UNIVERSITY OF PASSAU

Faculty of Computer Science and Mathematics

Lecture

Nonlinear Optimization

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Preface

This scriptum has been developed during the optimization lectures given at Augsburg University 2015-2022. The contents of Chapters 3-8 are based on hand-written notes of myself taken as a student at a lecture given by Helmut Maurer (University of Münster) in 2003. Chapter 9 is based on an article of Julian Schwarz and myself. Chapters 10.1-10.3 are loosely based on lecture notes of Hans-Joachim Oberle (Hamburg University). Chapter 10.4. on the Frank-Wolfe method is based on an article by Sebastian Pokutta (ZIB).
Special thanks go to Lukas Graf and Georg Kraus who suggested several corrections to the scriptum during the course held in SS 2020.

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Chapter 1

Introduction

1.1 Motivation and Terminology

We consider a normed vector space $(V, \|\cdot\|)$ for $K \subset V$ nonempty. We are given a function

$$f : K \to \mathbb{R}.$$

Our goal is to solve $\min\{f(x) \mid x \in K\}$. We sometimes write:

$$\begin{aligned} &\text{minimize } f(x) \\ &\quad \text{s.t.: } x \in K. \end{aligned} \tag{1.1}$$

An optimal solution of (1.1) is called global minimum.

> **Definition 1.1.** $x^* \in K$ is a global minimum of f over K, if
>
> $$f(x) \geq f(x^*) \text{ for all } x \in K.$$
>
> If
>
> $$f(x) > f(x^*) \text{ for all } x \in K, x \neq x^*,$$
>
> we speak of a strict global minimum. Global maxima are defined analogously.

- infinite dimensional optimization (e.g., V is a function space, L_1 or L_p space.),
- finite dimensional optimization ($V = \mathbb{R}^n$),
- continuous optimization ($\mathrm{int}(K) \neq \emptyset$),
- discrete optimization ($K \subseteq \mathbb{Z}^n$).

1.2 Examples and Applications

Example 1.2 (Optimal Supply). The goal is to buy an amount M of a certain commodity. We have the offers of n suppliers, where every supplier $i \in M$ has a maximum supply of

M_i units of the commodity. The prices of the i-th supplier are given by a function $f_i(x_i)$.

$$\begin{aligned} \min \sum_{i=1}^{n} f_i(x_i) \\ \text{s.t.}: \sum_{i=1}^{n} x_i = M \\ 0 \le x_i \le M_i, i = 1, \ldots, n. \end{aligned}$$

Example 1.3 (Regression). An experiment shows the following data $(t_i, y_i), i = 1, \ldots, m$. The hypothesis class under which this data is generated is given by a parameterized function $f(t, p)$. The goal is to choose parameters p in order to minimize the resulting error measured as:

$$\sum_{i=1}^{m} (y_i - f(t_i, p))^2$$

A more general least-squares problem is given as:

$$\begin{aligned} \min \sum_{i=1}^{q} \Phi_i(x)^2 \\ \text{s.t.}: \ g_j(x) \le 0, j = 1, \ldots, r \\ h_j(x) = 0, j = 1, \ldots, s. \end{aligned}$$

Example 1.4 (Optimal control). We search for a control function that steers a car with minimal "effort" in a given time frame $[0, t_f]$ from A to B. We use Newton's laws describing where $s(t)$ denotes the location at time t, $v(t)$ the speed at time t and $a(t)$ denotes the acceleration:

$$\dot{s}(t) = v(t), \dot{v}(t) = a(t).$$

Suppose we are in dimension 1 and there is a straight line between A nach B of length d. The coordinate of A is normalized to $s = 0$ and $s = d$ for B. We need to satisfy $s(0) = 0, v(0) = 0, s(t_f) = d, v(t_f) = 0$.

The control function corresponds to $a(t)$, where a positive sign is acceleration and negative sign is slow down. The effort accumulates quadratically in a:

$$\int_0^{t_f} a(t)^2 dt$$

We obtain the following optimal control problem:

$$\begin{aligned} \min \int_0^{t_f} a(t)^2 dt \\ \dot{s}(t) = v(t) \\ \dot{v}(t) = a(t) \\ s(0) = 0, s(t_f) = d \end{aligned}$$

$$v(0) = 0, v(t_f) = 0.$$

1.3 Finite Dimensional Optimization

In this lecture, we consider only finite dimensional problems, that is, $V := \mathbb{R}^n$.
The set of points in K that attain the minimum is denoted by $\arg\min(f, K)$. We get

$$\alpha = \min\{f(x) : x \in K\} \Leftrightarrow \arg\min(f, K) = \{x \in K \mid f(x) = \alpha\}.$$

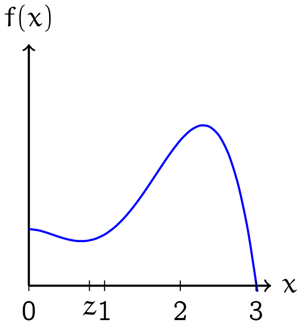


Figure 1.1: f attains on $K = [0, 3]$ its global minimum at $x^* = 3$. There is a further local minimum at z.

Definition 1.5. A point $x^* \in K$ is a local Minimum of f over K, if there is $\rho > 0$ such that

$$f(x) \geq f(x^*) \text{ for all } x \in K \cap B_\rho(x^*),$$

where

$$B_\rho(x) = \left\{y \in \mathbb{R}^n \;\middle|\; \|x - y\| < \rho\right\}$$

denotes the open ball around x with radius $\rho > 0$. If

$$f(x) > f(x^*) \text{ for all } x \in K \cap B_\rho(x^*), x \neq x^*,$$

we speak of a strict local minimum.

Usually K is represented via functional inequalities or equalities. In this case, we obtain:

$$K = \left\{x \in \mathbb{R}^n \;\middle|\; h_i(x) = 0, \;\; i \in I_1 = \{1, \dots, m\}, \; g_j(x) \leq 0, \;\; j \in I_2 = \{1, \dots, p\}\right\}, \qquad (1.2)$$

where all functions satisfy $f, h_i, g_j \in C^2$ for all $i \in I_1, j \in I_2$.
We obtain the following classes of optimization problems:

- unrestricted optimization: $m = p = 0$
- restricted optimization: $m > 0$ oder $p > 0$
- linear optimization: f linear, g_j, h_i affin linear

- quadratic optimization: $f(x) = x^T A x + b^T x + c$, g_j, h_i affin linear
- convex optimization: f convex, g_j convex, h_i affin linear
- L_2-problems: the function f has the form

  $$f(x) = \sum_{i=1}^{n} w_i \left(f_i(x)\right)^2,$$

  with smooth functions f_i and weights $w_i > 0$.
- minimax problems: f has the form

  $$f(x) = \max\{f_i(x), i = 1, \ldots, m\}$$

  with smooth functions f_i.

The following questions are the key drivers for the content of this lecture:

- When do optimal solutions exist?
- Are they unique?
- Can we derive useful necessary and sufficient optimality conditions?
- What about algorithms for solving such problems?
- What is the dependence of optimal solutions on problem parameters?

We recap a fundamental result due to Weierstrass.

> **Theorem 1.6** (Weierstrass). Let $K \subset \mathbb{R}^n$ be nonempty and compact and $f : K \to \mathbb{R}$ continuous. Then, there is $x^* \in K$ with
>
> $$f(x^*) \leq f(x) \text{ for all } x \in K.$$

Proof. As f is continuous, the image $f(K)$ of the compactum K is bounded in $\mathbb{R}$ and the infimum

$$A := \inf\{f(x) | x \in K\} \in \mathbb{R}$$

exists. Hence, there is a sequence $x_n \in K, n \in \mathbb{N}$ with

$$\lim_{n\to\infty} f(x_n) = A.$$

As $x_n, n \in \mathbb{N}$ is bounded, we can use the theorem of Bolzano/Weierstrass giving a convergent subsequence $x_{n_k}, k \in \mathbb{N}$ with

$$\lim_{k\to\infty} x_{n_k} =: x^* \in K.$$

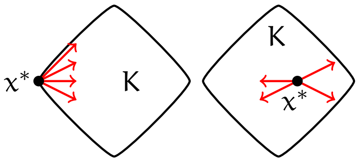


Figure 1.2: The red arcs represent feasible directions in $D_K(x^*)$.

With continuity of f we get

$$f(x^*) = \lim_{k\to\infty} f(x_{n_k}) = A.$$

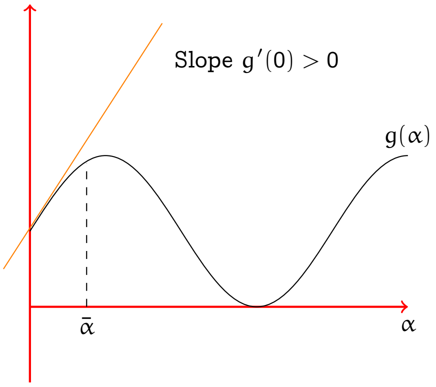
Thus, f attains at x^* its minimum over K . $\square$

1.4 Differentiable Classic Optimization

1.4.1 Variational Inequalities

Definition 1.7 (Feasible Directions). Let $x \in K \subseteq \mathbb{R}^n$ with $K \neq \emptyset$. The vector $d \in \mathbb{R}^n$ is a feasible direction at x, if there is $\bar{\alpha} > 0$ such that $x + \alpha d \in K$ for all $0 \leq \alpha \leq \bar{\alpha}$.

We denote by $D_K(x)$ the set of feasible directions at x. It is easy to see that $D_K(x)$ is a pointed cone containing 0 (cf. 2.1), hence, $D_K(x)$ is known as the cone of feasible directions.



For continuous optimization problems (1.1) we obtain the following necessary optimality conditions.

