the problem again “from scratch”.

We again consider an LP in standard form

$$\begin{aligned} \text{minimize } z(x) &= cx \\ Ax &= b \\ x &\geq 0, \end{aligned} \tag{LP}$$

and an optimal basis B .

7.1 Changing the Right-Hand Side b

We consider a change of the right-hand side of the form

$$b' = b + d \text{ for some } d \in \mathbb{R}^m.$$

Using the optimal basis B of the unperturbed problem, we obtain the following basic solution of the perturbed problem:

$$x'_B := A_B^{-1}(b + d) = x_B + A_B^{-1}d \text{ and } x'_N = 0.$$

If $x'_B \geq 0$, the modified basic solution remains optimal, since the reduced costs have not changed:

$$r'_N = r_N = c_B A_B^{-1} A_N - c_N \leq 0.$$

The new objective value is given by

$$z' = c_B A_B^{-1} b' = z + c_B A_B^{-1} d.$$

If, however, $x'_B \not\geq 0$, the corresponding basic solution is primal infeasible, but **dual** feasible. We can therefore apply the dual simplex method directly with basis matrix A_B .

7.2 Changing the Objective Function c

We replace c in the LP by $c' = c + d$. The new objective value for the given basis B is

$$z' = (c + d)x = (c_B + d_B)A_B^{-1}b = z + d_B A_B^{-1}b.$$

Whether the current basic solution remains optimal depends on the modified reduced costs. These are given by

$$\begin{aligned} r'_N &= (c_B + d_B)A_B^{-1}A_N - c_N - d_N \\ &= r_N + d_B A_B^{-1}A_N - d_N. \end{aligned}$$

If $r'_N \leq 0$, the current basic solution remains optimal, even though z changes. Otherwise, there is a positive reduced cost $r'_j > 0$, and we can apply the primal simplex method with B as the current feasible basis.

7.3 Adding a New Variable

We introduce a new variable x_{n+1} with objective coefficient c_{n+1} into the LP. This also adds a new column a^{n+1} to the matrix A . We can treat x_{n+1} as an additional nonbasic variable and extend the current tableau T by one column. In the following, tableau columns are denoted by $(a')^j$.

$$\begin{aligned} (a')^{n+1} &= A_B^{-1}a^{n+1} \\ r'_{n+1} &= c_B A_B^{-1}a^{n+1} - c_{n+1} = \lambda a^{n+1} - c_{n+1}. \end{aligned}$$

If $r'_{n+1} \leq 0$, the basis B remains optimal. Otherwise, we apply the primal simplex method: either x_{n+1} enters the basis, or the ratio test shows that the modified LP is unbounded.

7.4 Changing a Column of A

We replace the column a^j of A by $a^j + d$. There are two cases.

Case 1: $j \in N$.

Since the basis matrix A_B remains unchanged, the new j -th tableau column $(a')^j$ is given by

$$(a')^j = A_B^{-1}(a^j + d). \quad (7.1)$$

The basis remains primal feasible, but the reduced costs may change:

$$r'_j = c_B A_B^{-1}(a^j + d) - c_j.$$

Either the current basis remains optimal, or we apply the primal simplex method to the modified tableau, starting with the feasible basis B .

Case 2: $j \in B$. In this case, both primal and dual feasibility of the current basis may be lost. Moreover, the modified basis matrix A'_B may become singular.

To address some of these difficulties, we append an additional column of the form

$$s := a^j + d$$

to A . We obtain a problem with an additional variable, which we denote by x'_j . Its objective coefficient is again c_j . We extend the tableau by the column

$$\begin{pmatrix} r'_j \\ s' \end{pmatrix} := \begin{pmatrix} c_B A_B^{-1} s - c_j \\ A_B^{-1} s \end{pmatrix}.$$

If the entry s'_j in the row of s' indexed by x_j is nonzero, we can use it as a pivot element and exchange the old basic variable x_j for the new variable x'_j . We then remove the old variable x_j and rename x'_j as x_j . Note, however, that the current basis B need be neither primal nor dual feasible after this step. If either feasibility condition holds, we continue with the corresponding simplex method. If neither holds, we must apply Phase I of the primal simplex method to obtain an initial feasible basis.

If $s'_j = 0$, replacing x_j by x'_j would not yield a basis. In this case, we restart the simplex method from the beginning.

We give an example of changing a column.

Example 7.1 (Changing a Column). Consider the following problem:

Primal problem (P):

$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ & -x_1 - x_2 \leq -2 \\ & -2x_1 - x_2 \leq -3 \\ & x_i \geq 0, i = 1, 2 \end{aligned}$$

We introduce slack variables y_3 and y_4 and obtain the tableau

	x_1	x_2	
z	0	-1	-2
y_3	-2	-1	-1
y_4	-3	-2	-1

The current basis is dual feasible. After a few pivot steps of the [dual simplex method](#), we obtain the following optimal tableau:

		x_2	y_3
z	2	-1	-1
y_4	1	1	-2
x_1	2	1	-1

The optimal basis is $B = (4, 1)$.

We now replace the nonbasic column a^2 by

$$a^2 + d := \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Using $r_2 = -1$ and

$$\lambda = c_B A_B^{-1} = (0, 1) \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} = (-1, 0)$$

we obtain

$$r'_2 = r_2 + \lambda d = -1 - \alpha.$$

Thus, $r'_2 \leq 0$ if and only if $\alpha \geq -1$. Hence, for $\alpha \geq -1$, the basic solution associated with $B = (4, 1)$ is optimal for all linear programs of the form

$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ & -x_1 + (\alpha - 1)x_2 \leq -2 \\ & -2x_1 + (\beta - 1)x_2 \leq -3 \\ & x_i \geq 0, i = 1, 2. \end{aligned} \quad (\text{LP}(\alpha, \beta))$$

We next consider a change of the basic column a^1 . We obtain

$$s := a^1 + d = \begin{pmatrix} -1 \\ -2 \end{pmatrix} + \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$$

with tableau column

$$s' = A_B^{-1}s = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha - 1 \\ \alpha - 2 \end{pmatrix} = \begin{pmatrix} -\alpha \\ 1 - \alpha \end{pmatrix},$$

and

$$r'_1 = c_B s' - c_1 = (0, 1) \begin{pmatrix} -\alpha \\ 1 - \alpha \end{pmatrix} - 1 = -\alpha.$$

We therefore append the vector $\begin{pmatrix} -\alpha \\ -\alpha \\ 1 - \alpha \end{pmatrix}$ as a new column to the tableau of the original

optimal basis B:

		x_2	y_3	x'_1	
z	2	-1	-1	$-\alpha$	
y_4	1	1	-2	$-\alpha$	
x_1	2	1	-1	$(1 - \alpha)$	

The method described above applies if $\alpha \neq 1$, since the pivot element is then nonzero. Pivoting on $(1 - \alpha)$ gives the tableau

		x_2	y_3	x_1	
z	$\frac{2}{1-\alpha}$	$\frac{2\alpha-1}{1-\alpha}$	$\frac{-1}{1-\alpha}$	$\frac{\alpha}{1-\alpha}$	
y_4	$\frac{1+\alpha}{1-\alpha}$	$\frac{1}{1-\alpha}$	$\frac{\alpha-2}{1-\alpha}$	$\frac{\alpha}{1-\alpha}$	
x'_1	$\frac{2}{1-\alpha}$	$\frac{1}{1-\alpha}$	$\frac{-1}{1-\alpha}$	$\frac{1}{1-\alpha}$	

As described above, we now delete the x_1 column and obtain

		x_2	y_3	
z	$\frac{2}{1-\alpha}$	$\frac{2\alpha-1}{1-\alpha}$	$\frac{-1}{1-\alpha}$	
y_4	$\frac{1+\alpha}{1-\alpha}$	$\frac{1}{1-\alpha}$	$\frac{\alpha-2}{1-\alpha}$	
x'_1	$\frac{2}{1-\alpha}$	$\frac{1}{1-\alpha}$	$\frac{-1}{1-\alpha}$	

We easily see that, with respect to B, the tableau T' is

- dual feasible if and only if $\alpha \leq 1/2$,
- primal feasible if and only if $-1 \leq \alpha < 1$.

If either condition holds, we can continue with the corresponding simplex method. It also follows directly that, for

$$-1 \leq \alpha \leq 1/2$$

the current basis B remains optimal. Thus, the program

$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ (\alpha - 1)x_1 - x_2 &\leq -2 \\ (\alpha - 2)x_1 - x_2 &\leq -3 \\ x_i &\geq 0, i = 1, 2 \end{aligned}$$

has the optimal solution, for $-1 \leq \alpha \leq 1/2$,

$$x_1^* = \frac{2}{1 - \alpha}, x_2^* = 0.$$

The objective value is $z^* = \frac{2}{1 - \alpha}$.

Chapter 8

Parametric Linear Optimization

We now study how optimal solutions of LPs can be described as functions of the input parameters.

8.1 Perturbing the Objective Function c

Consider the following parametric LP:

$$\begin{aligned} \min \quad & cx \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0, \end{aligned} \tag{LP(c)}$$

In this formulation, c is allowed to vary.

We obtain the following theorem.

Theorem 8.1. Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$ such that

$$P^+(A, b) := \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$$

is a nonempty polyhedron. Define

$$BP(A, b) := \{c \in \mathbb{R}^n \mid cx \text{ is bounded below on } P^+(A, b)\}.$$

For $c \in BP(A, b)$, let

$$M(c) := \min\{cx \mid x \in P^+(A, b)\}.$$

Then the following statements hold:

1. $BP(A, b)$ is a convex cone.
2. There are finitely many vectors $v_1, \dots, v_N \in \mathbb{R}^n$ such that

$$M(c) = \min\{cv_i \mid i = 1, \dots, N\}$$

for all $c \in BP(A, b)$.

3. $M(c)$ is **concave** on $BP(A, b)$.

Proof. (1): Clearly, $BP(A, b)$ is a cone. Let $c, c' \in BP(A, b)$ and $\alpha \in (0, 1)$. For all $x \in P^+(A, b)$, we have

$$(\alpha c + (1 - \alpha)c')x = \alpha cx + (1 - \alpha)c'x \geq \alpha M(c) + (1 - \alpha)M(c')$$

and hence $\alpha c + (1 - \alpha)c' \in BP(A, b)$.

(2): By the fundamental theorem of linear optimization, every linear objective function that is bounded below attains its minimum at one of the finitely many vertices of $P^+(A, b)$. We can therefore choose the v_i to be the vertices of $P^+(A, b)$.

(3): By (2), $M(c)$ is the minimum of concave (in fact, linear) functions and is therefore concave. Compare the analogous statement about convex functions in the earlier chapter on convexity. □

Corollary 8.2. The statements of Theorem 8.1 also hold for polyhedra of the form

$$P^-(A, b) := \{x \in \mathbb{R}^n | Ax = b, x \geq 0\} \text{ and } P^-(A, b) := \{x \in \mathbb{R}^n | Ax \geq b, x \geq 0\}.$$

We obtain an analogous result for maximization problems:

$$\max\{cx | x \in P^+(A, b)\}.$$

Theorem 8.3. Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$ such that

$$P^+(A, b) := \{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\}$$

is a nonempty polyhedron. Define

$$BP^*(A, b) := \{c \in \mathbb{R}^n | cx \text{ is bounded above on } P^+(A, b)\}.$$

For $c \in BP^*(A, b)$, let

$$M^*(c) := \max\{cx | x \in P^+(A, b)\}.$$

Then the following statements hold:

1. $BP^*(A, b)$ is a convex cone.
2. There are finitely many vectors $v_1, \dots, v_N \in \mathbb{R}^n$ such that

$$M^*(c) = \max\{cv_i | i = 1, \dots, N\}$$

for all $c \in BP^*(A, b)$.

3. $M^*(c)$ is **convex** on $BP^*(A, b)$.

8.2 Perturbing the Right-Hand Side b

We next consider linear programs of the form

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0, \end{aligned} \tag{LP(b)}$$

Here, b is allowed to vary. We use the formulation $Ax \geq b$ to simplify the presentation, as will become clear below. Let

$$P^-(A, b) := \{x \in \mathbb{R}^n \mid Ax \geq b, x \geq 0\}.$$

Theorem 8.4. Let A be an $m \times n$ matrix and $c \in \mathbb{R}^n$ such that the linear objective function cx is bounded below on at least one nonempty polyhedron of the form $P^-(A, b)$. Define

$$AP(A, c) := \{b \in \mathbb{R}^m \mid P^-(A, b) \neq \emptyset, cx \text{ is bounded below on } P^-(A, b)\}.$$

For $b \in AP(A, c)$, let

$$L(b) := \min\{cx \mid x \in P^-(A, b)\}.$$

Then the following statements hold:

1. $AP(A, c)$ is a convex cone.
2. There are finitely many vectors $u_1, \dots, u_N \in \mathbb{R}^m$ such that

$$L(b) = \max\{b^T u_i \mid i = 1, \dots, N\}$$

for all $b \in AP(A, c)$.

3. $L(b)$ is **convex** on $AP(A, c)$.