Theorem 1.8 (Variational Inequality). Let $K \subseteq \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Let x^* be a local minimum of f over K and $d \in D_K(x^*)$. Then

$$\nabla f(x^*)^T d \geq 0.$$

Proof. Since $d \in D_K(x^*)$, there is $\bar{\alpha} > 0$ such that $x^*(\alpha) := x^* + \alpha d \in K$ for all $0 \leq \alpha \leq \bar{\alpha}$. We define a 1-dimensional function $g(\alpha) := f(x^*(\alpha))$. For a local minimum x^* (w.r.t. $B_\rho(x^*)$), we have

$$g(\alpha) \geq g(0) \text{ for all } \alpha \in [0, \min\{\bar{\rho}, \bar{\alpha}\}],$$

where

$$\bar{\rho} := \sup\{\alpha \geq 0 | x^* + \alpha d \in B_{\rho/2}(x^*)\}.$$

Thus,

$$\lim_{\alpha \to +0} \frac{g(\alpha) - g(0)}{\alpha} \geq 0.$$

With the differentiability of f we get

$$0 \leq \lim_{\alpha \to +0} \frac{g(\alpha) - g(0)}{\alpha} = g'(0) = \nabla f(x^*)^T d.$$

☐

> **Theorem 1.9.** Let $K \subseteq \mathbb{R}^n$ and $f : K \to \mathbb{R}$ be continuously differentiable. If $x^* \in \mathrm{int}(K)$ is a local minimum of f over K, then:
>
> $$\nabla f(x^*) = 0. \tag{1.3}$$
>
> In particular, we have (1.3) for every local minimum of an unconstrained optimization problem.

Proof. With Theorem 1.8 we get for every $d \in D_K(x^*)$: $\nabla f(x^*)^T d \geq 0$. With $x^* \in \mathrm{int}(K)$, we get $D_K(x^*) = \mathbb{R}^n$. ☐

> **Remark 1.10.** Note that the concept of feasible directions of Definition 1.7 is not useful for sets given my algebraic manifolds. Here, we need curved directions leading to concepts of the tangent cone and linearized cone that we will see later.

1.4.2 Convex Optimization

We consider now a differentiable convex function f over a convex set $K \subset \mathbb{R}^n$.

> **Definition 1.11.** A set $K \subset \mathbb{R}^n$ is convex, if for all $x, y \in K$ the segment between x and y lies in K, that is,
>
> $$\lambda x + (1 - \lambda) y \in K \text{ for all } \lambda \in [0, 1].$$

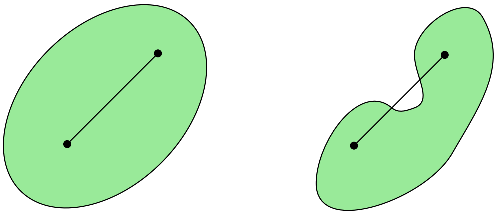


Figure 1.3: Left: convex set. Right: non-convex set.

Definition 1.12. Let $K \subset \mathbb{R}^n$ be convex. A function $f : K \to \mathbb{R}$ is convex, if for all $x, y \in K$ and $\lambda \in [0,1]$ we have:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y). \tag{1.4}$$

f is strict convex, if for all $x \neq y$ and $\lambda \in (0,1)$ the above inequality is strict. The function f is called (strictly) concave, if $-f$ is (strictly) convex.

Theorem 1.13. Let $K \subset \mathbb{R}^n$ be convex, and let $f_1, f_2 : K \to \mathbb{R}$ be convex functions and let $\alpha > 0$. Then, the functions $\alpha f_1, f_1 + f_2$ and $\max\{f_1, f_2\}$ are convex over K.

Proof. Exercise. □

Differences, products and minima of convex functions are not always convex!

Definition 1.14. Let $K \subset \mathbb{R}^n$ be convex, and $f : K \to \mathbb{R}$. The set

$$\mathrm{Epi}(f) = \{(x, \alpha) \in K \times \mathbb{R} : f(x) \leq \alpha\}$$

is the epigraph vof f. For $\beta \in \mathbb{R}$, we term the set

$$L(f, \beta) = \{x \in K : f(x) \leq \beta\}$$

lower level set of f with level β.

Theorem 1.15. Let $K \subseteq \mathbb{R}^n$ and $f : K \to \mathbb{R}$. Then:

1. f is convex $\Leftrightarrow$ $\mathrm{Epi}(f)$ is convex.
2. f is convex $\Rightarrow$ $L(f, \beta)$ is convex for all $\beta \in \mathbb{R}$. The reverse need not be true.

Proof. Exercise. □

For convex differentiable functions we obtain the following characterization:

Theorem 1.16. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $f \in C^1$. Then:

1. f is convex over the convex set $K \subseteq \mathbb{R}^n$ iff for all $x, y \in K$:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x). \tag{1.5}$$

2. f is strict convex $\Rightarrow$ (1.5) is strict for all $x \neq y \in K$.

Proof. We first show $\Leftarrow$ for the first statement. Assume (1.5) holds for all $x, y \in K$. Choose arbitrary $x, y \in K$ and $\lambda \in (0,1)$. With convexity of K we get

$$z = \lambda x + (1-\lambda)y \in K. \tag{1.6}$$

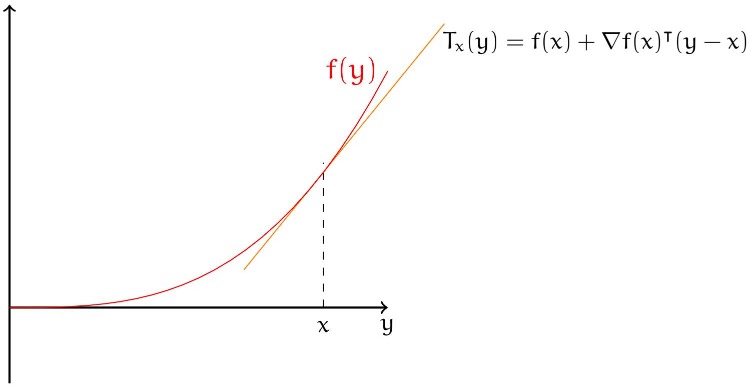


Figure 1.4: Illustration of inequality 1.5. $T_x(y)$ represents the tangent plane of f in x and we have $T_x(y) \leq f(y)$ for all $y \in K$.

With (1.5), we get for $x, y, z \in K$:

$$f(x) \geq f(z) + (x - z)^T \nabla f(z) \tag{1.7}$$
$$f(y) \geq f(z) + (y - z)^T \nabla f(z). \tag{1.8}$$

Multiply (1.7) with λ and (1.8) with $(1 - \lambda)$, add both inequalities and obtain:

$$\begin{aligned} \lambda f(x) + (1-\lambda) f(y) &\geq f(z) + \Big((\lambda(x - z) + (1-\lambda)(y - z) \Big)^T \nabla f(z) \\ &= f(z) + \Big(\lambda x + (1-\lambda) y - z) \Big)^T \nabla f(z) \\ &= f(z). \end{aligned}$$

With (1.6), the second expression of the second equation is 0. Thus, f is convex.
$\Rightarrow$: Let f be convex. We choose $x, y \in K$ and define $\psi : \mathbb{R} \to \mathbb{R}$ as

$$\psi(\lambda) = (1 - \lambda) f(x) + \lambda f(y) - f((1 - \lambda)x + \lambda y).$$

With convexity of f we get for all $\lambda \in [0, 1]$ that $\psi(\lambda) \geq 0$. Moreover $\psi(0) = 0$. We compute the derivative of ψ at 0 and get

$$0 \leq \lim_{t \to 0+} \frac{\psi(t) - \psi(0)}{t} = \dot{\Psi}(0) = -f(x) + f(y) - \nabla f(x)^T (y - x).$$

The second statement is easy.

□

We obtain a sufficient optimality criterion for convex optimization problems.

Theorem 1.17. Let $K \subset \mathbb{R}^n$ be convex and let $f : K \to \mathbb{R}$ be a differentiable convex function. Then, every local minimum of f over K is also a global Minimum.

Proof. Let x^* be a local minimum. With Theorem 1.8, we get for every $d \in D_K(x^*)$ the condition $\nabla f(x^*)^T d \geq 0$. Because K is convex, for any $y \in K$, we get $x^* + \lambda(y - x^*) =$

$\lambda y + (1-\lambda)x^* \in K$ for all $\lambda \in [0,1]$. Hence, $y - x^* \in D_K(x^*)$. We get

$$f(y) \geq f(x^*) + \nabla f(x^*)^T(y - x^*) \geq f(x^*),$$

where the first inequality follows from Theorem 1.16 and the second one from the variational inequality. ☐

For unrestricted convex problems, we get the following implication.

> **Corollary 1.18.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function. Then, every $x^* \in \mathbb{R}^n$ with $\nabla f(x^*) = 0$ is a <u>global</u> minimum of the associated unrestricted optimization problem.

Chapter 2

Convexity and Separating Hyperplanes

2.1 Convex Sets and Cones

Definition 2.1.

1. For $M \subset \mathbb{R}^n$ we define

$$\mathrm{co}(M) := \cap\{K \supset M | K \text{ convex}\}$$

as the convex hull of M. For $x^0, \dots, x^k \in \mathbb{R}^n$ we define

$$\mathrm{co}(x^0, \dots, x^k) = \mathrm{co}(\{x^0, \dots, x^k\}).$$

This set is known as the simplex spanned by the points $x^0, \dots, x^k$. If $x^1 - x^0, \dots, x^k - x^0$ are linearly independent, then the simplex is non-degenerate.

2. A subset $K \subset \mathbb{R}^n$ is a cone (pointed at 0), if for all $x \in K$ the half-ray through x lies in K, i.e.

$$\alpha x \in K \text{ for all } \alpha \geq 0.$$

3. Let $K \subset \mathbb{R}^n$ and $x \in K$. The cone

$$K(x) := \{\alpha(y - x) | y \in K, \alpha > 0\} = \bigcup_{\alpha > 0} \alpha(K - x)$$

is termed the conic hull of K w.r.t. x.

See Fig. 2.1 for an illustration.

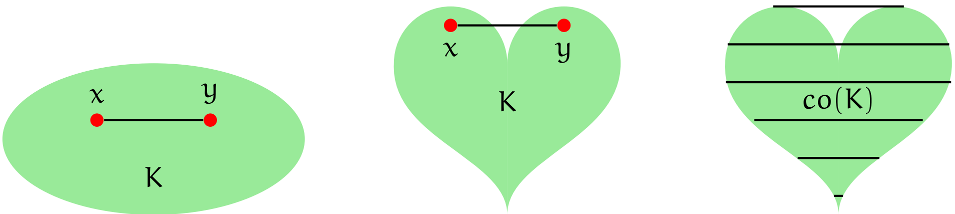


Figure 2.1: The first set is convex. The second heart is non-convex and the dashed set represents the convex hull.

Definition 2.2. Let $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$.

1. The set $H = \{x \in \mathbb{R}^n : a^\intercal x = b\}$ is called hyperplane.
2. The sets $H^- = \{x \in \mathbb{R}^n : a^\intercal x \leq b\}$ and $H^+ = \{x \in \mathbb{R}^n : a^\intercal x \geq b\}$ are Halfspaces.
3. Let A be a real-valued $m \times n$ matrix and $b \in \mathbb{R}^m$.
   $K = \{x \in \mathbb{R}^n | Ax \leq b\}$ is a polyhedron and $K = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ is a polyhedron in standard form.

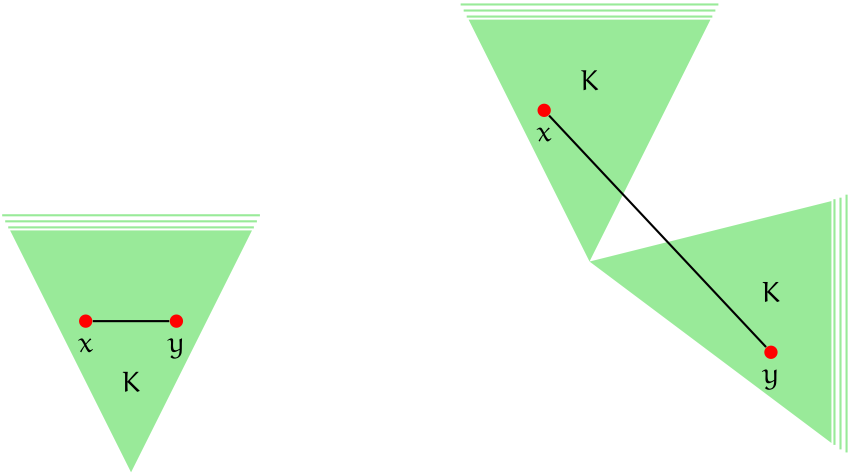


Figure 2.2: Left: convex cone. Right: non-convex cone.

2.2 Convex Combinations

Definition 2.3. Let $x^1, \dots, x^k \in \mathbb{R}^n$ und $\lambda_1, \dots, \lambda_k \in \mathbb{R}_{\geq 0}$ with $\lambda_1 + \cdots + \lambda_k = 1$.

- The vector $\sum_{i=1}^k \lambda_i x^i$ is called convex combination of $x^1, \dots, x^k$.

Theorem 2.4.
1. The intersection of convex sets is convex.
2. Every polyhedron is convex.
3. A convex combination of a finite points of a convex set lies in the respective set.
4. The convex hull of a set $K \subset \mathbb{R}^n$ is the set of all finitely generated convex combinations of points in K. The set of convex combinations of a finite point set is convex.

Proof. (1): Let $X_i, i \in I$ be convex sets and define $X := \cap_{i \in I} X_i$. For $x, y \in X$ we have $x, y \in X_i$ for all $i \in I$, hence $\lambda x + (1-\lambda)y \in X_i$ for all $i \in I$ and, hence, $\lambda x + (1-\lambda)y \in X$.

(2): A polyhedron is the intersection of finitely many convex halfspaces.

(3): We prove via induction over k, that every convex combination of k points in X lies in X. For $k = 1$ the statement is trivial and for $k = 2$ the statement follows from the convexity of X. For the step $k - 1 \to k$ consider a convex combination $\mu_1 x^1 + \cdots \mu_k x^k$. If $\mu_i = 0$ for some $i \in \{1, \dots, k\}$ we can use the induction hypothesis, hence, we can assume w.l.o.g. that $\mu_k \in (0, 1)$. Define

$$\nu_1 = \frac{\mu_1}{1-\mu_k}, \dots, \nu_{k-1} = \frac{\mu_{k-1}}{1-\mu_k} \geq 0, \sum_{l=1}^{k-1} \nu_l = 1.$$

Set

$$y := \sum_{l=1}^{k-1} \nu_l x^l$$

and observe that $y \in X$ follows by the induction hypothesis. We get

$$x = (1-\mu_k)y + \mu_k x^k \in X,$$

because the case $k = 2$ was shown already.

(4): Let L be the set of convex combinations of points in K.

$L \subseteq \mathrm{co}(K)$: With (1) the set $\mathrm{co}(K)$ is convex, hence, all convex combinations of points in $\mathrm{co}(K)$ lie again in $\mathrm{co}(K)$. With the definition of the convex hull, we get $K \subseteq \mathrm{co}(K)$, thus, $L \subseteq \mathrm{co}(K)$.

$\mathrm{co}(K) \subseteq L$: Let $x, y \in L$ with

$$x = \sum_{i=1}^{k} \alpha_i x^i \text{ and } y = \sum_{j=1}^{l} \beta_j y^j, \text{ where } \alpha_i, \beta_j \geq 0, \sum_{i=1}^{k} \alpha_i = 1, \sum_{j=1}^{l} \beta_j = 1.$$

For $\lambda \in [0, 1]$ we get

$$z = \lambda x + (1-\lambda)y = \sum_{i=1}^{k} \lambda \alpha_i x^i + \sum_{j=1}^{l} (1-\lambda)\beta_j y^j,$$

and thus $z \in L$ because

$$0 \leq \lambda\alpha_i \leq 1\, \forall i,\ 0 \leq (1-\lambda)\beta_j \leq 1\, \forall j \text{ and}$$

$$\sum_{i=1}^{k} \lambda\alpha_i + \sum_{j=1}^{l} (1-\lambda)\beta_j = \lambda \sum_{i=1}^{k} \alpha_i + (1-\lambda) \sum_{j=1}^{l} \beta_j = \lambda + 1 - \lambda = 1,$$

which shows that z is a convex combination of points in K. Thus L is convex. Obviously $K \subseteq L$, since every $x^p \in K$ can be written as

$$x^p = \sum_{j \in J} \lambda_j x^j \text{ with } \lambda_p = 1 \text{ and } \lambda_j = 0 \text{ for } i \neq p.$$

Per definition we get $\operatorname{co}(K) \subseteq L$, because L is a convex set containing K. ☐

Corollary 2.5. The set of convex combinations of $x^1, \dots, x^k \in \mathbb{R}^n$ is the smallest (w.r.t. inclusion) convex subset of $\mathbb{R}^n$, which contains $x^1, \dots, x^k$.

Proof. Let X be the set of convex combinations of $x^1, \dots, x^k \in \mathbb{R}^n$. Define

$$Y := \operatorname{co}(x^1, \dots, x^k) = \bigcap_{\substack{C \subseteq \mathbb{R}^n \\ C \text{ convex} \\ \{x^1,\dots,x^k\} \subset C}} C. \tag{2.1}$$

Y ist well-defined because $\mathbb{R}^n$ is one candidate C. With Theorem 2.4(1.) Y is convex as intersection of convex sets. Y is also the smallest convex set containing $x^1, \dots, x^k$. Since X is convex (see Theorem 2.4(4.) we get $Y \subseteq X$. Let $x \in X$. With the definition of X we get $x = \sum_{i=1}^k \lambda_i x^i$. As by assumption $x^1, \dots, x^k \in Y$ we get with Theorem 2.4(3.), that $x \in Y$. ☐

Theorem 2.6 (Charathéodory). For $K \subset \mathbb{R}^n$, $\operatorname{co}(K)$ is equal to the set of all convex combinations which require at most $(n+1)$ points of K.

Proof. Let $x \in \operatorname{co}(K)$. With Theorem 2.4 (4) there are $x^1, \dots, x^k \in K$ with

$$x = \sum_{i=1}^k \lambda_i x^i \text{ mit } \lambda_i \geq 0 \text{ for all } i = 1, \dots, k \text{ and } \sum_{i=1}^k \lambda_i = 1.$$

If $k \leq n+1$ we are done. If $k > n+1$, then we show that for the representation of x we can ignore one of the k points: Define the $(k-1)$ vectors $y^i = x^i - x^k, i = 1, ..., k-1$. For $k > n+1$, the points y^i are linearly dependent, i.e., there are $\alpha_1, \dots, \alpha_{k-1}$ with $\alpha_j \neq 0$ for at least one $j \in \{1, \dots, k-1\}$ with

$$\begin{aligned} &\sum_{i=1}^{k-1} \alpha_i y^i = 0 \\ \Leftrightarrow &\sum_{i=1}^{k-1} \alpha_i (x^i - x^k) = 0 \\ \Leftrightarrow &\sum_{i=1}^{k-1} \alpha_i x^i + (-\sum_{i=1}^{k-1} \alpha_i) x^k = 0. \end{aligned}$$

With $\alpha_k = -\sum_{i=1}^{k-1} \alpha_i$ we get

$$\sum_{i=1}^k \alpha_i x^i = 0 \text{ and } \sum_{i=1}^k \alpha_i = 0.$$

Because $\alpha_j \neq 0$ for at least one $j \in \{1, \dots, k-1\}$, the following value is well-defined:

$$i_0 = \arg \min_{i \in \{1,\dots,k\}} \{\frac{\lambda_i}{\alpha_i} | \alpha_i > 0\} = \frac{\lambda_{i_0}}{\alpha_{i_0}}.$$

We get

$$\lambda_i - \frac{\lambda_{i_0}}{\alpha_{i_0}} \alpha_i \geq 0 \ \forall i \ \text{ and } \sum_{i=1}^{k} \lambda_i - \frac{\lambda_{i_0}}{\alpha_{i_0}} \alpha_i = 1.$$

Moreover

$$x = \sum_{i=1}^{k} \lambda_i x^i = \sum_{i=1}^{k} \left(\lambda_i x^i - \frac{\lambda_{i_0}}{\alpha_{i_0}} \alpha_i x^i \right) = \sum_{i=1}^{k} (\lambda_i - \alpha_i \frac{\lambda_{i_0}}{\alpha_{i_0}}) x^i.$$

Here, we have $\lambda_{i_0} - \alpha_{i_0} \frac{\lambda_{i_0}}{\alpha_{i_0}} = 0$, and, hence, x can represented as a convex combination of at most $k-1$ points. ☐

2.3 Separating Hyperplanes

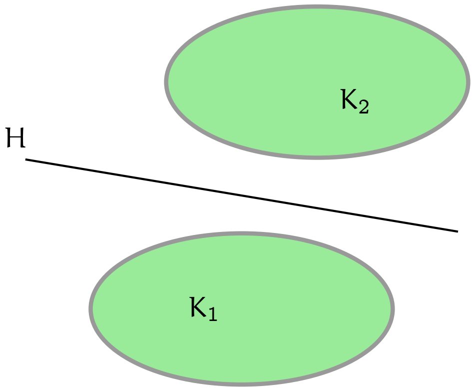


Figure 2.3: Illustration of the (strict) separation of two disjoint convex sets.