Proof. We consider the following dual LP of $(LP(b))$:

$$\begin{aligned} \max \quad & \lambda b \\ \text{s.t.} \quad & \lambda A \leq c \\ & \lambda \geq 0, \end{aligned} \tag{DP(b)}$$

This is a parametric LP with objective vector b and fixed right-hand side c . By strong duality,

$$AP(A, c) = BP^*(A^T, c^T),$$

where

$$BP^*(A^T, c^T) = \{b \in \mathbb{R}^m \mid \lambda b \text{ is bounded above on } P^+(A^T, c^T)\}.$$

Moreover, writing $M^*(b)$ for the optimal value of this dual problem, we obtain

$$L(b) = M^*(b),$$

and the claim follows from Theorem 8.3. □

The preceding theorems are of theoretical interest, but do not yet provide a practical method for computing a parametric solution. We now illustrate such a method by an example.

Consider the following linear program:

$$\begin{aligned} \min \quad & -x_1 - 3x_2 \\ & x_1 + x_2 \leq 6 \\ & -x_1 + 2x_2 \leq 6 \\ & x_i \geq 0, i = 1, 2 \end{aligned}$$

We introduce slack variables y_3, y_4 and solve the problem as before.

We obtain the optimal basis $B = (1, 2)$ with tableau

		y_3	y_4
z	-14	$-5/3$	$-2/3$
x_1	2	$2/3$	$-1/3$
x_2	4	$1/3$	$1/3$

We now perturb the objective function in direction $d = (2, 1)$:

$$c' = c + td = (-1, -3) + t(2, 1) \text{ with } t \geq 0.$$

The objective coefficients and perturbation components of the slack variables remain zero. For the given basis $B = (1, 2)$, the objective vector c' yields the reduced costs

$$\begin{aligned} r'_N &= c'_B A_B^{-1} A_N - c'_N \\ &= r_N + t(d_B A_B^{-1} A_N - d_N) \\ &:= r_N + td'. \end{aligned}$$

In our example, $r_N = (-5/3, -2/3)$ and

$$d' = (2, 1) \begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{pmatrix} - (0, 0) = (5/3, -1/3).$$

Thus,

$$\mathbf{r}'_{\mathbf{N}} = (-5/3 + 5/3t, -2/3 - 1/3t).$$

The current basis and its basic solution remain optimal as long as $\mathbf{r}'_{\mathbf{N}} \leq 0$. This holds for $0 \leq t \leq 1$. Hence,

		y_3	y_4
z	$-14 + 8t$	$-5/3 + 5/3t$	$-2/3 - 1/3t$
x_1	2	$2/3$	$-1/3$
x_2	4	$1/3$	$1/3$

is an optimal tableau for $LP(t)$ for all $t \in [0, 1]$. The entry $-14 + 8t$ is obtained from

$$\mathbf{c}'_{\mathbf{B}} \mathbf{A}_{\mathbf{B}}^{-1} \mathbf{b} = \mathbf{c}_{\mathbf{B}} \mathbf{A}_{\mathbf{B}}^{-1} \mathbf{b} + t \mathbf{d}_{\mathbf{B}} \mathbf{A}_{\mathbf{B}}^{-1} \mathbf{b} = -14 + t(2, 1) \begin{pmatrix} 2 \\ 4 \end{pmatrix} = -14 + 8t.$$

Since $\mathbf{r}'_3 > 0$ for $t > 1$, we bring y_3 into the basis using the tableau at $t = 1$. As $\mathbf{r}'_3 = 0$ at this point, optimality is preserved.

We replace the tableau

		y_3	y_4
z	-6	0	-1
x_1	2	$2/3$	$-1/3$
x_2	4	$1/3$	$1/3$

by

		x_1	y_4
z	-6	0	-1
y_3	3	$3/2$	$-1/2$
x_2	3	$-1/2$	$1/2$

which is still optimal.

We recompute the reduced costs $\mathbf{r}'_{\mathbf{N}'}$, as a function of $t \geq 1$ for the basis $\mathbf{B}' = (3, 2)$ and $\mathbf{N}' = (1, 4)$, obtaining

$$\mathbf{r}'_{\mathbf{N}'} = \mathbf{c}'_{\mathbf{B}'} \mathbf{A}_{\mathbf{B}'}^{-1} \mathbf{A}_{\mathbf{N}'} - \mathbf{c}'_{\mathbf{N}'},$$

$$\begin{aligned}
&= ((0, -3) + t(0, 1)) \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} - ((-1, 0) + t(2, 0)) \\
&= (3/2 - 1/2t, -3/2 + 1/2t) - (-1 + 2t, 0) \\
&= (5/2 - 5/2t, -3/2 + 1/2t).
\end{aligned}$$

Now $r'_{N'} \leq 0$ for $1 \leq t \leq 3$. Using

$$c'_{B'x_{B'}} = (c_{B'} + t d_{B'}) A_{B'}^{-1} b = ((0, -3) + t(0, 1)) \begin{pmatrix} 3 \\ 3 \end{pmatrix} = -9 + 3t$$

we obtain the tableau

		x_1	y_4
z	$-9 + 3t$	$5/2 - 5/2t$	$-3/2 + 1/2t$
y_3	3	$3/2$	$-1/2$
x_2	3	$-1/2$	$1/2$

This tableau is optimal for $1 \leq t \leq 3$. At $t = 3$, we have $r'_4 = 0$, so we bring y_4 into the basis for $t \geq 3$:

		x_1	x_2
z	0	-5	0
y_3	6	1	1
y_4	6	-1	2

Recomputing the reduced costs for $B'' = (3, 4)$ gives

$$\begin{aligned}
r'_{N''} &= c'_{B''} A_{B''}^{-1} A_{N''} - c'_{N''} \\
&= ((0, 0) + t(0, 0)) \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} - ((-1, -3) + t(2, 1)) \\
&= (0, 0) - (-1 + 2t, -3 + t) \\
&= (1 - 2t, 3 - t).
\end{aligned}$$

Since $r'_{N''} \leq 0$ for all $t \geq 3$, the current basis B'' remains optimal throughout this range. The tableau is

		x_1	x_2
z	0	$1 - 2t$	$3 - t$
y_3	6	1	1
y_4	6	-1	2

We have now computed the optimal solutions and objective values as functions of t for all $t \geq 0$.

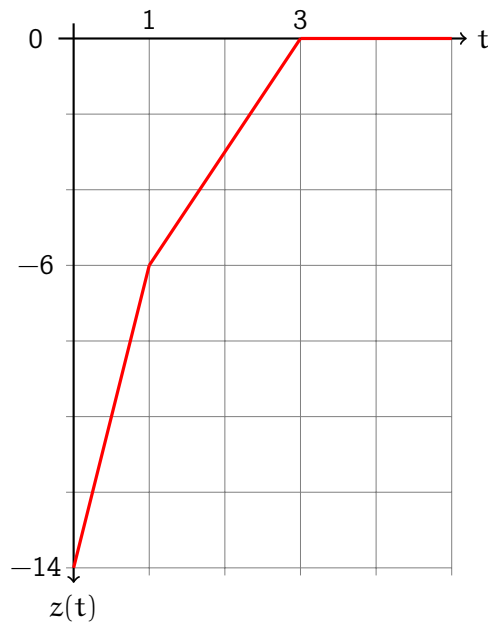


Figure 8.1: Optimal value function $z(t)$.

A similar procedure applies to perturbations of the right-hand side and/or the coefficient matrix; see the exercises.

Chapter 9

Complexity of Linear Optimization

In this chapter we introduce the complexity-theoretic foundations needed to analyze a polynomial-time algorithm for LPs, the so-called [ellipsoid method](#). We first show that the simplex method can have exponential running time when a [poor](#) pivot rule is used.

9.1 Klee-Minty Cubes

In the chapter on the simplex method, we saw that Dantzig's pivot rule need not terminate and may cycle. For the lexicographic pivot rule, on the other hand, we proved that an optimum is reached after finitely many steps, provided one exists. This raises the question of whether a suitable pivot rule guarantees a [polynomial](#) running time. This question is open in the setting discussed here, and we will return to it below.

We first construct an LP with an improving simplex path that visits exponentially many vertices. The construction shows that a suitable choice of pivots can force exponential running time; it does not assert this for every pivot rule.

For this purpose, we define the following [Klee-Minty](#) cubes.

Definition 9.1. The d -dimensional Klee-Minty cube is defined by the inequalities

$$\epsilon \leq x_1 \leq 1 \tag{9.1}$$

$$\epsilon x_{j-1} \leq x_j \leq 1 - \epsilon x_{j-1} \text{ for } j = 2, \dots, d, \tag{9.2}$$

where $0 < \epsilon < 1/2$. Choosing $\epsilon = 0$ gives the ordinary d -dimensional cube.

We give an example for $d = 3$ and $\epsilon = 1/4$:

$$1 \leq 4x_1 \leq 4 \tag{9.3}$$

$$x_1 \leq 4x_2 \leq 4 - x_1 \tag{9.4}$$

$$x_2 \leq 4x_3 \leq 4 - x_2 \tag{9.5}$$

Figure 9.1 gives a schematic illustration of the vertices and edges of the Klee-Minty cube. The indicated path through all eight vertices in the right-hand diagram corresponds to an execution of the simplex method with a suitable pivot rule for the LP with $d = 3$ and objective $-x_3$.

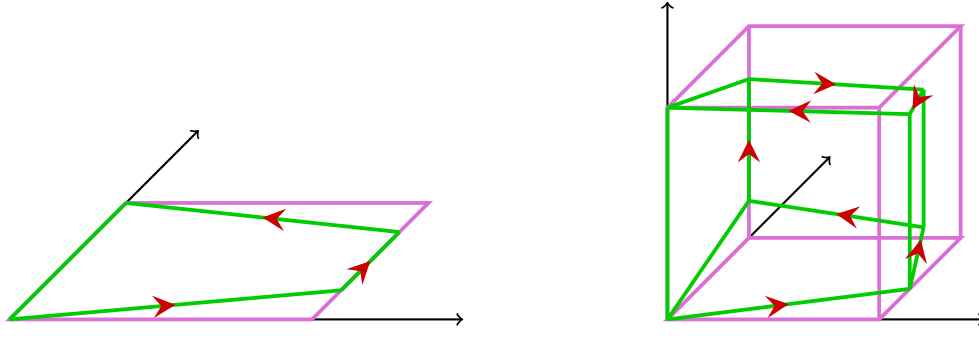


Figure 9.1: The Klee-Minty cube and the corresponding simplex path in dimensions $d = 2$ and $d = 3$.

We now generalize this observation to d -dimensional Klee-Minty cubes.

$$\begin{aligned}
 & \min -x_d \\
 & x_1 - r_1 = \epsilon \\
 & x_1 + s_1 = 1 \\
 & x_j - \epsilon x_{j-1} - r_j = 0 \text{ for } j = 2, \dots, d \\
 & x_j + \epsilon x_{j-1} + s_j = 1 \text{ for } j = 2, \dots, d \\
 & x, r, s \geq 0,
 \end{aligned} \tag{KM(d)}$$

Again, $0 < \epsilon < 1/2$.

The following lemma characterizes the feasible bases of $(\text{KM}(d))$.

Lemma 9.2. The feasible bases of $(\text{KM}(d))$ are in one-to-one correspondence with the subsets of $\{1, \dots, d\}$ and are all non-degenerate. Identifying indices with the corresponding variables, a basis has the form

$$B_S := \{x_1, \dots, x_d\} \cup \{r_j | j \in S\} \cup \{s_j | j \notin S\}. \tag{9.6}$$

Proof. Let B be a feasible basis of $(\text{KM}(d))$, with corresponding basic solution x . The inequalities $x_1 \geq \epsilon$ and $x_j \geq \epsilon x_{j-1}$ for $j = 2, \dots, d$ imply

$$x_j \geq \epsilon^j > 0 \text{ for } j = 1, \dots, d.$$

Hence, all indices $1, \dots, d$ belong to the basis.

Moreover, for every j , at least one of the variables r_j, s_j must belong to the basis. Suppose that $r_j = s_j = 0$. For $j = 1$, this gives $x_1 = \epsilon = 1$, a contradiction. For $j > 1$, the equalities $x_j = \epsilon x_{j-1}$ and $x_j + \epsilon x_{j-1} = 1$ imply

$$2\epsilon x_{j-1} = 1.$$

Since $x_{j-1} \leq 1$ and $\epsilon < 1/2$, this is again a contradiction. Thus, for every x_j , at least one of r_j, s_j is positive and therefore basic. We have found at least $2d$ positive basic variables.

Since $m = 2d$, there are exactly $2d$ basic variables. Hence, exactly one of r_j, s_j is positive for every j , and the basis has the form (9.6). It is also non-degenerate.

Conversely, a feasible basic solution is easily computed from a given set S ; see the exercises. $\square$

Together with Theorem 4.22, this yields the following consequence.

Corollary 9.3. The Klee-Minty cube described in $(KM(d))$ has exactly 2^d vertices.

We record a few further observations that will be useful below.

Corollary 9.4. Let B_S be a feasible basis of $(KM(d))$ as in (9.6). The corresponding basic solution satisfies:

- $x_1 = 1$ if $1 \in S$, and $x_1 = \epsilon$ otherwise.
- $x_j = 1 - \epsilon x_{j-1}$ for $j \in S$ with $j \geq 2$.
- $x_j = \epsilon x_{j-1}$ for $j \notin S$ with $j \geq 2$.

Corollary 9.4 allows us to compute the basic solution corresponding to a given basis B_S recursively.

We next consider the d -th coordinate of x , which determines the simplex path.