Definition 2.7. Two sets $K_1, K_2 \subset \mathbb{R}^n$ are called separable, if there are $c \in \mathbb{R}$ and row vector $\lambda \in \mathbb{R}^n, \lambda \neq 0$ with

$$\lambda x \leq c \leq \lambda y \text{ for all } x \in K_1, y \in K_2.$$

The hyperplane $H = \{x \in \mathbb{R}^n | \lambda x = c\}$ is called separating hyperplane (cf. Fig. 2.3); The sets K_1, K_2 are strictly separable via H, if $K_1 \cup K_2$ is not contained in H. The hyperplane H defines two halfspaces

$$H^+ = \{x \in \mathbb{R}^n | \lambda x \geq c\}, \ H^- = \{x \in \mathbb{R}^n | \lambda x \leq c\}.$$

The sets K_1, K_2 are separable, if either $K_1 \subseteq H^+, K_2 \subseteq H^-$ or $K_1 \subseteq H^-, K_2 \subseteq H^+$.

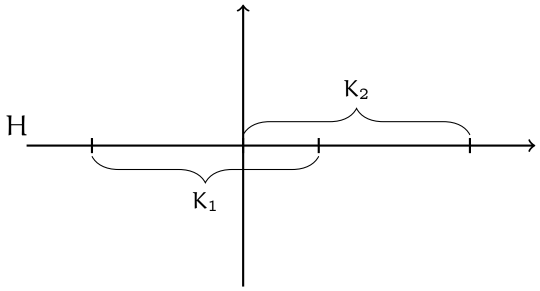


Figure 2.4: The two sets $K_1, K_2 \subset \mathbb{R}^2$ are separable but not strictly. The separating hyperplane is given as $H = \{x \in \mathbb{R}^2 | x_2 = 0\}$.

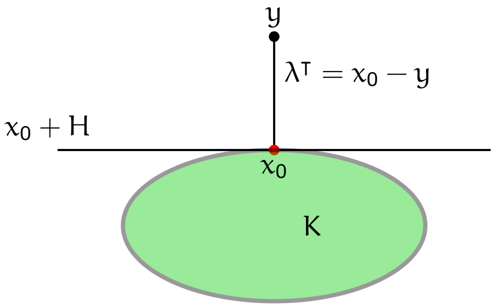


Figure 2.5: Illustration of the proof of Theorem 2.8. If $x_0 = y$ then $H + y$ is a separating supporting hyperplane.

> **Theorem 2.8.** Let $K \subset \mathbb{R}^n$ be a non-empty, convex set an let $y \notin \overset{o}{K} := \mathrm{int}(K)$. Then, $\{y\}$ and K are separable, i.e., there is a row vector $\lambda \in \mathbb{R}^n \setminus \{0\}$ with
>
> $$\lambda y \leq \lambda x \text{ for all } x \in K.$$
>
> If $\overset{o}{K} \neq \emptyset$ then $\{y\}$ and $\overset{o}{K}$ are strictly separable and we get
>
> $$\lambda y < \lambda x \text{ for all } x \in \overset{o}{K}.$$

Proof. 1. Case: $y \notin \bar{K}$, where $\bar{K}$ denotes the topological closure of K. With $\|x\|$ we denote as usual the Euklidian norm. Set

$$d := \inf_{x \in \bar{K}} \|x - y\| > 0.$$

The function $f(x) := \|y - x\|$ is continuous and attains on $\bar{K} \cap \{x \in \mathbb{R}^n | \, \|x - y\| \leq 2d\}$ its minimum (Theorem of Weierstrass). As $\bar{K}$ is closed, there is $x_0 \in \bar{K}$ with $d = \|y - x_0\|$. With convexity of $\bar{K}$ one can further show that the point x_0 is unique (cf. Fig. 2.5).
We show $\lambda := (x_0 - y)^T \neq 0$ satisfies the conditions of the theorem. Let $x \in K$. With convexity of $\bar{K}$ we get

$$x_0 + \alpha(x - x_0) \in \bar{K} \text{ für } 0 \leq \alpha \leq 1.$$

Hence,

$$\|x_0 + \alpha(x - x_0) - y\|^2 \geq \|x_0 - y\|^2$$

and therefore

$$2\alpha(x_0 - y)^T(x - x_0) + \alpha^2 \|x - x_0\|^2 \geq 0.$$

Division by$\alpha > 0$ yields for $\alpha \downarrow 0$

$$(x_0 - y)^T(x - x_0) \geq 0$$

and therefore we get using $\lambda^T = (x_0 - y)$ and $d = \|\lambda\|$

$$\lambda x \geq \lambda x_0 = \lambda y + d^2 > \lambda y.$$

Thus, $H := \{x \in \mathbb{R}^n | \lambda x = \lambda x_0\}$ is a hyperplane separating $\{y\}$ and K.

<u>2. Case:</u> $y \in \partial K = \bar{K} - \overset{o}{K}$.

For $y \in \partial K$ there is a sequence $\{y_k\}, y_k \notin \bar{K}$, with $y = \lim_{k\to\infty} y_k$. For y_k we can choose according to Case 1. a row vector $\lambda_k \neq 0$ with

$$\lambda_k y_k \leq \lambda_k x \text{ for all } x \in K.$$

W.l.o.g. we can set $\|\lambda_k\| = 1$ and hence we can assume that the bounded sequence $\{\lambda_k\}$ converges with $\lambda = \lim_{k\to\infty} \lambda_k, \|\lambda\| = 1$. Taking the limit on both sides yields

$$\lambda y \leq \lambda x \text{ for all } x \in K.$$

The statement

$$\lambda y < \lambda x \text{ for all } x \in \overset{o}{K}$$

follows immediately. ☐

In case $y \in \partial K$ we call the hyperplane <u>supporting</u>. From the separating hyperplane theorem 2.8 we get:

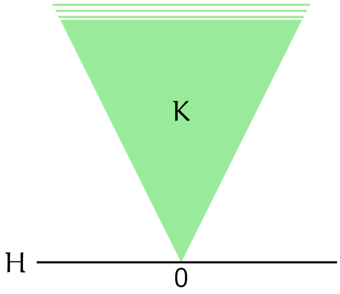


Figure 2.6: Separation of a convex cone via a hyperplane through 0.

Theorem 2.9. Let $K \subset \mathbb{R}^n$ be a non-empty convex and closed cone and suppose $y \notin K$. Then, there is $\lambda \in \mathbb{R}^n \setminus \{0\}$ with

$$\lambda y < 0 \leq \lambda x \text{ for all } x \in K.$$

Proof. With $\bar{K} = K$ there is – using the first statement of Theorem 2.8 – a row vector $\lambda \in \mathbb{R}^n \setminus \{0\}$ such that for all $x \in K$ we have $\lambda y < \lambda x$. From $0 \in K$ we get $\lambda y < \lambda 0 = 0$. Suppose there is $x \in K$ with $\lambda x < 0$. Then,

$$\lambda(\alpha x) = \alpha(\lambda x) \to_{\alpha \to \infty} -\infty,$$

contradicting boundedness of λK via λy from below. Thus, we get

$$\forall x \in K : \lambda y < 0 \leq \lambda x.$$

□

We get as an implication the following theorem of the alternatives:

Theorem 2.10 (Lemma of Farkas). Let B a $k \times n$ matrix and $d \in \mathbb{R}^k$. Then, exactly one of the following statements is true

1. $Bx = d, x \geq 0$ admits a solution $x \in \mathbb{R}^n$.
2. $\lambda B \geq 0, \lambda d < 0$ admits a solution $\lambda \in \mathbb{R}^k$.

Proof. The cone

$$K := \{Bx | x \geq 0\} \subset \mathbb{R}^k$$

non-empty convex and closed. Exactly one of the statements is true

(a) $d \in K$.

(b) $d \notin K$.

Statement (a) is statement (1) of the theorem. In case (b) we get with Theorem 2.8 the existence of some $\lambda \in \mathbb{R}^k$ with

$$\lambda d < 0 \leq \lambda z \text{ for all } z \in K.$$

Also $\lambda Bx \geq 0$ for all $x \geq 0$, i.e., $\lambda B \geq 0$. This is Statement (2) of the theorem. Note that (a) and (1) (and (b) and (2)) are equivalent, and, thus, (1) and (2) cannot be true simultaneously. □

Chapter 3

Nonlinear Optimization under Constraints

3.1 The Problem

We are given $f : \mathbb{R}^n \to \mathbb{R}$ and a subset $S \subset \mathbb{R}^n$. The general optimization problem has the form:

$$\boxed{\min \{f(x) | x \in S\}. \qquad (3.1)}$$

Recall that maximization

$$\max \{f(x) | x \in S\}$$

is equivalent to minimization

$$\min \{-f(x) | x \in S\}.$$

> **Definition 3.1** (Local Minimum). $\bar{x} \in S$ is called local minimum of (3.1), if there is an open neighbourhood $U \subset \mathbb{R}^n$ of $\bar{x}$ with
>
> $$f(\bar{x}) \le f(x) \text{ for all } x \in S \cap U.$$
>
> $\bar{x} \in S$ is called strong local minimum of (3.1), if there is an open neighbourhood $U \subset \mathbb{R}^n$ of $\bar{x}$ with
>
> $$f(\bar{x}) < f(x) \text{ for all } x \in S \cap U, x \neq \bar{x}.$$

> **Definition 3.2** (Global Minimum). $\bar{x} \in S$ is a global minimum of (3.1), if
>
> $$f(\bar{x}) \le f(x) \text{ for all } x \in S.$$

$\bar{x} \in S$ is strict global minimum of (3.1), if

$$f(\bar{x}) < f(x) \text{ for all } x \in S, x \neq \bar{x}.$$

Usually S is described by functional equations and inequalities. A general form reads as

$$\boxed{S := \{x \in \mathbb{R}^n | g(x) \in K\}, \qquad (3.2)}$$

where $g : \mathbb{R}^n \to \mathbb{R}^m$ and $K \subset \mathbb{R}^m$ is convex. The problem (3.1) then reads as

$$\boxed{\min \{f(x) | g(x) \in K\}. \qquad (3.3)}$$

Let k be the dimension of the affine hull of $K \subset \mathbb{R}^m$, $(0 \leq k \leq m)$. W.l.o.g., we can replace K with

$$K \times \{0_{m-k}\}, K \subset \mathbb{R}^k, \overset{o}{K} \neq \emptyset.$$

Accordingly, we can replace $g = (g_1, \dots, g_m)^T$ with

$$g := (g_1, \dots, g_k)^T \text{ und } h := (g_{k+1}, \dots, g_m)^T.$$

The feasible set S is represented via a system of inclusions and equalities $g(x) \in K, \overset{o}{K} \neq \emptyset$ and $h(x) = 0$:

$$S := \{x \in \mathbb{R}^n | g(x) \in K, h(x) = 0, \overset{o}{K} \neq \emptyset\}.$$

The problem (3.3) is then equivalent to

$$\boxed{\min \{f(x) | g(x) \in K, h(x) = 0, \overset{o}{K} \neq \emptyset\}. \qquad (3.4)}$$

3.2 Formulation for the Standard Cone

If we choose in (3.4) for K the standard cone

$$K := \mathbb{R}^k_- = \{x \in \mathbb{R}^k | x_i \leq 0, i = 1, \dots, k\},$$

we get the standard problem of nonlinear optimization:

$$\min\{f(x)|g(x)\le 0, h(x)=0\}. \tag{3.5}$$

The inequalities $g(x)\le 0$ need to be component-wise valid. Equivalently

$$\begin{aligned}
&\min\ f(x)\\
&\text{s.t.:}\\
&g_i(x)\le 0,\ i=1,\dots,k\\
&g_j(x)=0,\ j=k+1,\dots,m.
\end{aligned} \tag{3.6}$$

For $k=0$, i.e. $g(x)=0$ the problem is only meaningful if $m\le n$. Sign-constraints

$$x_i\ge 0, i=1,\dots,r\ (r\le n),$$

or variable bounds

$$a_i\le x_i\le b_i, i=1,\dots,r,\ (r\le n)$$

are modeled as inequalities.

We study differentiable optimization problems, thus, f and g are assumed to be continuously differentiable in an open neighbourhood of $\bar{x}$. The first derivtives $f'(x)$ or $g'(x)$ at $\bar{x}$ are given as

$$f'(\bar{x})=\left(\frac{\partial f}{\partial x_1}(\bar{x}),\dots,\frac{\partial f}{\partial x_n}(\bar{x})\right)$$

and the $m\times n$-matrix

$$g'(\bar{x})=\begin{bmatrix}\frac{\partial g_1}{\partial x_1}(\bar{x}) & \cdot\ \cdot\ \cdot & \frac{\partial g_1}{\partial x_n}(\bar{x})\\ \vdots & & \vdots\\ \frac{\partial g_m}{\partial x_1}(\bar{x}) & \cdot\ \cdot\ \cdot & \frac{\partial g_m}{\partial x_n}(\bar{x})\end{bmatrix}.$$

We will derive in the following necessary and sufficient optimality conditions for the standard optimization problem.

Chapter 4

Tangent Cone and Regularity

In order to set up a theory of necessary and sufficient optimality conditions we introduce the tangent cone $T(S, \bar{x})$ of a set $S \subset \mathbb{R}^n$ at $\bar{x} \in S$.

Example 4.1. Let $S = \mathbb{R}^n$ and consider

$$\min\{f(x) | x \in S\}.$$

Let $\bar{x}$ be a local minimum of f over S, see Fig. 4.1. By definition of a local minimum of f if we move away from $\bar{x}$ along a feasible direction the objective may not decrease. Suppose we speak of linear feasible directions $d \in \mathbb{R}^n$ at $\bar{x} \in S$, that is, there is $\bar{\alpha} > 0$ with $x = \bar{x} + \alpha d \in S$ for all $\alpha \leq \bar{\alpha}$.

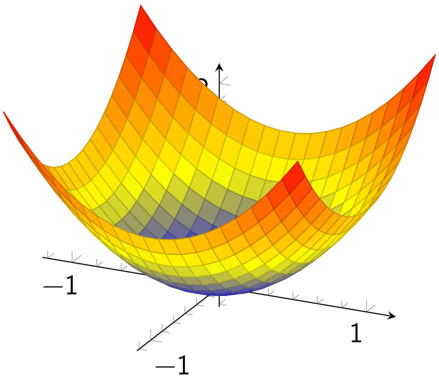
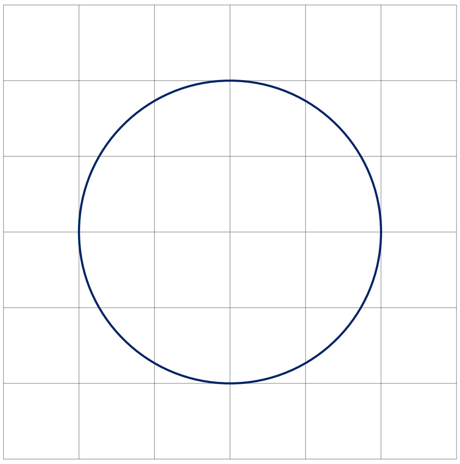


Figure 4.1: Example of a local min. at $\bar{x} = 0$.

For $\bar{x} \in \overset{o}{S}$, every $d \in \mathbb{R}^n$ is a feasible direction. For $\bar{x} \in \bar{S}$ the concept of linearly feasible directions is not enough.

Example 4.2. Let $S = \{x \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\}$ and consider

$$\min\{2x_1 + x_2^2 | x \in S\}.$$



Here $\overset{o}{S} = \emptyset$ and for no $\bar{x} \in S$, there is $\bar{\alpha} > 0$ with $x = \bar{x} + \alpha d \in S$ for all $\alpha \leq \bar{\alpha}$. Hence, we need a more general concept: infinitesimally feasible directions at $\bar{x}$.

4.1 Motivation of the Theory

$$\begin{aligned} &\text{minimize } f(x) \\ &\quad \text{s.t. } h_i(x) = 0, \quad i \in \{1, \dots, k\}, \\ &\qquad g_j(x) \leq 0, \quad j \in \{k+1, \dots, m\}. \end{aligned} \tag{4.1}$$

with $x \in \mathbb{R}^n$and smooth functions f, h_i, g_j for all i, j. Notation: $h = (h_1, \dots, h_k), g = (g_{k+1}, \dots, g_m)$

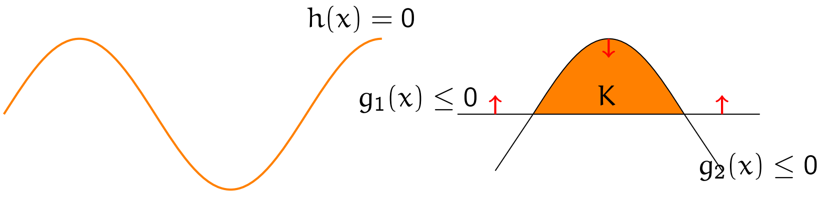


Figure 4.2: Illustration of S.

Let us now explain the idea of infinitesimally feasible directions. Consider the case $h(x) = 0$ as in Fig. 4.3. Under some assumptions (regularity as introduced later) the tangent plane $T_h(x) = \{v : \nabla h(x)^\intercal \cdot v = 0\}$ conaints the set of infinitesimally feasible directions at x (proof later).

Consider $f(x_1, x_2) = -x_1 - x_2$ with $-\nabla f(x_1, x_2) = (1, 1)$.

The intuition is as follows: Suppose there is a force tracking x with a rope along the hypersurface $h(x) = 0$ in the direction of the gradient $-\nabla f(x)$. The rope tracks the moving point continuously along $h(x) = 0$ in the direction of the force $-\nabla f(x)$. Note that there are two forces acting: the rope tracks x along the descent direction $-\nabla f(\bar{x})$

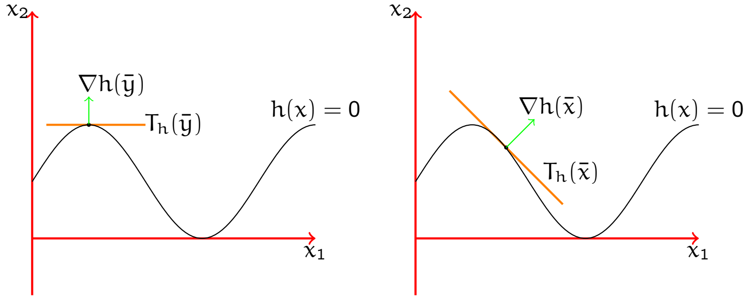


Figure 4.3: The set S is defined via $h(x) = 0$. The tangent plane is illustrated for $\bar{y}$ and $\bar{x}$.

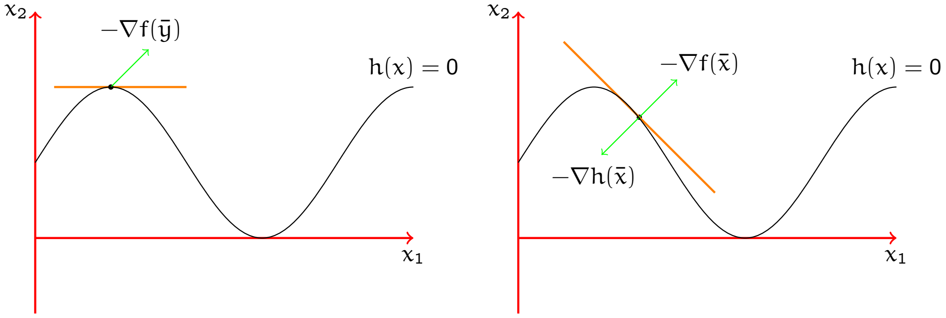


Figure 4.4: Gradients of h and f in a local minimum.

of f and the force $-\nabla h(\bar{x})$ keeps $\bar{x}$ on the hypersurface. The movement will stop if the forces $-\nabla f(x)$ and $\nabla h(x)$ act in opposite direction and an equilibrium of the two forces is reached.