Lemma 9.5. Let $S, S' \subseteq \{1, \dots, d\}$, and let $x^S, x^{S'}$ be the corresponding vertices of the Klee-Minty cube. If $d \in S \setminus S'$, then $x_d^S > x_d^{S'}$. If, in addition, $S' = S \setminus \{d\}$ and $d \geq 2$, then $x_d^{S'} = 1 - x_d^S$.

Proof. We use Corollary 9.4. For $d = 1$, the first claim follows from $x_1^S = 1 > \epsilon = x_1^{S'}$. For $d \geq 2$, we have $d \in S$, $x_{d-1}^S \leq 1$, and $\epsilon < 1/2$, so that $x_d^S = 1 - \epsilon x_{d-1}^S > 1/2$. On the other hand, $d \notin S'$ implies $x_d^{S'} = \epsilon x_{d-1}^{S'} < 1/2$. Thus, $x_d^S > x_d^{S'}$. If also $S' = S \setminus \{d\}$, recursive application of Corollary 9.4 gives $x_j^S = x_j^{S'}$ for $j = 1, \dots, d-1$. Hence,

$$x_d^{S'} = \epsilon x_{d-1}^{S'} = \epsilon x_{d-1}^S = 1 - (1 - \epsilon x_{d-1}^S) = 1 - x_d^S. \quad \square$$

We prove another lemma.

Lemma 9.6. Number the $D = 2^d$ subsets of $\{1, \dots, d\}$ so that

$$x_d^{S_1} \leq x_d^{S_2} \leq \dots \leq x_d^{S_D}. \quad (9.7)$$

Then all inequalities are strict, and the vertices x^{S_i} and $x^{S_{i+1}}$ are adjacent for $i = 1, \dots, D-1$.

Proof. We use induction on $d \in \mathbb{N}$. For $d = 1$, take $S_1 = \emptyset$ and $S_2 = \{1\}$. Then

$$x_1^{S_1} = \epsilon \text{ and } x_1^{S_2} = 1,$$

and the claim holds. For the induction step $d \rightarrow d+1$, note that $S_1, \dots, S_D$ are also subsets of $\{1, \dots, d+1\}$. They therefore describe 2^d of the 2^{d+1} vertices of the $(d+1)$ -dimensional Klee-Minty cube. By Corollary 9.4, their $(d+1)$ -st coordinates satisfy

$$x_{d+1}^{S_i} = \epsilon x_d^{S_i}$$

The induction hypothesis (9.7) gives

$$x_{d+1}^{S_1} < x_{d+1}^{S_2} < \dots < x_{d+1}^{S_D}. \quad (9.8)$$

The remaining D subsets of $\{1, \dots, d+1\}$ have the form

$$\bar{S}_i := S_i \cup \{d+1\} \text{ for } i = 1, \dots, D.$$

Since $d+1 \in \bar{S}_i \setminus S_j$, Lemma 9.5 implies

$$x_{d+1}^{\bar{S}_i} > x_{d+1}^{S_j} \text{ for } i, j = 1, \dots, D.$$

Moreover, $S_i = \bar{S}_i \setminus \{d+1\}$ gives

$$x_{d+1}^{\bar{S}_i} = 1 - x_{d+1}^{S_i} \text{ for } i = 1, \dots, D.$$

Together with (9.8), we obtain

$$x_{d+1}^{S_1} < x_{d+1}^{S_2} < \dots < x_{d+1}^{S_D} < x_{d+1}^{\bar{S}_D} < x_{d+1}^{\bar{S}_{D-1}} < \dots < x_{d+1}^{\bar{S}_1}.$$

By the induction hypothesis, x^{S_i} and $x^{S_{i+1}}$ are adjacent for $i = 1, \dots, D-1$. This also implies that $x^{\bar{S}_i}$ and $x^{\bar{S}_{i+1}}$ are adjacent: the bases B_{S_i} and $B_{S_{i+1}}$ differ by one variable exchange, and the same holds for $B_{\bar{S}_i}$ and $B_{\bar{S}_{i+1}}$. Finally, x^{S_D} and $x^{\bar{S}_D}$ are adjacent, since $B_{\bar{S}_D}$ is obtained from B_{S_D} by replacing s_{d+1} with r_{d+1} . $\square$

Combining these lemmas yields the following result.

Theorem 9.7. For every $d > 1$, there is an LP in standard form with $2d$ equations, $3d$ variables, and integer coefficients of absolute value at most 4, together with a pivot rule for which the simplex method can require $2^d - 1$ iterations to find the optimum.

Proof. Set $\epsilon := 1/4$ and multiply the equations in $(\text{KM}(d))$ by 4, so that all coefficients are integers. With objective function $-x_d$, the claim follows directly from Lemma 9.6. $\square$

We conclude this section with some remarks on pivot rules that find an optimum in polynomial time. We first define the [diameter of a polyhedron](#).

- Definition 9.8.**
1. The distance $d(x, y)$ between two vertices x, y of a polyhedron P is the minimum number of edges in a path from x to y , where a path is a sequence of adjacent vertices.
 2. The diameter $D(P)$ of a polyhedron with at least one vertex is the maximum of $d(x, y)$ over all pairs of vertices (x, y) of P .
 3. $\Delta(n, m)$ is the maximum of $D(P)$ over all full-dimensional **polytopes** in $\mathbb{R}^n$ with at most m facets, where $m \geq n + 1$.
 4. $\Delta_u(n, m)$ is the maximum of $D(P)$ over all full-dimensional **pointed polyhedra** in $\mathbb{R}^n$ with at most m facets. Here, pointed means that the polyhedron contains no line.

The diameter of a polyhedron or polytope gives a **lower bound** on the worst-case number of simplex steps, **independently** of the pivot rule.

The famous **polynomial Hirsch conjecture** states:

Conjecture 9.9 (Polynomial Hirsch Conjecture).

$$\Delta(n, m) \leq \text{poly}(n, m).$$

The following **bounds** are known:

Lower bound: The unbounded Klee-Walkup polyhedron has dimension 4, eight facets, and diameter 5. Thus,

$$\Delta_u(4, 8) \geq 5 > 8 - 4.$$

Upper bound:

$$\Delta(n, m) \leq \Delta_u(n, m) \leq m^{1+\log_2(n)} = (2n)^{\log_2(m)}.$$

- The **Hirsch conjecture**, $\Delta(n, m) \leq m - n$, was disproved by Paco Santos in 2010 for $n = 43$ and $m = 86$.

Remark 9.10. There have been several successful attempts to explain the good empirical performance of the simplex method mathematically. For example, Borgwardt showed in a series of papers that, under certain assumptions, the **expected running time of the simplex method** is polynomial in a suitable probabilistic model.

9.2 Polynomial-Time Algorithms

In the following sections, we study the first known method for solving linear programs in **polynomial time**, due to the Russian mathematician Leonid Khachiyan (1979). We first need a formal definition of the **running time of an algorithm**.

To describe running times, we need a **model of computation**, **encoding conventions**, and related notions. We introduce only those concepts needed for linear programming.

In our model of computation, each number must be represented by finitely many bits. We therefore assume that all LPs have **rational coefficients**.

Given an inequality

$$ax \leq \alpha$$

with $a \in \mathbb{Q}^n$ and $\alpha \in \mathbb{Q}$, let p be the least common multiple of the denominators of the components of a and of α . The inequality

$$pax \leq p\alpha$$

then has **integer** coefficients and is clearly equivalent to the original inequality.

Assumption 9.11. All input data of the LPs under consideration are **integers**. Thus, for $\min\{cx | Ax \leq b, x \geq 0\}$ or $\max\{cx | Ax \leq b, x \geq 0\}$, we assume $c^T \in \mathbb{Z}^n$, $A \in \mathbb{Z}^{m \times n}$, and $b \in \mathbb{Z}^m$.

A computer, and hence our model of computation, uses **binary encodings**. Representing an integer n in binary requires $\lceil \log_2(|n| + 1) \rceil$ bits and one additional bit for the sign.

Definition 9.12. For $n \in \mathbb{Z}$, the quantity

$$\langle n \rangle := \lceil \log_2(|n| + 1) \rceil + 1$$

is called the **encoding length** of n . For a nonintegral rational number $r = p/q$ in lowest terms with $q > 0$, the encoding length is

$$\langle r \rangle = \langle p \rangle + \langle q \rangle.$$

The encoding length of a matrix $A = (a_{ij}) \in \mathbb{Q}^{m \times n}$, and analogously of a vector, is

$$\langle A \rangle = \sum_{i=1}^m \sum_{j=1}^n \langle a_{ij} \rangle.$$

We record some useful properties of the encoding length.

Lemma 9.13. 1. For every $r \in \mathbb{Q}$, we have

$$|r| \leq 2^{\langle r \rangle - 1} - 1.$$

2. For all $r, s \in \mathbb{Q}$, we have

$$\langle rs \rangle \leq \langle r \rangle + \langle s \rangle.$$

3. For every vector $x \in \mathbb{Q}^n$, we have

$$\|x\|_2 \leq \|x\|_1 \leq 2^{\langle x \rangle - n} - 1.$$

4. For every matrix $A \in \mathbb{Q}^{n \times n}$, we have

$$|\det(A)| \leq 2^{\langle A \rangle - n^2} - 1.$$

Proof. Statements (1) and (2) are left as exercises.

For (3), let $x \in \mathbb{Q}^n$. By (1),

$$1 + \|x\|_1 = 1 + \sum_{i=1}^n |x_i| \leq \prod_{i=1}^n (1 + |x_i|) \leq \prod_{i=1}^n 2^{\langle x_i \rangle - 1} = 2^{\langle x \rangle - n}.$$

The inequality

$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2} \leq \sum_{i=1}^n |x_i| = \|x\|_1$$

is immediate.

For (4), we obtain

$$1 + |\det(A)| \leq 1 + \prod_{j=1}^n \|a^j\|_2 \leq \prod_{j=1}^n (1 + \|a^j\|_2) \leq \prod_{j=1}^n 2^{\langle a^j \rangle - n} = 2^{\langle A \rangle - n^2}.$$

The first inequality is [Hadamard's inequality](#), and the third follows from (3). $\square$

In the following, we describe the running time of an algorithm in terms of the number of [elementary arithmetic operations](#). These are [addition](#), [subtraction](#), [multiplication](#), [division](#), and [comparison](#) of integers or rational numbers. We must also account for the size of the numbers involved, for example when multiplying large numbers.

Definition 9.14. The [running time of an algorithm](#) A on a problem instance Π , denoted by $L_A(\Pi)$, is the number of elementary bit operations performed during its execution.

Arithmetic operations on integers or rational numbers can be implemented using a number of bit operations polynomial in their encoding lengths. Thus, to establish polynomial running time, it suffices to bound both the number of arithmetic operations and the encoding lengths of all intermediate numbers by polynomials in the input size. Multiplying the operation count by the largest encoding length does not, in general, give the actual bit running time.

We can now define [polynomial running time](#).

Definition 9.15. Let A be an algorithm for a class of problems with instances Π . Let $\langle \Pi \rangle$ denote the encoding length of the data describing Π .

1. The function $f_A : \mathbb{N} \rightarrow \mathbb{N}$, defined by

$$f_A(n) := \max\{L_A(\Pi) \mid \langle \Pi \rangle \leq n\}$$

is called the [running-time function](#) of A .

2. The algorithm A has **polynomial running time**, or is a **polynomial-time algorithm**, if there is a polynomial $p : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$f_A(n) \leq p(n) \quad \forall n \in \mathbb{N}.$$

Remark 9.16. For solving LPs, we therefore seek an algorithm whose running time is bounded by a polynomial in $\langle A \rangle + \langle b \rangle + \langle c \rangle$.

9.3 Reduction to Feasibility Problems – LP Duality

The ellipsoid method, which we introduce below, is a feasibility algorithm: for a given full-dimensional polytope, it finds a feasible point or determines that the set is empty. We must therefore reduce linear optimization to such a feasibility problem.

Consider an LP of the form

$$\begin{aligned} \min & \quad cx \\ & Ax = b \\ & \quad x \geq 0, \end{aligned} \tag{LP}$$

Its dual LP is

$$\begin{aligned} \max & \quad \lambda b \\ & \quad \lambda A \leq c. \end{aligned} \tag{DP}$$

By strong duality, the two problems have optimal solutions with equal objective values if and only if both are **feasible**. Thus, every point of the polyhedron $P^{\leq}$ defined by

$$\begin{pmatrix} c & -b^T \\ -A & 0 \\ A & 0 \\ 0 & A^T \\ -I & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda^T \end{pmatrix} \leq \begin{pmatrix} 0 \\ -b \\ b \\ c^T \\ 0 \end{pmatrix}. \tag{9.9}$$

yields an optimal solution of (LP) by taking its x -component.

The resulting polyhedron $P^{\leq}$ is often not full-dimensional, i.e., $\text{int}(P^{\leq}) = \emptyset$. This can occur even when the original primal feasible region was full-dimensional.

9.4 Reduction by Binary Search

We use bounds on the “size” of vertices to reduce linear optimization to problems over polytopes. These polytopes need not be full-dimensional; we address this issue later.

Throughout this reduction, we assume $\text{rank}(A) = m$. Gaussian elimination detects inconsistent equations and removes redundant ones in polynomial time. If no equations remain, the problem over $x \geq 0$ can be solved directly.

Theorem 9.17. For every vertex $v = (v_1, \dots, v_n)$ of a polyhedron $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ with $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $\text{rank}(A) = m$, we have

$$v_i \leq 2^{\langle A \rangle + \langle b \rangle - m^2} \text{ for all } i = 1, \dots, n.$$

Proof. If v is a vertex of P , there is a nonsingular $m \times m$ submatrix A_B of A such that v_B is the unique solution of the linear system

$$A_B x_B = b.$$

By Cramer's rule,

$$v_i = \frac{\det A_B^i}{\det A_B} = \frac{|\det A_B^i|}{|\det A_B|} \text{ for } i \in B, \quad v_i = 0 \text{ for } i \in N,$$

where A_B^i is obtained from A_B by replacing the column corresponding to i with b . Since $v_i \geq 0$, we have

$$\frac{\det A_B^i}{\det A_B} = \frac{|\det A_B^i|}{|\det A_B|}.$$

Recall Laplace's expansion of the determinant along the i -th row of an $n \times n$ matrix F :

$$\det F = \sum_{j=1}^n (-1)^{i+j} f_{ij} \det F_{ij}, \quad (9.10)$$

Here, F_{ij} is the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j of F . Since A_B is nonsingular and integral, (9.10) implies

$$|\det A_B| \geq 1.$$

Moreover, A_B is a submatrix of A , and Lemma 9.13(4) yields

$$|\det A_B^i| \leq 2^{\langle A \rangle + \langle b \rangle - m^2}.$$

□

This theorem has the following consequence.

Theorem 9.18. The primal problem (LP) has a finite optimal solution if and only if the two LPs

$$\begin{aligned} \min cx \\ Ax = b \\ x_i \leq 2^{\langle A \rangle + \langle b \rangle - m^2} \text{ for } i = 1, \dots, n \\ x \geq 0 \end{aligned} \quad (\text{LP-Polytope})$$

$$\begin{aligned} \min cx \\ Ax = b \\ x_i \leq 2^{\langle A \rangle + \langle b \rangle - m^2} + 1 \text{ for } i = 1, \dots, n \\ x \geq 0. \end{aligned} \quad (\text{LP-Polytope}+1)$$

have finite optimal solutions with the same objective value. Problem (LP) is unbounded if and only if the optimal values of (LP-Polytope) and (LP-Polytope+1) differ. It is infeasible if and only if either of these two problems is infeasible.

We have reduced a general LP in standard form to LPs over a polytope.

It therefore suffices to solve LPs over polytopes.

Theorem 9.19. Consider (LP) with $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $c \in \mathbb{Z}^n$, and suppose that $P^=(A, b)$ is a nonempty polytope. Then the optimal value of cx belongs to the finite set

$$\mathcal{S} := \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, |p| \leq n 2^{\langle A \rangle + \langle c \rangle + \langle b \rangle - m^2}, 1 \leq q \leq 2^{\langle A \rangle - m^2} \right\} \quad (9.11)$$

Proof. By the fundamental theorem of linear optimization, it suffices to consider vertices v of $P^=(A, b)$:

$$cv = \sum_{i=1}^n c_i v_i.$$

As in the preceding proof, we obtain

$$\sum_{i=1}^n c_i v_i = \sum_{i \in B} c_i \frac{|\det A_B^i|}{|\det A_B|} =: \frac{p}{q} \in \mathbb{Q}.$$

Set

$$q := |\det A_B|$$

and

$$p := \sum_{i \in B} c_i |\det A_B^i|$$

Then

$$|p| \leq n 2^{\langle A \rangle + \langle c \rangle + \langle b \rangle - m^2}.$$

□

This theorem allows us to perform a **binary search** over the finite set $\mathcal{S}$.

Binary Search

Input: An LP of the form $\min\{cx \mid Ax = b, x \geq 0\}$ with $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$, and bounded feasible region.

Output: An optimal solution x^* , or the conclusion that $P^=(A, b) = \emptyset$.

1. Check whether

$$P^=(A, b) = \emptyset.$$

If so, STOP: the LP is infeasible.

2. Choose $s \in \mathcal{S}$ such that the sets $\mathcal{S}' = \{t \in \mathcal{S} \mid t \leq s\}$ and $\mathcal{S}'' = \{t \in \mathcal{S} \mid t > s\}$ satisfy

$$|\mathcal{S}'| \leq |\mathcal{S}''| \leq |\mathcal{S}'| + 1.$$

3. Check whether

$$P_s := P^=(A, b) \cap \{x \in \mathbb{R}^n \mid cx \leq s\} = \emptyset.$$

4. If $P_s = \emptyset$, set $\mathcal{S} \leftarrow \mathcal{S}''$; otherwise, set $\mathcal{S} \leftarrow \mathcal{S}'$.

5. If $|\mathcal{S}| = 1$, then its unique element s satisfies

$$s = \min\{cx \mid x \in P^=(A, b)\},$$

Call the feasibility algorithm once more to obtain a point in P_s for this final value of s , and return it as an optimal solution. Otherwise, return to Step (2).

Theorem 9.20. **Binary Search** is correct and terminates after at most

$$\log_2(n) + 2\langle A \rangle + \langle c \rangle + \langle b \rangle - 2m^2 + 3$$

executions of Step (3).

Proof. Each iteration reduces the cardinality of $\mathcal{S}$ by approximately one half. Thus, at most

$$N' = \lceil \log_2(|\mathcal{S}|) \rceil$$

splits into two subsets of nearly equal size are needed to obtain a singleton. By Theorem 9.19,

$$|\mathcal{S}| \leq \left(2n \cdot 2^{\langle A \rangle + \langle c \rangle + \langle b \rangle - m^2} + 1\right) \cdot 2^{\langle A \rangle - m^2} \leq 2^2 \cdot n \cdot 2^{2\langle A \rangle + \langle c \rangle + \langle b \rangle - 2m^2}$$

Hence, the number of executions of Step (3) is bounded by

$$\begin{aligned} N' &\leq \log_2(|\mathcal{S}|) + 1 \leq \log_2 \left(2^2 \cdot n \cdot 2^{2\langle A \rangle + \langle c \rangle + \langle b \rangle - 2m^2} \right) + 1 \\ &= \log_2 n + \log_2 2^2 + \log_2 \left(2^{2\langle A \rangle + \langle c \rangle + \langle b \rangle - 2m^2} \right) + 1 \\ &= \log_2 n + 2 + 2\langle A \rangle + \langle c \rangle + \langle b \rangle - 2m^2 + 1 \end{aligned}$$

This is polynomial in $\langle A \rangle + \langle c \rangle + \langle b \rangle$. $\square$

This proves a polynomial bound on the number of feasibility calls. It does not yet prove polynomial running time: explicitly listing $\mathcal{S}$ may take exponential time, and selecting its median efficiently requires a separate argument. We can avoid this issue by using the primal-dual formulation (9.9).

We obtain the following result.

Theorem 9.21. There is a polynomial-time algorithm for solving (LP) if there is a polynomial-time algorithm that decides whether a polyhedron $P^\leq(A, b)$ with integer data is empty and, if it is nonempty, computes a point in $P^\leq(A, b)$.

Proof. First test primal feasibility. If the primal problem is infeasible, report this. Otherwise, test dual feasibility. If the dual is infeasible, the feasible primal problem is unbounded. If both are feasible, strong duality implies that (9.9) is feasible, and any point in this system yields a primal optimum. Each of these three systems has encoding length polynomial in the original input size. $\square$

For $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$, the **ellipsoid method** initially solves the following problem under the assumption that P is bounded:

1. Find $x \in P := P^\leq(A, b) = \{x \in \mathbb{R}^n | Ax \leq b\}$, or
2. determine that $\overset{\circ}{P} = \emptyset$.

9.5 Geometry of Ellipsoids

Definition 9.22. A set $E \subset \mathbb{R}^n$ is called an ellipsoid if there are a vector $a \in \mathbb{R}^n$ and a symmetric positive definite $n \times n$ matrix A such that

$$E = E(A, a) := \{x \in \mathbb{R}^n | (x - a)^\top A^{-1} (x - a) \leq 1\}. \quad (9.12)$$

Note that the inverse of a positive definite matrix is again positive definite. Using the **ellipsoid norm** on $\mathbb{R}^n$ defined by a symmetric positive definite $n \times n$ matrix A ,

$$\|x\|_A := \sqrt{x^\top A^{-1} x}$$

we obtain the equivalent representation

$$E(A, \mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{a}\|_A \leq 1\}. \quad (9.13)$$

Thus, E is the unit ball centered at $\mathbf{a}$ with respect to the norm $\|\cdot\|_A$.

Moreover, $E(A, \mathbf{a})$ is the image of the Euclidean unit ball

$$B_1(0) := \{\mathbf{u} \in \mathbb{R}^n \mid \|\mathbf{u}\|_2 \leq 1\}$$

under an [affine transformation](#).

Theorem 9.23. We have

$$E(A, \mathbf{a}) = f(B_1(0)) := A^{1/2}B_1(0) + \mathbf{a},$$

where $A^{1/2}$ is the unique symmetric positive definite square root of A , satisfying

$$A^{1/2}A^{1/2} = A.$$

Proof. We obtain

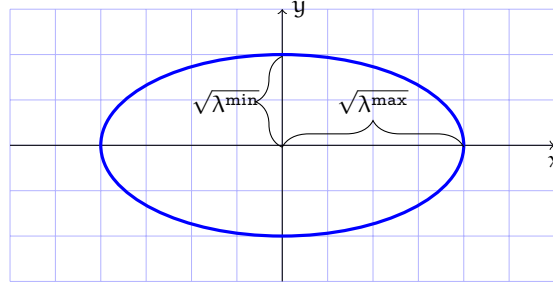
$$\begin{aligned} A^{1/2}B_1(0) + \mathbf{a} &= \{A^{1/2}\mathbf{y} \in \mathbb{R}^n : \mathbf{y} \in B_1(0)\} + \mathbf{a} \\ &= \{\mathbf{x} \in \mathbb{R}^n : (A^{1/2})^{-1}\mathbf{x} \in B_1(0)\} + \mathbf{a} \\ &= \{\mathbf{x} \in \mathbb{R}^n : ((A^{1/2})^{-1}\mathbf{x})^\top ((A^{1/2})^{-1}\mathbf{x}) \leq 1\} + \mathbf{a} \\ &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top (A^{1/2}A^{1/2})^{-1}\mathbf{x} \leq 1\} + \mathbf{a} \\ &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top A^{-1}\mathbf{x} \leq 1\} + \mathbf{a} \\ &= \{\mathbf{x} + \mathbf{a} \in \mathbb{R}^n : \mathbf{x}^\top A^{-1}\mathbf{x} \leq 1\} \\ &= \{\mathbf{z} \in \mathbb{R}^n : (\mathbf{z} - \mathbf{a})^\top A^{-1}(\mathbf{z} - \mathbf{a}) \leq 1\} \\ &= E(A, \mathbf{a}). \end{aligned}$$

□

Further geometric properties of an ellipsoid follow from the algebraic properties of A :

1. The diameter of $E(A, \mathbf{a})$, defined as the length of its longest [axis](#), is $2\sqrt{\lambda^{\max}}$, where $\lambda^{\max}$ is the largest eigenvalue of A .
2. The width, defined as the length of its shortest [axis](#), is $2\sqrt{\lambda^{\min}}$, where $\lambda^{\min}$ is the smallest eigenvalue of A .
3. The [principal axes](#) are given by the eigenvectors of A .

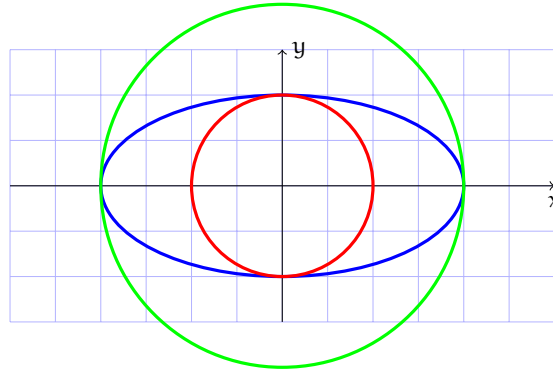
These observations imply that $B_{\sqrt{\lambda^{\max}}}(\mathbf{a})$ is the smallest Euclidean ball containing $E(A, \mathbf{a})$, and $B_{\sqrt{\lambda^{\min}}}(\mathbf{a})$ is the largest Euclidean ball contained in $E(A, \mathbf{a})$.

Figure 9.2: The ellipsoid $E(A, 0)$ centered at the origin.

The ellipsoid in Figure 9.2 is defined by

$$A = \begin{pmatrix} 16 & 0 \\ 0 & 4 \end{pmatrix}.$$

Its eigenvalues are $\lambda_1 = 16$ and $\lambda_2 = 4$, with eigenvectors $v_1 = (1, 0)^\top$ and $v_2 = (0, 1)^\top$. The diameter of $E(A, 0)$ is $2\sqrt{\lambda_{\max}} = 8$, and its width is $2\sqrt{\lambda_{\min}} = 4$.

Figure 9.3: The ellipsoid $E(A, 0)$ with its smallest enclosing ball and largest inscribed ball.

The **volume** of ellipsoids will play a central role in the following.

Definition 9.24. Let $K \subset \mathbb{R}^n$ be Lebesgue measurable. The **volume** of K is defined by

$$\text{Vol}(K) := \int_{x \in K} dx.$$

For ellipsoids, we obtain the following formula.

Theorem 9.25. The volume of an ellipsoid $E(A, a)$ is

$$\text{Vol}(E(A, a)) = \sqrt{\det A} \cdot V_n,$$

where V_n is the volume of the unit ball $B_1(0)$ in $\mathbb{R}^n$.