Formally, for a local minimum at $\bar{x}$ for any movement from $\bar{x}$ along v for some v in the tangent plane, $\nabla f(\bar{x})^T v$ may only be nonnegative, that is, $\nabla f(\bar{x})^T v \geq 0$. As with $v \in T_h(\bar{x})$ we get $-v \in T_h(\bar{x})$ we follow $\nabla f(\bar{x})^T v = 0$. Hence, $\nabla f(\bar{x})^T$ and $\nabla h(\bar{x})^T$ are linearly dependent (without proof) and in $\bar{x}$ we have the condition

$$f'(\bar{x}) + \lambda h'(\bar{x}) = 0.$$

4.2 Tangent Cone and Variational Inequality

Definition 4.3 (Tangent Cone). The tangent cone $T(S, \bar{x})$ of $S \subset \mathbb{R}^n$ in $\bar{x}$ is defined as:

$$T(S, \bar{x}) := \left\{ v \in \mathbb{R}^n | \, \exists \, x_i \in S, t_i > 0, \text{ with } \lim_{i \to \infty} t_i = 0, v = \lim_{i \to \infty} \frac{x_i - \bar{x}}{t_i} \right\}. \tag{4.2}$$

The definition $v \in T(S, \bar{x})$ is equivalent to

$$x_i = \bar{x} + t_i v + r_i, r_i \in \mathbb{R}^n, \lim_{i \to \infty} \frac{r_i}{t_i} = 0. \tag{4.3}$$

With the Landau calculus, we have $r_i = o(t_i)$. In particular:

$$\lim_{i \to \infty} x_i = \bar{x}, \quad \lim_{i \to \infty} \frac{\|x_i - \bar{x}\|}{t_i} = \|v\| \, .$$

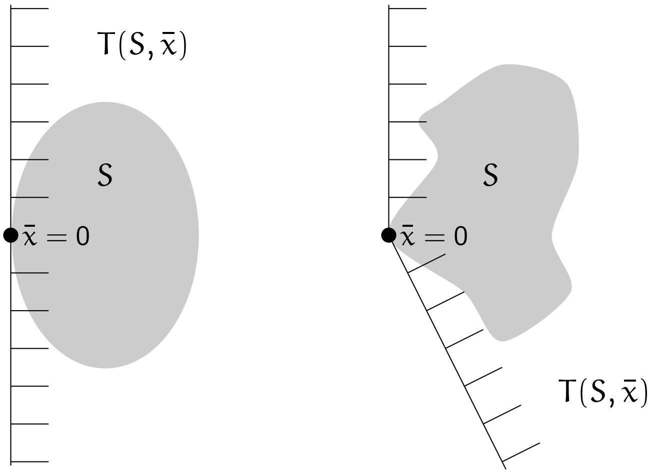


Figure 4.5: Tangent cone pointed at 0.

Notation.

Let us recap some terminology for sets in $\mathbb{R}^n$. For $K \subset \mathbb{R}^n$ we zse:

$\mathrm{aff}(K)$: the smallest affine subspace in $\mathbb{R}^n$ containing K,

$\mathrm{span}(K)$: the smallest linear subspace in $\mathbb{R}^n$ containing K,

$\mathrm{int}(K) = \overset{o}{K}$: topological interior of K wrt. $\mathbb{R}^n$,

K^i: relative topological interior of K wrt. $\mathrm{aff}(K)$,

$c\ell(K) =: \bar{K}$: topological closure of K,

$\partial K = \bar{K} \setminus \overset{o}{K}$: boundary of K.

Conic hull: Let $K \subset \mathbb{R}^n$ and $x \in K$. The cone

$$K(x) := \{\alpha(y - x) | y \in K, \alpha > 0\} = \bigcup_{\alpha > 0} \alpha(K - x)$$

is called <u>conical hull</u> of K wrt. x. Per definition $K(x)$ corresponds to the conic hull of $K - x$.

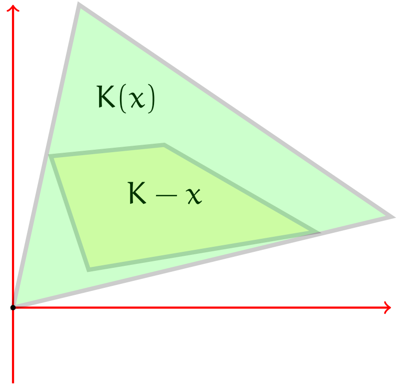


Figure 4.6: Illustration of the conic hull $K(x)$.

Lemma 4.4. For every $S \subset \mathbb{R}^n$ and $\bar{x} \in S$ the set $T(S, \bar{x})$ is a closed cone pointed at 0.

Proof. Obviously $T(S, \bar{x})$ is a cone pointed at 0. It remains to show that $T(S, \bar{x})$ is closed. Let $v \in c\ell(T(S, \bar{x}))$ and $(v_l)_{l\in\mathbb{N}}$ a sequence with

$$v_l \to v,\ v_l \in T(S, \bar{x}),\ \text{for which w.l.o.g. } \|v_l - v\| \leq \frac{1}{l}.$$

We need to show that $v \in T(S, \bar{x})$. Per definition of the tangent cone for every $l \in \mathbb{N}$ there are sequences $(x_{l,k})_{k\in\mathbb{N}}$ and $(t_{l,k})_{k\in\mathbb{N}}$ with

$$x_{l,k} \in S, t_{l,k} > 0,\ \text{with } \lim_{k\to\infty} t_{l,k} = 0, v_l = \lim_{k\to\infty} \frac{x_{l,k} - \bar{x}}{t_{l,k}}. \tag{4.4}$$

Hence, for all $l \in \mathbb{N}$ there is an index k_l with

$$\left\| \frac{x_{l,k_l} - \bar{x}}{t_{l,k_l}} - v_l \right\| \leq \frac{1}{l},\ \|x_{l,k_l} - \bar{x}\| \leq \frac{1}{l} \text{ und } t_{l,k_l} \leq \frac{1}{l}.$$

For $l \to \infty$ we get $x_{l,k_l} \to \bar{x}$ and $t_l \to 0$. Moreover, wit the triangle inequality we get

$$\left\| \frac{x_{l,k_l} - \bar{x}}{t_{l,k_l}} - v \right\| \leq \left\| \frac{x_{l,k_l} - \bar{x}}{t_{l,k_l}} - v_l \right\| + \|v_l - v\| \leq \frac{2}{l}.$$

It follows that $\frac{x_{l,k_l} - \bar{x}}{t_{l,k_l}} \to v$ and hence $v \in T(S, \bar{x})$. ☐

For convex sets $K \subset \mathbb{R}^n$ we can easily compute the tangent cone.

Lemma 4.5. Let $K \subset \mathbb{R}^n$ be convex and let $\bar{x} \in K$. Then, $T(K, \bar{x})$ is the closure of the conical hull of K in $\bar{x}$, i.e.,

$$T(K, \bar{x}) = c\ell\left(\cup_{\alpha>0}\alpha(K - \bar{x})\right) =: \overline{K(\bar{x})}.$$

Proof. $T(K, \bar{x}) \supset \overline{K(\bar{x})}$: Let $x \in K$ and $\alpha > 0$. We need to show that $\alpha(x - \bar{x}) \in T(K, \bar{x})$.

As K convex and $x \in K$, we get

$$\begin{aligned} x_i &= \bar{x} + \frac{\alpha}{i}(x - \bar{x}) \in K, \text{ for } i \geq \alpha \\ \Rightarrow \frac{x_i - \bar{x}}{1/i} &= \alpha(x - \bar{x}) \\ \Rightarrow \alpha(x - \bar{x}) &= \lim_{i \to \infty} \frac{x_i - \bar{x}}{1/i} = v \in T(K, \bar{x}). \end{aligned}$$

As $T(K, \bar{x})$ is closed, we get $T(K, \bar{x}) \supset \overline{K(\bar{x})}$.

$T(K, \bar{x}) \subset \overline{K(\bar{x})}$: Let $v \in T(K, \bar{x})$ with

$$x_i = \bar{x} + t_i v + r_i, r_i \in \mathbb{R}^n, \lim_{i \to \infty} \frac{r_i}{t_i} = 0.$$

Then,

$$\underbrace{\frac{x_i - \bar{x}}{t_i}}_{\in K(\bar{x})} = v + \frac{r_i}{t_i} \Rightarrow v = \lim_{i \to \infty} \frac{x_i - \bar{x}}{t_i} \in \overline{K(\bar{x})}.$$

□

We derive a fundamental necessary optimality criterion.

Theorem 4.6 (Variational Inequality). Let $\bar{x}$ be a local minimum of

$$\min \{f(x) | x \in S\}$$

Then,

$$f'(\bar{x})v \geq 0 \text{ for all } v \in T(S, \bar{x}).$$

Proof. Let $v \in T(S, \bar{x})$. With $x_i = \bar{x} + v t_i + r_i$ we get

$$\begin{aligned} f(x_i) &= f(\bar{x}) + f'(\bar{x})(x_i - \bar{x}) + o(\|x_i - \bar{x}\|) \\ &= f(\bar{x}) + f'(\bar{x}) v t_i + o(t_i). \end{aligned}$$

We obtain

$$\lim_{i \to \infty} \frac{f(x_i) - f(\bar{x})}{t_i} = \lim_{i \to \infty} \left(f'(\bar{x})v + \frac{o(t_i)}{t_i} \right) = f'(\bar{x})v.$$

Since $\bar{x}$ is a local minimum, we get for i large enough

$$f(x_i) \geq f(\bar{x}).$$

Thus,

$$0 \leq \lim_{i \to \infty} \frac{f(x_i) - f(\bar{x})}{t_i} = f'(\bar{x})v.$$

□

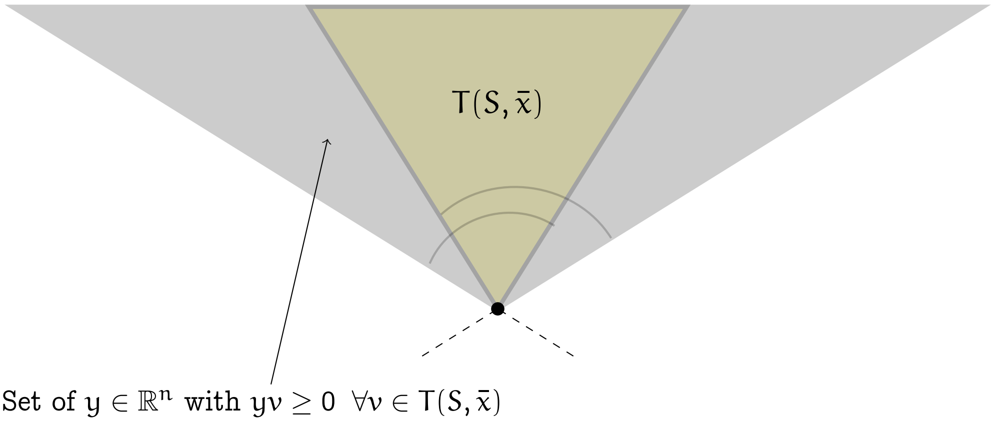


Figure 4.7: Set of vectors y for which $yv \geq 0$ for all $v \in T(S, \bar{x})$.

4.3 Linearized Cone

We consider the case that S is given as

$$S = \{x \in \mathbb{R}^n | g(x) \in K\},$$

where $g : \mathbb{R}^n \to \mathbb{R}^m$ is C^1 and $K \subset \mathbb{R}^m$ is convex. As the tangent cone is hard to compute, we introduce the linearized cone of S in $\bar{x}$.

As motivation for that cone, let us linearize the inclusion $g(x) \in K$ in $\bar{x}$, i.e., for $v \in \mathbb{R}^n$ we consider

$$g(\bar{x}) + g'(\bar{x})v \in K,$$

hence

$$g'(\bar{x})v \in K - g(\bar{x}) \subset \cup_{\alpha > 0} \alpha(K - g(\bar{x})) =: K(g(\bar{x})).$$

This leads directly to the definition of the linearized cone of S in $\bar{x}$:

$$L(S, \bar{x}) = \{v \in \mathbb{R}^n | g'(\bar{x})v \in K(g(\bar{x}))\}. \tag{4.5}$$

Lemma 4.7. $L(S, \bar{x})$ is a convex cone pointed at 0.

Proof. We have $0 \in L(S, \bar{x})$, because $0 \in K(g(\bar{x}))$, which in turn follows from $g(\bar{x}) \in K$. The cone property $v \in L(S, \bar{x}) \Rightarrow \alpha v \in L(S, \bar{x})$ for all $\alpha \geq 0$ is also satisfied. We need to show convexity. Let $u, v \in L(S, \bar{x})$ and $\lambda \in (0, 1)$. For $w = \lambda\, u + (1 - \lambda)v$, we get

$$g'(\bar{x})w = \lambda \underbrace{g'(\bar{x})u}_{\in K(g(\bar{x}))} + (1 - \lambda) \underbrace{g'(\bar{x})v}_{\in K(g(\bar{x}))} \in K(g(\bar{x}))$$

where the last inclusion follows by convexity of $K(g(\bar{x}))$ (see exercise). $\square$

Exercise 4.8. Let $K \subset \mathbb{R}^m$ convex and $g : \mathbb{R}^n \to \mathbb{R}^m$ a C^1 mapping. Let $x \in \mathbb{R}^n$ with $g(x) \in K$. Show that the conical hull of K w.r.t. $g(x)$, that is, $K(g(x)) = \cup_{\alpha>0}\alpha(K - g(x))$, is convex.

Note that $L(S, \bar{x})$ depends not only on the set S but also on the chosen mapping g in order to represent S; see the following example.

Example 4.9. Let $S := \{0\} \subset \mathbb{R}, \bar{x} = 0$. With $K = \{0\} \subset \mathbb{R}, g(x) = x^2$ we get that S can be represented as $S = \{x \in \mathbb{R} | g(x) \in K\}$. With $g'(\bar{x}) = 0$ we have

$$L(S, \bar{x}) = \{v \in \mathbb{R} | g'(\bar{x})v = 0\} = \mathbb{R}.$$

If we choose $g(x) = x$, we still get $S = \{x \in \mathbb{R} | g(x) \in K\}$. But with $g'(\bar{x}) = 1$ we get $L(S, \bar{x}) = \{0\}$.

Another special case is $K = \{0\}$. Here

$$S = \{x \in \mathbb{R}^n | g(x) = 0\}$$

is an equation-defined mannifold and S in $\bar{x}$ is the linear subspace

$$L(S, \bar{x}) = \{v \in \mathbb{R}^n | g'(\bar{x})v = 0\}.$$

Another important special case is the standard cone $K = \mathbb{R}^k_- \times \{0_{m-k}\}$. Here, S is given as

$$S = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{l} g_i(x) \le 0, i = 1, \dots, k, \\ g_i(x) = 0, i = k+1, \dots, m \end{array} \right\}.$$

Let

$$I(\bar{x}) := \{i \in \{1, \dots, k\} | g_i(\bar{x}) = 0\}$$

be the set for which the inequalities $g_i(\bar{x}) \le 0$ are active. We denote this set as the set of active indices.

We get that $K(g(\bar{x}))$ has the following form:

$$K(g(\bar{x})) = \left\{ y \in \mathbb{R}^m \,\middle|\, \begin{array}{l} y_i \le 0, i \in I(\bar{x}), \\ y_i = 0, i = k+1, \dots, m \end{array} \right\}$$

and hence the linearized cone can be computed as

$$L(S, \bar{x}) = \left\{ v \in \mathbb{R}^n \,\middle|\, \begin{array}{l} g_i'(\bar{x})v \le 0, i \in I(\bar{x}), \\ g_i'(\bar{x})v = 0, i = k+1, \dots, m \end{array} \right\}. \tag{4.6}$$

We obtain the following relationship between $T(S, \bar{x})$ and $L(S, \bar{x})$.

Lemma 4.10. If $K(g(\bar{x}))$ is closed, then

$$T(S, \bar{x}) \subset L(S, \bar{x}).$$

Proof. Let $v \in T(S, \bar{x})$ with

$$v = \lim_{i \to \infty} \frac{(x_i - \bar{x})}{t_i}, \ x_i \in S, \ t_i > 0.$$

As in the proof of Thm. 4.6 we get

$$g'(\bar{x})v = \lim_{i \to \infty} \frac{g(x_i) - g(\bar{x})}{t_i}.$$

With $x_i \in S \Rightarrow g(x_i) \in K$ and $t_i > 0$ using the definition of $K(g(\bar{x}))$ we get $\frac{g(x_i)-g(\bar{x})}{t_i} \in K(g(\bar{x}))$. Closedness of $K(g(\bar{x}))$ implies

$$g'(\bar{x})v \in K(g(\bar{x})), \ \text{i.e.} \ v \in L(S, \bar{x}).$$

☐

The reverse inclusion $L(S, \bar{x}) \subset T(S, \bar{x})$ is not valid in general as shown by the following example.

Example 4.11. Let $S := \{0\} \subset \mathbb{R}, \bar{x} = 0, T(S, \bar{x}) = \{0\}$.

1. With $K = \{0\} \subset \mathbb{R}, g(x) = x^2$ the set S is given as $S = \{x \in \mathbb{R} | g(x) \in K\}$. With $g'(\bar{x}) = 0$ we get
$$L(S, \bar{x}) = \{v \in \mathbb{R} | g'(\bar{x})v = 0\} = \mathbb{R}$$
and therefore $T(S, \bar{x}) \subsetneq L(S, \bar{x})$.
2. With $K = \{0\} \subset \mathbb{R}, g(x) = x$ we can represent S as $S = \{x \in \mathbb{R} | g(x) \in K\}$. With $g'(\bar{x}) = 1$ we get $L(S, \bar{x}) = \{0\}$ and therefore $T(S, \bar{x}) = L(S, \bar{x})$.

4.4 Regularity Conditions

For obtaining $L(S, \bar{x}) \subset T(S, \bar{x})$ one needs to impose additional conditions on g and K. Such conditions are known as constraint qualifications or regularity conditions. For a motivation, consider the case

$$S = \{x \in \mathbb{R}^n | g(x) = 0\}.$$

Definition 4.12. $\bar{x} \in S$ is called regular, if

$$\operatorname{Im} g'(\bar{x}) = \mathbb{R}^m \tag{4.7}$$

where $\operatorname{Im} g'(\bar{x})$ is the image of the linear mapping $x \mapsto g'(\bar{x})x$ for $x \in \mathbb{R}^n$.

$\bar{x}$ is regular iff the gradients $g_i'(\bar{x})^T, i = 1, \dots, m$ are linearly independent, or, equivalently

$$g'(\bar{x})g'(\bar{x})^T \text{ is non-singular.}$$

For later, we recap that exactly one of the following statements is true:

1. $\bar{x}$ is regular.
2. there is $\lambda \in \mathbb{R}^m, \lambda \neq 0$ with $\lambda g'(\bar{x}) = 0$.

Theorem 4.13. Let $S = \{x \in \mathbb{R}^n | g(x) = 0\}$ and let $\bar{x} \in S$ be regular. Then,

1. For $v \in \mathbb{R}^n$ with $g'(\bar{x})v = 0$ there is $\epsilon > 0$ and a curve $x : [-\epsilon, \epsilon] \to S$ with
$$x(0) = \bar{x}, \ \dot{x}(0) = \lim_{t \to 0} \frac{x(t) - \bar{x}}{t} = v.$$
2. We have $T(S, \bar{x}) = L(S, \bar{x}) = \{v \in \mathbb{R}^n | g'(\bar{x})v = 0\}$.