Proof. Using

$$E(A, a) = f(B_1(0)) = A^{1/2}B_1(0) + a,$$

and applying a change of variables gives

$$\int_{y \in f(B_1(0))} dy = \int_{x \in B_1(0)} |\det A^{1/2}| dx = \sqrt{\det A} \cdot V_n. \quad \square$$

The volume V_n has the representation

$$V_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} \approx \frac{1}{\sqrt{\pi n}} \left(\frac{2e\pi}{n} \right)^{n/2}, \quad (9.14)$$

where

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0,$$

denotes the Gamma function.

We will repeatedly need estimates for V_n . The following coarse bounds suffice.

Lemma 9.26.

$$n^{-n} \leq V_n \leq 2^n.$$

Proof. The unit ball $B_1(0)$ contains the set $B^1 := \{x \in \mathbb{R}^n | 0 \leq x_i \leq 1/n, i = 1, \dots, n\}$, whose volume is $\text{Vol}(B^1) = n^{-n}$. Moreover, $B_1(0)$ is contained in $B^2 := \{x \in \mathbb{R}^n | \|x\|_\infty \leq 1\}$, and $\text{Vol}(B^2) = 2^n$. $\square$

We next compare the **volume ratio** of two ellipsoids. By Theorem 9.25, an invertible affine transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $x \mapsto Dx + d$, satisfies

$$\text{Vol}(T(E(A, a))) = |\det D| \sqrt{\det A} V_n.$$

Thus, for two ellipsoids $E(A, a)$ and $E(B, b)$,

$$\frac{\text{Vol}(E(A, a))}{\text{Vol}(E(B, b))} = \frac{\text{Vol}(T(E(A, a)))}{\text{Vol}(T(E(B, b)))}. \quad (9.15)$$

We will also need to optimize a linear function over an ellipsoid. Recall that a linear function cx , with $c^\top \neq 0$, attains its maximum over the unit ball $B_1(a)$ at $a + c^\top / \|c^\top\|_2$.

Theorem 9.27. Let $E(A, a) \subset \mathbb{R}^n$ be an ellipsoid and $c^\top \neq 0$. Set

$$b := \frac{Ac^\top}{\sqrt{cAc^\top}}, \quad z_{\max} := a + b, \quad z_{\min} := a - b.$$

These points attain the maximum and minimum of cx over $E(A, a)$, respectively. In particular,

$$\begin{aligned} cz_{\max} &:= \max\{cx \mid x \in E(A, a)\} = ca + \sqrt{cAc^\top} = ca + \|c^\top\|_{A^{-1}}, \\ cz_{\min} &:= \min\{cx \mid x \in E(A, a)\} = ca - \sqrt{cAc^\top} = ca - \|c^\top\|_{A^{-1}}. \end{aligned} \quad (9.16)$$

Proof. Set $Q := A^{1/2}$. We have

$$Q^{-1}E(A, a) = B_1(0) + Q^{-1}a = B_1(Q^{-1}a)$$

and hence

$$\begin{aligned} \max\{cx \mid x \in E(A, a)\} &= \max\{cQQ^{-1}x \mid Q^{-1}x \in Q^{-1}E(A, a)\} \\ &= \max\{cQy \mid y \in B_1(Q^{-1}a)\} \\ &= cQ \frac{1}{\|Qc^\top\|_2} Qc^\top + cQQ^{-1}a \\ &= c \frac{1}{\sqrt{cAc^\top}} Ac^\top + ca \\ &= ca + \sqrt{cAc^\top}. \end{aligned}$$

The maximizing point is $a + Ac^\top / \sqrt{cAc^\top} = z_{\max}$. Applying the same argument to $-c$ gives the minimum at $z_{\min}$. $\square$

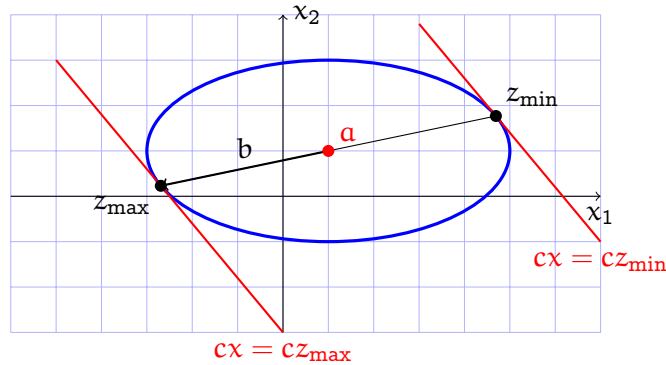


Figure 9.4: The maximum and minimum of a linear function over $E(A, a)$, with $z_{\max} = a + b$ and $z_{\min} = a - b$.

We record a classical result without proof: every convex, compact, full-dimensional set in $\mathbb{R}^n$ has a unique enclosing ellipsoid of minimum volume and a unique inscribed ellipsoid of maximum volume.

We first recall the notions of [affine independence](#) and [dimension](#) for subsets of $\mathbb{R}^n$.

Definition 9.28. Points $x_1, \dots, x_k \in \mathbb{R}^n$ are [affinely independent](#) if the vectors $x_2 - x_1, \dots, x_k - x_1$ are linearly independent. For $S \subseteq \mathbb{R}^n$, the [affine rank](#) $\text{arank}(S)$ is the cardinality of a largest affinely independent subset of S . The dimension of S is defined by

$$\dim(S) = \text{arank}(S) - 1.$$

A set $S \subseteq \mathbb{R}^n$ is [full-dimensional](#) if $\dim(S) = n$.

Theorem 9.29 (Löwner, John). Every convex, compact, full-dimensional set $K \subset \mathbb{R}^n$ has a unique enclosing ellipsoid $E = E(A, a)$ of minimum volume.

This ellipsoid is called the [Löwner-John ellipsoid](#) of K .

9.6 The Ellipsoid Method

We now give a geometric description of the ellipsoid method.

Given a polytope $P := P^{\leq}(C, d)$, we want to find $x \in P$ or show that $\overset{\circ}{P} = \emptyset$.

We first inspect every zero row of C . An inequality $0 \leq d_i$ with $d_i < 0$ proves infeasibility immediately; otherwise, it can be deleted. All rows used to define an ellipsoid update are therefore nonzero.

1. Construct an ellipsoid $E_0 = E(A_0, a_0)$ with $P \subseteq E_0$, and set $k \leftarrow 0$.
2. If the volume of E_k is “too small”, STOP: $\overset{\circ}{P} = \emptyset$.
3. Check whether $a_k \in P$.
4. If $a_k \in P$, STOP.
5. If $a_k \notin P$, there is a violated inequality

$$ca_k > d_i,$$

where c is the i -th row of C . Let

$$E'_k := E_k \cap \{x \in \mathbb{R}^n \mid cx \leq ca_k\}$$

be the [half-ellipsoid](#) of E_k containing P . Construct its Löwner-John ellipsoid and denote it by E_{k+1} .

6. Set $k \leftarrow k + 1$ and return to Step (2).

To make this geometric procedure precise, we must specify how to construct the initial ellipsoid and the Löwner-John ellipsoid of a half-ellipsoid.

The initial ellipsoid will be a ball centered at the origin.

If $P = P^{\leq}(C, d)$ contains explicit inequalities of the form

$$\ell_i \leq x_i \leq u_i \text{ for } i = 1, \dots, n,$$

we can choose

$$R := \max \left\{ 1, \sqrt{\sum_{i=1}^n (\max\{|u_i|, |\ell_i|\})^2} \right\}$$

as its radius. More generally, the following holds.

Lemma 9.30. Let $P = \{x \in \mathbb{R}^n | Cx \leq d\}$ be a polytope with integer data C, d . Then $P \subseteq B_R(0)$ for $R := \sqrt{n} 2^{\langle C \rangle + \langle d \rangle - n^2}$.

Proof. As in the proof of Theorem 9.17, every vertex v of P satisfies

$$|v_i| \leq 2^{\langle C \rangle + \langle d \rangle - n^2} \text{ for all } i = 1, \dots, n.$$

Thus, its Euclidean norm satisfies

$$\|v\|_2 = \sqrt{\sum_{i=1}^n v_i^2} \leq \sqrt{n} \max\{|v_i|\} \leq \sqrt{n} 2^{\langle C \rangle + \langle d \rangle - n^2}.$$

Every vertex of P is therefore contained in $B_R(0)$. Since P is a polytope, $P \subseteq B_R(0)$. $\square$

We next specify how, in iteration k , an ellipsoid E_k and a violated inequality

$$ca_k > d_i$$

determine the Löwner-John ellipsoid of

$$E'_k := E_k \cap \{x \in \mathbb{R}^n | cx \leq ca_k\}$$

For $n \geq 2$, define

$$b := \frac{1}{\sqrt{cA_k c^T}} A_k c^T \tag{9.17}$$

$$a_{k+1} := a_k - \frac{1}{n+1} b \tag{9.18}$$

$$A_{k+1} := \frac{n^2}{n^2 - 1} \left(A_k - \frac{2}{n+1} b b^T \right). \tag{9.19}$$

The resulting ellipsoid $E_{k+1} = E(A_{k+1}, a_{k+1})$ is the required Löwner-John ellipsoid. The construction has the following geometric interpretation. Since $c \neq 0$, the maximum and

minimum of cx over E_k can be computed explicitly by Theorem 9.27. The new center a_{k+1} lies on the line segment between a_k and $z_{\min}$. This segment has length $\|b\|_2$, and a_{k+1} is obtained from a_k by taking a step of length $\|b\|_2/(n+1)$ in direction $-b$. The point $z_{\min}$ lies on the boundaries of both E_{k+1} and E'_k .

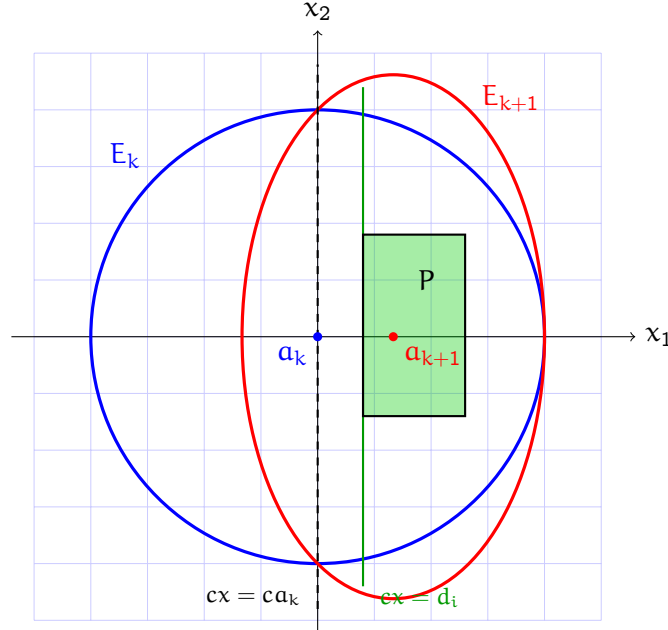


Figure 9.5: An ellipsoid update for $A_k = I$, $a_k = 0$, and $c = (-1, 0)$. The new center is $(1/3, 0)^\top$, and the new semiaxes have lengths $2/3$ and $2/\sqrt{3}$.

We must show that the ellipsoid method terminates after polynomially many steps. We use a volume argument: each iteration constructs an ellipsoid of strictly smaller volume. We now analyze the *rate* of this decrease.

Lemma 9.31. Let $E_k = E(A_k, a_k)$ be an ellipsoid in dimension $n \geq 2$, and let $E_{k+1} = E(A_{k+1}, a_{k+1})$ be the Löwner-John ellipsoid constructed by (9.17)–(9.19) for $c \neq 0$. Then

$$\frac{\text{Vol}(E_{k+1})}{\text{Vol}(E_k)} = \left(\left(\frac{n}{n+1} \right)^2 \left(\frac{n^2}{n^2-1} \right)^{n-1} \right)^{1/2} < e^{-1/(2(n+1))} < 1.$$

Proof. We first consider the case where E_k is the unit ball and

$$c = (-1, 0, \dots, 0).$$

Using (9.17)–(9.19), we obtain

$$b = (-1, 0, \dots, 0)^\top$$

$$a_{k+1} = a_k - \frac{1}{n+1}b = \left(\frac{1}{n+1}, 0, \dots, 0 \right)^\top$$

$$\begin{aligned} A_{k+1} &= \frac{n^2}{n^2-1} \left(I - \frac{2}{n+1} (-1, 0, \dots, 0)^\top (-1, 0, \dots, 0) \right) \\ &= \text{diag} \left(\frac{n^2}{(n+1)^2}, \frac{n^2}{n^2-1}, \dots, \frac{n^2}{n^2-1} \right). \end{aligned}$$

We can also verify directly that the new ellipsoid contains the required half-ball. Write $x = (t, y)$, where $0 \leq t \leq 1$ and $\|y\|_2^2 \leq 1 - t^2$. Then

$$\frac{((n+1)t-1)^2}{n^2} + \frac{n^2-1}{n^2} \|y\|_2^2 \leq 1 + \frac{2(n+1)}{n^2} (t^2 - t) \leq 1.$$

Thus, $x \in E_{k+1}$. The affine transformation below gives the same containment for an arbitrary ellipsoid and nonzero normal c .

The volume ratio of $E_k = E(I, 0)$ and E_{k+1} is

$$\begin{aligned} \frac{\text{Vol}(E_{k+1})}{\text{Vol}(E_k)} &= \frac{\sqrt{\det A_{k+1}} V_n}{\sqrt{\det A_k} V_n} = \sqrt{\det A_{k+1}} = \left(\frac{n^{2n}}{(n+1)^2 (n^2-1)^{n-1}} \right)^{1/2} \\ &= \left(\frac{n}{n+1} \right) \left(\frac{n^2}{n^2-1} \right)^{\frac{n-1}{2}}. \end{aligned}$$

Rearranging and estimating gives

$$\begin{aligned} \left(\frac{n}{n+1} \right) \left(\frac{n^2}{n^2-1} \right)^{\frac{n-1}{2}} &= \left(1 + \frac{-1}{n+1} \right) \left(1 + \frac{1}{n^2-1} \right)^{\frac{n-1}{2}} \\ &\leq \left(e^{\frac{-1}{n+1}} \right) \left(e^{\frac{1}{n^2-1}} \right)^{\frac{n-1}{2}} \\ &= \left(e^{\frac{-1}{n+1}} \right) \left(e^{\frac{1}{2(n+1)}} \right) \\ &= e^{\frac{-1}{2(n+1)}}, \end{aligned} \tag{9.20}$$

Here, (9.20) follows from $1 + x \leq e^x$ for all $x \in \mathbb{R}$. One can verify this by observing that $f(x) := 1 + x - e^x$ attains its global maximum at $x = 0$. The inequality is strict for the nonzero arguments used above.

We now consider arbitrary E_k and $c \neq 0$. A suitable affine transformation of E_k and E_{k+1} reduces this to the case already considered.

Let $F := B_1(0)$, and let F' be the ellipsoid obtained from F by (9.17)–(9.19) with $c = (-1, 0, \dots, 0)$, as in the first case.

By Theorem 9.23,

$$E_k(A_k, a_k) = A_k^{1/2} B_1(0) + a_k = A_k^{1/2} F + a_k$$

From linear algebra, there is an orthogonal matrix Q that maps $A_k^{1/2} c^\top$ to a positive multiple of $(-1, 0, \dots, 0)^\top$, i.e.,

$$(-1, 0, \dots, 0)^\top = \frac{1}{\|A_k^{1/2} c^\top\|_2} Q A_k^{1/2} c^\top.$$

The required bijective affine transformation is

$$T(x) = A_k^{1/2} Q^T x + a_k,$$

with inverse

$$T^{-1}(x) = Q A_k^{-1/2} (x - a_k).$$

We verify that

$$T(F) = E_k \text{ and } T(F') = E_{k+1}.$$

The [relative](#) volume of two ellipsoids is invariant under invertible affine transformations; see (9.15). Hence,

$$\frac{\text{Vol}(E_{k+1})}{\text{Vol}(E_k)} = \frac{\text{Vol}(T(F'))}{\text{Vol}(T(F))} = \frac{\text{Vol}(F')}{\text{Vol}(F)} = \left(\frac{n}{n+1} \right) \left(\frac{n^2}{n^2-1} \right)^{\frac{n-1}{2}} \leq e^{\frac{-1}{2(n+1)}}.$$

□

We must also show that only polynomially many iterations are needed to determine whether P has nonempty interior. For this purpose, we derive a lower bound on the volume of a full-dimensional polytope.

Lemma 9.32. Let $P = \{x \in \mathbb{R}^n \mid Cx \leq d\}$ be a full-dimensional polytope with integer data C, d . Then

$$\text{Vol}(P) \geq 2^{-(n+1)\langle C \rangle + n^3}.$$

Proof. A full-dimensional polytope in $\mathbb{R}^n$ contains $n+1$ affinely independent [vertices](#). Their convex hull is a simplex $S := \text{co}(\{v_0, \dots, v_n\}) \subseteq P$, so $\text{Vol}(P) \geq \text{Vol}(S)$. The volume of a simplex is given by

$$\text{Vol}(S) = \frac{1}{n!} \left| \det \begin{pmatrix} 1 & \dots & 1 \\ v_0 & \dots & v_n \end{pmatrix} \right|.$$

This follows from the change-of-variables formula, since S is an affine image of the unit simplex $S_n := \{x \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i \leq 1\}$, whose volume is $\text{Vol}(S_n) = 1/n!$.

For every vertex v_i of P , its j -th component has the form $\det B_{ij} / \det B_i$, where B_i is a nonsingular submatrix of C and B_{ij} is a submatrix of (C, d) . Using $\det(AB) = \det A \cdot \det B$ for square matrices of the same size, we obtain

$$\left| \det \begin{pmatrix} 1 & \dots & 1 \\ v_0 & \dots & v_n \end{pmatrix} \right| = \left| \det \begin{pmatrix} \begin{pmatrix} \det B_0 & \dots & \det B_n \\ \det B_{01} & \dots & \det B_{n1} \\ \dots \\ \det B_{0n} & \dots & \det B_{nn} \end{pmatrix} \text{diag} \left(\frac{1}{\det B_0}, \dots, \frac{1}{\det B_n} \right) \end{pmatrix} \right|$$

$$= \left| \frac{1}{\det B_0 \cdots \det B_n} \right| \left| \det \begin{pmatrix} \det B_0 & \cdots & \det B_n \\ u_0 & \cdots & u_n \end{pmatrix} \right|,$$

where $u_0, \dots, u_n \in \mathbb{Z}^n$.

The second factor is a positive integer and hence at least 1. It therefore suffices to bound $|1/(\det B_0 \cdots \det B_n)|$ from below. Lemma 9.13 gives

$$|\det B_j| \leq 2^{\langle B_j \rangle - n^2} - 1 \leq 2^{\langle C \rangle - n^2} \text{ for all } j = 0, \dots, n,$$

and therefore

$$\left| \det \begin{pmatrix} 1 & \cdots & 1 \\ v_0 & \cdots & v_n \end{pmatrix} \right| \geq \left| \frac{1}{\det B_0 \cdots \det B_n} \right| \geq \left(2^{\langle C \rangle - n^2} \right)^{-(n+1)}.$$

Finally, $n! \leq 2^{n^2}$ implies

$$\text{Vol}(P) \geq \text{Vol}(S) \geq \frac{1}{n!} \left(2^{\langle C \rangle - n^2} \right)^{-(n+1)} \geq 2^{-(n+1)\langle C \rangle + n^3}.$$

□

It remains to estimate the running time of the ellipsoid method. By Lemma 9.31, each iteration reduces the volume by a factor of at most $e^{-1/(2(n+1))}$. We must determine how many iterations are needed to reduce the volume of the initial ball $B_R(0)$, where $R := \sqrt{n} 2^{\langle C \rangle + \langle d \rangle - n^2}$, below $2^{-(n+1)\langle C \rangle + n^3}$.

Lemma 9.33. Let $P = \{x \in \mathbb{R}^n | Cx \leq d\}$ be a full-dimensional polytope with integer data C, d and $n \geq 2$, and let $E_0 = B_R(0)$ with $R := \sqrt{n} 2^{\langle C \rangle + \langle d \rangle - n^2}$ be the initial ellipsoid. After $N := (2n+2)((n+1)\langle C \rangle + n(\langle C \rangle + \langle d \rangle))$ ellipsoid updates, we have

$$\text{Vol}(E_N) < 2^{-N/(2n+2)+nT} \leq \text{Vol}(P),$$

where $T := \langle C \rangle + \langle d \rangle$.

Proof. We have $E_0 = B_R(0) \subseteq \{x \in \mathbb{R}^n | \|x\|_\infty \leq R\}$, with $R := \sqrt{n} 2^{\langle C \rangle + \langle d \rangle - n^2}$; see Lemma 9.30.

Thus,

$$\begin{aligned} \text{Vol}(E_0) &\leq 2^n R^n = n^{n/2} 2^{n\langle C \rangle + n\langle d \rangle - n^3 + n} = 2^{n\langle C \rangle + n\langle d \rangle - n^3 + n \log_2(n)/2} \\ &= 2^{n(\langle C \rangle + \langle d \rangle - n^2 + 1 + \log_2(n)/2)} < 2^{n(\langle C \rangle + \langle d \rangle)} := 2^{nT}, \end{aligned}$$

where $T := \langle C \rangle + \langle d \rangle$.

Each step reduces the ellipsoid volume by a factor of at most $e^{-1/(2(n+1))}$. Hence,

$$\text{Vol}(E_N) < e^{-N/(2n+2)} \text{Vol}(E_0) < 2^{-N/(2n+2)+nT}.$$

It suffices to choose N large enough that

$$\begin{aligned}
 2^{-N/(2n+2)+nT} &\leq 2^{-(n+1)\langle C \rangle + n^3} \\
 \Leftrightarrow -N/(2n+2) + nT &\leq -(n+1)\langle C \rangle + n^3 \\
 \Leftrightarrow N/(2n+2) &\geq (n+1)\langle C \rangle - n^3 + nT \\
 \Leftrightarrow N &\geq (2n+2)((n+1)\langle C \rangle - n^3 + nT).
 \end{aligned}$$

Our choice of N satisfies this inequality. □

9.7 Polynomial Running Time of the Ellipsoid Method

We can now state the ellipsoid method formally for a polytope $P = \{x \in \mathbb{R}^n | Cx \leq d\}$ with integer data. We assume $n \geq 2$; the one-dimensional case reduces to finding a feasible interval. As above, we first delete satisfied inequalities with zero normal and report infeasibility if such an inequality is violated. We choose a slightly larger iteration bound that is positive even when the feasible set is empty.

1. Initialization:

- (a) $k := 0$,
- (b) $N := (2n + 2) ((n + 1)\langle C \rangle + n(\langle C \rangle + \langle d \rangle))$,
- (c) $A_0 := R^2 I$ with $R := \sqrt{n} 2^{\langle C \rangle + \langle d \rangle - n^2}$,
- (d) $a_0 := 0$

2. General step:

- (a) If $k = N$, STOP: $\overset{\circ}{P} = \emptyset$. If P is promised to be either empty or full-dimensional, conclude that $P = \emptyset$.
- (b) If $a_k \in P$, STOP.
- (c) If $a_k \notin P$, choose a violated inequality

$$ca_k > \gamma.$$

Set:

$$\begin{aligned} b &:= \frac{1}{\sqrt{cA_k c^\top}} A_k c^\top \\ a_{k+1} &:= a_k - \frac{1}{n+1} b \\ A_{k+1} &:= \frac{n^2}{n^2 - 1} \left(A_k - \frac{2}{n+1} b b^\top \right) \\ k &:= k + 1. \end{aligned}$$

- (d) Return to Step (2).

Correctness follows from the preceding lemmas. To estimate the running time, we count the elementary operations needed for initialization and for the general step.

Initialization requires $O(n \cdot m)$ operations.

In Step (2), testing a_k against the system $Cx \leq d$ requires $O(m \cdot n)$ elementary operations. Computing b requires $O(n^2)$ arithmetic operations and one square root. Updating a_k requires $O(n)$ operations, and updating A_k requires $O(n^2)$ operations.

The number of ellipsoid updates is at most

$$N := (2n + 2) ((n + 1)\langle C \rangle + n(\langle C \rangle + \langle d \rangle)) = O(n^2(\langle C \rangle + \langle d \rangle)).$$

This gives an upper bound of $O(m \cdot n^4(\langle C \rangle + \langle d \rangle))$ on the total number of operations, which is **polynomial** in the input size. In this idealized count, square roots are treated as single operations. The implementation section below addresses rational approximation and the encoding lengths of intermediate numbers.

9.8 Polytope Feasibility without Full Dimensionality

So far, we have assumed that the polytopes under consideration are full-dimensional. We now show how to remove this assumption.

Let $P = \{x \in \mathbb{R}^n | Cx \leq d\}$ be a polytope with $C \in \mathbb{Z}^{m \times n}$ and $d \in \mathbb{Z}^m$.

Define a slightly perturbed polytope by

$$P(\epsilon) := \{x \in \mathbb{R}^n | Cx \leq d + \epsilon e\}, \quad (9.21)$$

where $e = (1, \dots, 1)^\top$ and

$$\epsilon := \frac{2^{-(\langle C \rangle + \langle d \rangle + 1)}}{n + 1}. \quad (9.22)$$

We show that $P = \emptyset$ if and only if $P(\epsilon) = \emptyset$, and that $P(\epsilon)$ is full-dimensional whenever $P \neq \emptyset$. To decide feasibility, it therefore suffices to apply the ellipsoid method to $P(\epsilon)$.

Lemma 9.34. Let $P = \{x \in \mathbb{R}^n | Cx \leq d\}$ be a polytope with $C \in \mathbb{Z}^{m \times n}$ and $d \in \mathbb{Z}^m$. Define $P(\epsilon)$ by (9.21), with ϵ as in (9.22). Then $P = \emptyset$ if and only if $P(\epsilon) = \emptyset$. Moreover, if $P \neq \emptyset$, then $P(\epsilon)$ is full-dimensional.

Proof. We first prove $P = \emptyset \Rightarrow P(\epsilon) = \emptyset$. The converse is immediate.

If $P = \emptyset$, then the LP

$$\min\{0 \cdot x | Cx \leq d\}$$

is infeasible, and its dual

$$\max\{-\lambda d | \lambda C = 0, \lambda \geq 0\}$$

is unbounded. Indeed, $\lambda = 0$ is dual feasible, so Theorem 6.7 implies dual unboundedness. We impose the additional hyperplane constraint

$$\lambda d = -1.$$

This yields the nonempty polyhedron in $\mathbb{R}^m$ described by

$$\lambda C = 0$$

$$\lambda d = -1$$

$$\lambda \geq 0.$$

Set $T := \langle C \rangle + \langle d \rangle$. Delete dependent equations from this nonempty system, and let its row rank be $r \leq n + 1$. Choose a basic feasible solution λ . Its basic components are quotients of determinants of integral $r \times r$ matrices. The denominator is nonzero and hence has absolute value at least 1. For the numerator, Cramer's rule replaces one column by a vector whose only possible nonzero entry is -1 . Expanding along that column leaves a determinant of a submatrix of $(C, d)^T$. Lemma 9.13 therefore gives

$$0 \leq \lambda_i \leq 2^T \text{ for all } i.$$

For $r = 1$, the numerator is 0 or ± 1 , and the same bound holds.

Since λ is a basic feasible solution, at most $n + 1$ components can be positive, as there are $n + 1$ equality constraints. Hence,

$$\sum_{i=1}^m \lambda_i \leq (n + 1)2^T.$$

By our choice of ϵ ,

$$\lambda(d + \epsilon e) = -1 + \epsilon \sum_{i=1}^m \lambda_i \leq -1 + \epsilon(n + 1)2^T = -\frac{1}{2} < 0.$$

The dual problem associated with $P(\epsilon)$ is therefore still unbounded, since every $t\lambda$ with $t > 0$ remains feasible. Thus, $P(\epsilon) = \emptyset$.

It remains to show that $P(\epsilon)$ is full-dimensional if $P \neq \emptyset$.

Let $x \in P$. The polytope

$$S := \{y \in \mathbb{R}^n \mid x_i - \bar{\epsilon} \leq y_i \leq x_i + \bar{\epsilon} \text{ for all } i = 1, \dots, n\}$$

is contained in $P(\epsilon)$, where

$$\bar{\epsilon} := \frac{\epsilon}{\max\{1, \max_i \sum_j |c_{ij}|\}}.$$

For $y \in S$, we have

$$Cy \leq Cx + \bar{\epsilon}\tilde{C}e \leq d + \epsilon e,$$

where $\tilde{C}$ is obtained from C by replacing each entry c_{ij} by $|c_{ij}|$, and the all-ones vectors have the appropriate dimensions.

The volume of S is $\text{Vol}(S) = (2\bar{\epsilon})^n > 0$. □

If P is nonempty and bounded, the only vector v satisfying $Cv \leq 0$ is $v = 0$. Since perturbing the right-hand side does not change these inequalities, $P(\epsilon)$ is also bounded.

If P is empty, the lemma shows that $P(\epsilon)$ is empty as well.

Before applying the ellipsoid method to $P(\epsilon)$, we clear the denominators of its right-hand side. We then compute the initial radius and iteration bound from this new integer system.

The perturbed input still has polynomial encoding length.

9.9 Recovering an Exact Feasible Point

A point of $P(\epsilon)$ need not belong to P . The preceding argument gives an exact **feasibility decision**, which we can use to recover a point of the original polytope.

Suppose that $P \neq \emptyset$. Starting with $F = P$, consider the original inequalities $c_i x \leq d_i$ one at a time. Test whether

$$F \cap \{x \in \mathbb{R}^n | c_i x = d_i\} \neq \emptyset.$$

If so, retain this equality in the description of F ; otherwise, leave F unchanged. Equalities are represented by two inequalities when calling the feasibility algorithm.

At the end, F is still a nonempty polytope. Every inequality for which equality was rejected is strict at every point of the final F , since subsequent steps only restrict the set. Choose a vertex of F . Its tight constraints have rank n , so the retained equalities must have rank n . Their unique solution, computed by Gaussian elimination, is therefore a point of P . This requires at most m additional feasibility decisions, each on a system of polynomial encoding length.

Finally, an arbitrary rational feasibility system can be converted to standard form by splitting unrestricted variables and adding slack variables. After removing redundant equations, Theorem 9.17 supplies bounds that preserve at least one feasible basic solution whenever the system is feasible. Adding these bounds reduces the problem to the polytope case. Together with the primal-dual reduction, this yields a polynomial-time algorithm for linear optimization, provided the ellipsoid updates are implemented with the precision described below.

9.10 Implementation

Initialization uses a square root to compute R , which may be irrational. However, only R^2 is needed to define the initial ellipsoid, so this causes no difficulty.

The square root in Step (2) requires more care. A common way to avoid irrational numbers is to truncate after a prescribed number of digits. This may, however, violate the invariant that the new ellipsoid contains the polytope. To preserve this invariant, we replace the Löwner-John ellipsoid by a slightly larger ellipsoid that can be represented using rational numbers of polynomial encoding length. The accuracy of this representation must be balanced against the resulting increase in volume. This can be done, but the details are technical and are omitted here.

Chapter 10

Integer Linear Optimization

In this final chapter, we give a brief introduction to linear optimization over subsets $S \subseteq \mathbb{Z}^n$. The essential difference from linear programming is the requirement that a feasible solution $x \in S$ have only **integer** components. This requirement removes the geometric and topological properties of the feasible region that we used to develop duality theory and prove correctness of the simplex method.

We consider the following class of problems:

$$\begin{aligned} & \text{minimize } cx \\ & Ax \leq b \\ & x \in \mathbb{Z}_+^n, \end{aligned} \tag{IP}$$

where $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, and $c^\top \in \mathbb{Q}^n$. The feasible set is

$$S := \{x \in \mathbb{R}_+^n : Ax \leq b\} \cap \mathbb{Z}_+^n.$$

We write $z_{\text{IP}} := \inf\{cx : x \in S\}$, with value $+\infty$ if $S = \emptyset$ and $-\infty$ if the objective is unbounded below. If the value is finite, rationality of c implies that it is attained: after multiplication by a common denominator, the objective values are integers. In this case we can write $z_{\text{IP}} = \min\{cx : x \in S\}$. Below, an empty minimization problem is understood to have value $+\infty$.

Example 10.1. We return to an example from the introduction to linear optimization. Let

$$S := \{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\} \cap \mathbb{Z}^n,$$

where the inequalities $Ax \leq b, x \geq 0$ are given by

$$2x_1 + x_2 \leq 8$$

$$4x_1 + 5x_2 \leq 20$$

$$6x_1 + 15x_2 \leq 45$$

$$x_1, x_2 \geq 0$$

The set S is easily identified in the figure below.

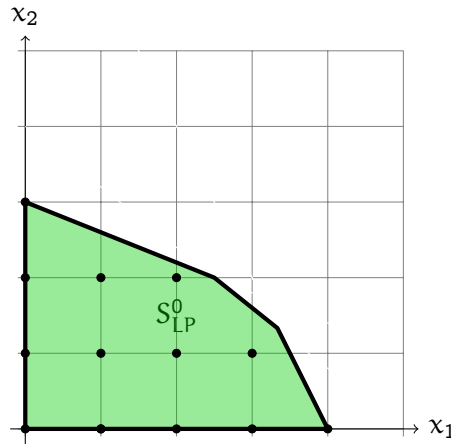


Figure 10.1: The black points form the integer feasible set S ; the shaded polygon is its LP relaxation S_{LP}^0 .

10.1 Total Dual Integrality and Integer Programs

For some integer programs, the LP relaxation already provides an integer optimum. We now give a sufficient condition based on the dual problem. Throughout this section, c is a row vector, as in the preceding chapters.

10.1.1 Totally Dual Integral Systems

Consider a rational system $Ax \leq b$ with $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$, and the primal-dual pair

$$\begin{aligned} (P_c) \quad & \max\{cx \mid Ax \leq b\}, \\ (D_c) \quad & \min\{\lambda b \mid \lambda A = c, \lambda \geq 0\}. \end{aligned} \tag{10.1}$$

Here, $x \in \mathbb{R}^n$ is unrestricted. A condition $x \geq 0$ must be included as the rows $-Ix \leq 0$ of the system when applying the following definition.

Definition 10.2. The system $Ax \leq b$ is totally dual integral (TDI) if, for every $c \in \mathbb{Z}^n$ for which (P_c) has a finite optimum, (D_c) has an optimal solution $\lambda \in \mathbb{Z}_+^m$.

Thus, TDI requires the existence of an integer dual optimum, not integrality of every dual feasible point. It is a property of the entire inequality system, including b and the chosen representation. The expression “TDI matrix” can therefore be misleading.

Example 10.3. Both systems

$$-x \leq 0, \quad x \leq 1 \quad \text{and} \quad -x \leq 0, \quad 2x \leq 2$$

describe $[0, 1]$. The first is TDI: for any integer $c \geq 0$, the multiplier of $x \leq 1$ can be c ; for $c < 0$, use multiplier $-c$ for $-x \leq 0$. The second is not TDI. For $c = 1$, its dual is

$$\min\{2\lambda_2 \mid -\lambda_1 + 2\lambda_2 = 1, \lambda_1, \lambda_2 \geq 0\}.$$

Its unique optimum is $(0, 1/2)$. Hence even an interval with integer endpoints can have a description that is not TDI.

10.1.2 Integer Right-Hand Sides and Integer Optima

Theorem 10.4 (Integrality consequence of TDI). Suppose $Ax \leq b$ is TDI and $b \in \mathbb{Z}^m$. Then every vertex of $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is integral. If P is nonempty and pointed, every linear objective with a finite optimum over P has an integer optimal vertex.

Proof. For every integer row vector h with a finite maximum over P , TDI and strong duality give

$$\max_{x \in P} hx = \lambda b \in \mathbb{Z}$$

for some integer optimal multiplier λ . Let v be a vertex. Choose n independent active rows and multiply each by a positive integer to obtain integer rows $a'_1, \dots, a'_n$. This scaling is used only to construct objective vectors; we keep the original TDI system. Set $h := \sum_i \alpha_i a'_i$. Since all coefficients are positive, v uniquely maximizes hx over P . For any coordinate j , write $e_j^T = \sum_i \alpha_i a'_i$. Choose an integer N such that $N + \alpha_i > 0$ for every i . Then v also uniquely maximizes $(Nh + e_j^T)x$. Both objective vectors are integral, so

$$v_j = (Nh + e_j^T)v - Nhv \in \mathbb{Z}.$$

This holds for each j . The final assertion follows by choosing an optimal vertex of the pointed polyhedron. $\square$

For a pointed rational polyhedron, the existence of an optimal vertex whenever the objective is bounded follows by conversion to standard form and Theorem 4.19, followed by

Corollary 4.17. In particular, suppose the **complete** system $Ax \leq b$, $x \geq 0$ is TDI, b is integral, and the relaxation has a finite optimum. Then

$$\min\{cx \mid Ax \leq b, x \in \mathbb{Z}_+^n\} = \min\{cx \mid Ax \leq b, x \geq 0\}. \quad (10.2)$$

This conclusion also holds for nonintegral c : all vertices are integral independently of the objective. An arbitrary optimal LP point may still be fractional if there are several optimal vertices; one should select an optimal vertex.

Remark 10.5. The assumption on b is essential. The system $-x \leq 0$, $x \leq 1/2$ is TDI by the same multiplier argument as above, but its maximum for $c = 1$ is $1/2$, whereas the corresponding IP has value 0.

10.1.3 Relation to Totally Unimodular Matrices

Definition 10.6. A matrix A is totally unimodular (TU) if the determinant of every square submatrix belongs to $\{-1, 0, 1\}$.

In particular, its entries lie in $\{-1, 0, 1\}$. Transposing a TU matrix, changing the sign of a row, or adjoining rows of I or $-I$ preserves total unimodularity.

Theorem 10.7. If A is TU, then $Ax \leq b$ is TDI for every rational b . Consequently, for integral b , all vertices of $\{x \mid Ax \leq b\}$ are integral.

Proof. Fix integral c for which the primal has a finite optimum. The dual feasible set is

$$Q = \{\lambda^\top \geq 0 \mid A^\top \lambda^\top = c^\top\}.$$

It is in standard form and has an optimal basic feasible solution. Remove dependent equations. Each basic component is obtained by solving a nonsingular square submatrix of $A^\top$ against an integer right-hand side. Its determinant is 1 or -1 , so Cramer's rule gives integer components. This proves TDI; the preceding theorem gives the consequence for integral b . $\square$

The converse fails. The system $-x \leq 0$, $x \leq 1$, $2x \leq 2$ is TDI: the first two rows already supply integer dual optima, and the third multiplier can be zero. Its matrix is not TU because it contains the entry 2. Moreover, replacing only the last right-hand side by 1 destroys TDI. For objective $c = 1$, the maximum is $1/2$, attained dually by multiplier $1/2$ on $2x \leq 1$. No integer multiplier vector has this objective value. Thus the same matrix can occur in both TDI and non-TDI systems.

10.1.4 An Application: Bipartite Matching

Let $G = (U \dot{\cup} W, E)$ be a bipartite graph. A matching is a set of edges no two of which share an endpoint. With a weight w_e for each edge, its maximum-weight matching problem is

$$\max \left\{ \sum_{e \in E} w_e x_e \mid \sum_{e \ni v} x_e \leq 1 \ (v \in U \cup W), \ x_e \in \mathbb{Z}_+ \ (e \in E) \right\}. \quad (10.3)$$

The degree constraints imply $x_e \leq 1$, so integer feasible points are precisely incidence vectors of matchings.

Let M be the vertex-edge incidence matrix. Multiply the rows in W by -1 . Each column then has one $+1$ and one -1 . To see that this matrix is TU, consider a square submatrix. If a column has at most one nonzero entry, expand its determinant along that column and use induction. Otherwise each column contains both signs, so the sum of the rows is zero and the determinant vanishes. Hence M , and also the complete matrix $(M^T, -I)^T$, are TU. The relaxation of (10.3) therefore has an integer optimal vertex.

For integral weights, TDI also gives integer optimal dual variables in

$$\min \left\{ \sum_{v \in U \cup W} y_v \mid y_u + y_v \geq w_{uv} \ (uv \in E), \ y \geq 0 \right\}. \quad (10.4)$$

Indeed, the multipliers of the rows $-x \leq 0$ are the nonnegative slacks in these dual inequalities. The equality of the two integer optima is an integer min-max relation.

Example 10.8. Take $U = \{u_1, u_2\}$, $W = \{v_1, v_2\}$, all four edges, and weights

	v_1	v_2
u_1	5	2
u_2	4	6

The matching $\{u_1v_1, u_2v_2\}$ has weight 11. The dual assignment

$$y_{u_1} = 5, \quad y_{u_2} = 4, \quad y_{v_1} = 0, \quad y_{v_2} = 2$$

is feasible: the four endpoint sums are 5, 7, 4, 6, respectively. Its value is also 11. Weak duality therefore certifies that the displayed matching is optimal for both the LP and the IP.

For an explicitly given rational TDI system with integral right-hand side, linear programming can thus replace a search over integer points. TDI alone neither supplies a recognition algorithm nor makes an exponentially large inequality description small. The general methods in the following sections remain necessary when these structural conditions are unavailable.

Remark 10.9. The integrality theorem is due to Edmonds and Giles. For a broader treatment, see A. Schrijver, [Theory of Linear and Integer Programming](#), Chapter 22. Lecture treatments are available in M. Goemans, Total Dual Integrality, and B. Gärtner, Discrete Optimization. For total unimodularity, see also Goemans' notes on polyhedral combinatorics.

We now introduce two fundamental approaches for solving such IPs: branch and bound and cutting-plane methods. These are exact approaches, although their running time need not be polynomial. Heuristic choices, such as which variable to branch on or which cut to add, affect their practical performance. For the algorithms below, we assume that the LP relaxation is bounded; the empty case is handled explicitly.

10.2 The Branch & Bound Method

Definition 10.10. A family of sets $\{S^i : i = 1, \dots, k\}$ is a [subdivision](#) of S if $\bigcup_{i=1}^k S^i = S$. It is a [partition](#) if, in addition, $S^i \cap S^j = \emptyset$ for all $i \neq j$.

Lemma 10.11. Let

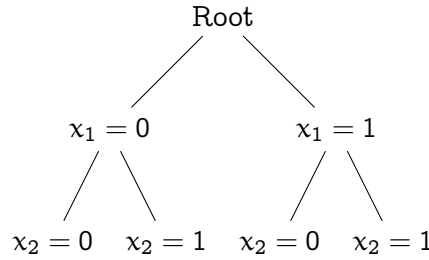
$$z_{IP}^i = \min\{cx : x \in S^i\}, \quad (IP^i)$$

where $\{S^i : i = 1, \dots, k\}$ is a subdivision of S . Then

$$z_{IP} = \min\{z_{IP}^i : i = 1, \dots, k\}.$$

Example 10.12. Consider the following optimization problem with **binary** variables:

$$\begin{aligned} &\text{minimize } cx && (\text{IP}) \\ &Ax \leq b \\ &x \in \{0, 1\}^n. \end{aligned}$$



The branch-and-bound method repeatedly subdivides the feasible set into subsets S^i and considers the IP restricted to each subset. The procedure can be represented by an **enumeration tree**. We start at the root S and subdivide it into sets S^i . Each subset corresponds to a child node connected to the root. We then apply the same procedure recursively to these children. If an optimal solution over S^i is known, no further subdivision of S^i is needed. The corresponding node of the enumeration tree is then said to be **pruned**.

Lemma 10.13. The enumeration tree can be pruned at a node S^i if one of the following conditions holds:

1. Infeasibility: $S^i = \emptyset$.
2. Optimality: An optimal solution of IP^i is known.
3. Dominance by a bound: $z_{\text{IP}}^i \geq \bar{z}_{\text{IP}}$, where $\bar{z}_{\text{IP}}$ is the value of any known feasible solution of the original IP.

Whenever a feasible integer solution is found, we keep it if its value improves the best known value $\bar{z}_{\text{IP}}$. In particular, this update must be made before pruning a node whose optimal solution has been found. Until the first feasible solution is known, we use $\bar{z}_{\text{IP}} = +\infty$.

We use Lemma 10.13 to develop an exact algorithm. Instead of solving the subproblems IP^i directly, we use **relaxations**.

Definition 10.14. A **relaxation** of IP^i is an optimization problem

$$\min\{z_R^i(x) \mid x \in S_R^i\}, \quad (\text{RP}^i)$$

satisfying

$$S^i \subseteq S_R^i \text{ and } z_R^i(x) \leq cx \text{ for all } x \in S^i.$$

Lemma 10.15. The enumeration tree can be pruned at node S^i if one of the following conditions holds:

1. RP^i is infeasible.
2. An optimal solution x_R^i of RP^i satisfies $x_R^i \in S^i$ and $z_R^i(x_R^i) = cx_R^i$.
3. Dominance by a bound: $z_R^i \geq \bar{z}_{IP}$, where z_R^i denotes the optimal value of RP^i and $\bar{z}_{IP}$ is the value of a known feasible solution of the original IP.

Proof. Condition (1) implies $S^i = \emptyset$. Condition (2) implies that x_R^i solves IP^i . Condition (3) gives $z_{IP}^i \geq z_R^i \geq \bar{z}_{IP}$, so the node cannot improve the best known solution. $\square$

10.3 LP Relaxations

We now construct an enumeration tree using LP relaxations.

At the root, we use the feasible region $S_{LP}^0 = \{x \in \mathbb{R}_+^n : Ax \leq b\}$ and the objective $z_R(x) = cx$.

Suppose that the LP relaxation at node i is

$$z_{LP}^i = \min\{cx : x \in S_{LP}^i\}, \quad (LP^i)$$

where $S_{LP}^i = \{x \in \mathbb{R}_+^n : A^i x \leq b^i\}$. If LP^i has an optimal solution, denote it by x^i . Node i can be pruned under any of the following conditions:

1. Infeasibility: $S_{LP}^i = \emptyset$.
2. Optimality: $x^i \in \mathbb{Z}_+^n$.
3. Dominance by a bound: $z_{LP}^i \geq \bar{z}_{IP}$, where $\bar{z}_{IP}$ is the value of a known feasible solution of the original IP.

It remains to describe how the children in the enumeration tree are constructed. The standard procedure is **branching on a variable**: if x^i is fractional, choose a fractional component x_j^i and create two subproblems by adding, respectively,

$$x_j \leq \lfloor x_j^i \rfloor \text{ and } x_j \geq \lfloor x_j^i \rfloor + 1.$$

Theorem 10.16. Let $P = \{x \in \mathbb{R}_+^n : Ax \leq b\}$ be a nonempty polytope. An enumeration tree constructed by branching on variables is finite. With

$$\omega_j = \lceil \max\{x_j : x \in P\} \rceil,$$

every path in the tree has at most $\sum_{j=1}^n \omega_j$ edges.

Proof. Once a constraint $x_j \leq d$, with $d \in \{0, \dots, \omega_j - 1\}$, has been added along a path, further branches on x_j can only add constraints of the form $x_j \leq d'$ with $d' \in \{0, \dots, d-1\}$ or $x_j \geq \bar{d}$ with $\bar{d} \in \{1, \dots, d\}$.

Each branch on x_j tightens an integer lower or upper bound and eliminates at least one possible integer value. Starting with $0 \leq x_j \leq \omega_j$, at most ω_j such branches can occur on a path. Summing over all variables gives the bound $\sum_{j=1}^n \omega_j$. $\square$

Example 10.17.

$$\begin{aligned} & \text{minimize } x_1 - 2x_2 & (\text{IP}) \\ & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x \in \mathbb{Z}_+^2, \end{aligned}$$

The LP relaxation at the root S_{LP}^0 has optimal solution $x^0 = (3/2, 5/2)$ with $cx^0 = -7/2$. Since $x_2^0 = 5/2$, we subdivide S^0 into $S^1 = \{x \in S^0 : x_2 \leq 2\}$ and $S^2 = \{x \in S^0 : x_2 \geq 3\}$. Solving the resulting LP relaxations gives $x^1 = (3/4, 2)$ with $cx^1 = -13/4$, while the second relaxation is infeasible. We therefore prune S^2 . Next, subdivide S^1 into $S^{11} = \{x \in S^1 : x_1 \leq 0\}$ and $S^{12} = \{x \in S^1 : x_1 \geq 1\}$. The corresponding LP optima are $x^3 = x^{11} = (0, 3/2)$ with $cx^{11} = -3$, and $x^4 = x^{12} = (1, 2)$ with $cx^{12} = -3$. Since x^{12} is integral, we set $\bar{z}_{\text{IP}} = -3$ and prune S^{11} by its bound.

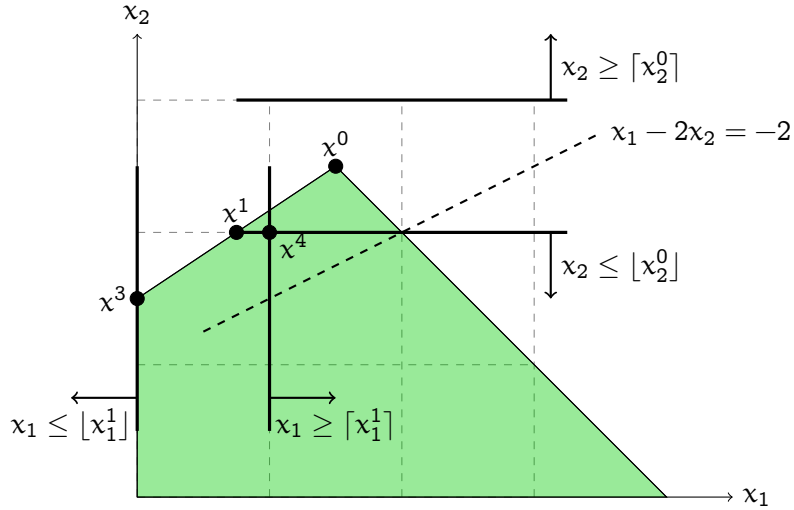


Figure 10.2: An example of branch and bound with LP relaxations.

10.4 Cutting-Plane Methods

Finally, we describe the **cutting-plane method**. Given an integer program (IP) with bounded LP relaxation, we first check whether that relaxation is feasible. If it is infeasible, so is (IP). Otherwise, we compute an optimal vertex and check whether it is integral. If so, we are done. Otherwise, we seek a valid inequality that **cuts off** this basic solution.

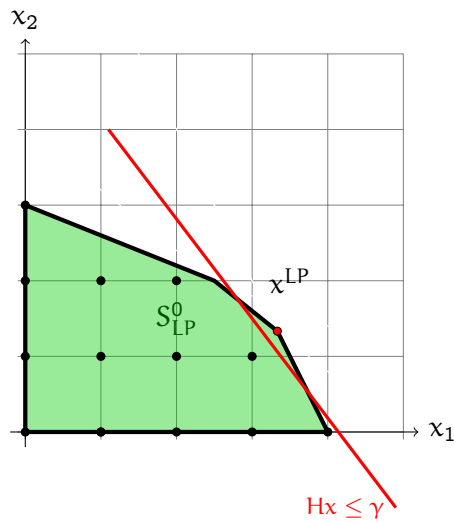


Figure 10.3: A cutting plane $Hx \leq \gamma$ removes the fractional vertex x^{LP} while retaining every black point in S .

This yields the following procedure for (IP) : $\min\{cx \mid x \in S\}$, where $S := \{x \in \mathbb{R}^n \mid Ax \leq b, x \in \mathbb{Z}_+^n\}$:

1. Set $k = 0$. Let P^0 be the LP relaxation of S , obtained by dropping the integrality constraints.

Repeat:

- (a) If $P^k = \emptyset$, STOP: the integer program is infeasible. Otherwise, compute an optimal vertex x^k of the LP relaxation over P^k .
- (b) If $x^k \in \mathbb{Z}^n$, STOP: an optimal solution has been found.
- (c) If $x^k \notin \mathbb{Z}^n$, find a separating inequality $Hx \leq \gamma$ such that $x^k \notin P^k \cap \{x \in \mathbb{R}^n \mid Hx \leq \gamma\} \supseteq S$.
- (d) $P^{k+1} \leftarrow P^k \cap \{x \in \mathbb{R}^n \mid Hx \leq \gamma\}$
- (e) $k \leftarrow k + 1$

Each cut retains S . Thus, termination gives either an optimal integer solution or a valid infeasibility conclusion. Since S is finite, $\text{co}(S)$ is compact. A fractional vertex of P^k cannot belong to $\text{co}(S)$, so a separating inequality exists; for $S = \emptyset$, any inequality excluding x^k is valid. Finding a cut requires an additional algorithm. Arbitrary cut choices need not give finite termination.