Proof. For (1): We define $F : \mathbb{R}^{m+1} \to \mathbb{R}^m$ via

$$F(y, t) := g(\bar{x} + tv + g'(\bar{x})^T y), \ y \in \mathbb{R}^m, t \in \mathbb{R}.$$

We have $F(0_m, 0) = g(\bar{x}) = 0$ and the partial derivatives of F wrt. y read as

$$\frac{\partial F}{\partial y}(0_m, 0) = g'(\bar{x})g'(\bar{x})^T.$$

This matrix is non-singular as $\bar{x}$ is regular. With the Implicit Function Theorem, there is $\epsilon > 0$ and a function $y : [-\epsilon, \epsilon] \to \mathbb{R}^m$ that is continuously differentiable in $t = 0$ with $y(0) = 0$ and

$$F(y(t), t) = 0 \text{ for } t \in [-\epsilon, \epsilon].$$

We obtain $\dot{y}(0) := \lim_{t \to 0} \frac{y(t) - y(0)}{t}$ and with the assumption $g'(\bar{x})v = 0$ we get

$$0 = \frac{dF(y(t), t)}{dt}\bigg|_{t=0} = F_y(y(0), 0)\dot{y}(0) + F_t(y(0), 0) = g'(\bar{x})g'(\bar{x})^T \dot{y}(0),$$

where we used $F_t(y(0), 0) = g'(\bar{x})v = 0$. Hence, $\dot{y}(0) = 0$. By setting

$$x(t) = \bar{x} + vt + g'(\bar{x})^T y(t), \ t \in [-\epsilon, \epsilon],$$

we obtain a cruve $x(t)$ with the desired properties: $g(x(t)) = 0$, i.e. $x(t) \in S, x(0) = \bar{x}$ and $\dot{x}(0) = v$, because $\dot{y}(0) = 0$.
For (2): The claim follows from (1) and Lemma 4.10. ☐

We consider now the general case.

$$S = \{x \in \mathbb{R}^n | g(x) \in K\}, \ K \text{ convex.} \tag{4.8}$$

Definition 4.14. Let S as in (4.8). $\bar{x} \in S$ is called regular, if

$$\operatorname{Im} g'(\bar{x}) - K(g(\bar{x})) = \mathbb{R}^m. \tag{4.9}$$

This condition generalizes (4.7) and means geometrically that the linear subspace $\operatorname{Im} g'(\bar{x})$ and the cone $K(g(\bar{x}))$ are transversal to each other.
In the following, we derive easier but equivalent conditions.

Lemma 4.15. Let K be convex. The following conditions are equivalent:

1. $\operatorname{Im} g'(\bar{x}) - K(g(\bar{x})) = \mathbb{R}^m$,
2. $0 \in \operatorname{int}\left(\operatorname{Im} g'(\bar{x}) - K(g(\bar{x}))\right)$,
3. $0 \in \operatorname{int}\left(\operatorname{Im} g'(\bar{x}) + g(\bar{x}) - K\right)$,

Proof. The proof is done in the following ordfer: 1. ⇒ 3. ⇒ 2. ⇒ 1.
1. ⇒ 3. : The set $\operatorname{Im} g'(\bar{x}) + g(\bar{x}) - K$ is convex and contains 0 because $g(\bar{x}) \in K$. Suppose $0 \notin \operatorname{int}\left(\operatorname{Im} g'(\bar{x}) + g(\bar{x}) - K\right)$. With the separating hyperplane theorem (Theorem 2.8) there is $\lambda \in \mathbb{R}^m, \lambda \neq 0$ with

$$\lambda\left(g'(\bar{x})v + g(\bar{x}) - y\right) \geq 0 \text{ for all } v \in \mathbb{R}^n, y \in K.$$

With the definition

$$K(g(\bar{x})) = \cup_{\alpha>0}\alpha\left(K - g(\bar{x})\right)$$

we get

$$\lambda\left(g'(\bar{x})v - y\right) \geq 0 \text{ for all } v \in \mathbb{R}^n, y \in K(g(\bar{x})). \tag{4.10}$$

The assumption $\operatorname{Im} g'(\bar{x}) - K(g(\bar{x})) = \mathbb{R}^m$ yields $\lambda = 0$ in contradiction to the choice of λ.
3. ⇒ 2. : Follows from $K - g(\bar{x}) \subset K(g(\bar{x}))$.
2. ⇒ 1. : Let $B_\epsilon(0) := \{y \in \mathbb{R}^m | \, \|y\| \leq \epsilon\}$. Per assumption there is $\epsilon > 0$ with

$$B_\epsilon(0) \subset \operatorname{Im} g'(\bar{x}) - K(g(\bar{x})).$$

Since $\operatorname{Im} g'(\bar{x}) - K(g(\bar{x}))$ is a cone, we get

$$\mathbb{R}^m = \cup_{\alpha \geq 0}\alpha B_\epsilon(0) \subset \operatorname{Im} g'(\bar{x}) - K(g(\bar{x})),$$

implying 1. ☐

These conditions are due to S. Robinson (1976).
Using the proof of 1. ⇒ 3. we get the following.

Corollary 4.16. $\bar{x} \in S$ is not regular if and only if there is $\lambda \in \mathbb{R}^m, \lambda \neq 0$ with

$$\lambda g'(\bar{x}) = 0,\ \lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})).$$

Exercise 4.17. Prove the statement of Corollary 4.16.

We specalize the conditions of Lemma 4.15 to the following case

$$S = \{x \in \mathbb{R}^n | g(x) \in K, h(x) = 0\},\ \overset{o}{K} \neq \emptyset, K \text{ convex}, \tag{4.11}$$

with $g = (g_1, \ldots, g_k)^T, h = (g_{k+1}, \ldots, g_m)^T$. Then, 2. from 4.15 is equivalent to

$$\operatorname{Im} h'(\bar{x}) = \mathbb{R}^{m-k} \text{ and there is } v \in \mathbb{R}^n \text{ with } h'(\bar{x})v = 0,\ g'(\bar{x})v \in \operatorname{int}(K(g(\bar{x}))). \tag{4.12}$$

and 3. is equivalent to

$$\operatorname{Im} h'(\bar{x}) = \mathbb{R}^{m-k} \text{ and there is } v \in \mathbb{R}^n \text{ with } h'(\bar{x})v = 0,\ g(\bar{x}) + g'(\bar{x})v \in \overset{o}{K}. \tag{4.13}$$

The conditions (4.12) and (4.13) are called local Slater-Conditions. For $K = \mathbb{R}^k_-$ we get from (4.12) the Mangasarian-Fromowitz-Conditions (cf. Mangasarian 1969).

Definition 4.18 (Mangasarian-Fromowitz). The gradients $g'_{k+1}(\bar{x})^T, \ldots, g'_m(\bar{x})^T$ are linearly independent and there is $v \in \mathbb{R}^n$ with

$$\begin{aligned} &g'_i(\bar{x})v < 0,\ i \in I(\bar{x}) \text{ (set of active indices)} \\ &g'_i(\bar{x})v = 0,\ i = k+1, \ldots, m. \end{aligned}$$

We now derive the central statement $L(S, \bar{x}) \subset T(S, \bar{x})$ under the aforementioned regularity conditions. We will do this for the case S being in the form (4.11).

Theorem 4.19. Let $\bar{x}$ be a regular point in

$$S = \{x \in \mathbb{R}^n | g(x) \in K, h(x) = 0\},\ \overset{o}{K} \neq \emptyset, K \text{ convex}.$$

Then:

1. For $v \in \mathbb{R}^n$ with $h'(\bar{x})v = 0$, and $g(\bar{x}) + g'(\bar{x})v \in \overset{o}{K}$ there is $\epsilon > 0$ and a curve

$$x : [0, \epsilon] \longrightarrow S$$

   with

$$x(0) = \bar{x},\ \dot{x}(0) = \lim_{t \downarrow 0} \frac{x(t) - \bar{x}}{t} = v.$$

2. We have $L(S, \bar{x}) \subset T(S, \bar{x})$.

Proof. For 1.: In case $k = m$ we set $x(t) = \bar{x} + tv$ for $t \in [0, \epsilon]$ and $\epsilon > 0$ small enough. If $k < m$ then using Thm. 4.13,(1.), there is $\delta > 0$ and a curve $x : [0, \delta] \to \mathbb{R}^n$ with

$$h(x(t)) = 0 \text{ for } t \in [-\delta, \delta],\ x(0) = \bar{x},\ \dot{x}(0) = v.$$

Using

$$g(\bar{x}) + g'(\bar{x})v \in \overset{o}{K},\ \lim_{t\to 0} \frac{g(x(t)) - g(\bar{x})}{t} = g'(\bar{x})v,$$

there is $\epsilon \leq \min\{\delta, 1\}$ with

$$g(\bar{x}) + \frac{g(x(t)) - g(\bar{x})}{t} \in K \text{ für } t \in [-\epsilon, \epsilon].$$

As $g(\bar{x}) \in K$ and with convexity of K, we get

$$g(x(t)) = (1-t)g(\bar{x}) + t\left(g(\bar{x}) + \frac{g(x(t)) - g(\bar{x})}{t}\right) \in K \text{ for } 0 \leq t \leq \epsilon.$$

Thus, $x(t) \in S$ for all $0 \leq t \leq \epsilon$.
For 2.: We repeat the definition of $L(S, \bar{x})$:

$$L(S, \bar{x}) = \{v \in \mathbb{R}^n | g'(\bar{x})v \in K(g(\bar{x})),\ h'(\bar{x})v = 0\}.$$

Let $v \in L(S, \bar{x})$. Per definition of $K(g(\bar{x}))$ there is $r > 0$ with

$$g'(\bar{x})v \in r(K - g(\bar{x})),\ h'(\bar{x})v = 0.$$

This implies

$$g(\bar{x}) + g'(\bar{x})\frac{v}{r} \in K,\ \ h'(\bar{x})v = 0.$$

Using regularity of $\bar{x}$ and consequently (4.13), there is $v_0 \in \mathbb{R}^n$ with

$$g(\bar{x}) + g'(\bar{x})v_0 \in \overset{o}{K},\ \ h'(\bar{x})v_0 = 0.$$

We use the convex combination

$$v_\alpha := (1-\alpha)v_0 + \alpha\frac{v}{r},\ \text{ for } 0 \leq \alpha < 1.$$

and get using the following statement

> **Exercise 4.20.** Let K be convex and $x \in \overset{o}{K}, y \in \bar{K}$. Then,
>
> $$(1-\alpha)x + \alpha y \in \overset{o}{K},\ \text{für } 0 \leq \alpha < 1.$$

that

$$g(\bar{x}) + g'(\bar{x})v_\alpha \in \overset{o}{K},\ \ h'(\bar{x})v_\alpha = 0 \text{ for } 0 \leq \alpha < 1.$$

With part 1. we get

$$v_\alpha \in T(S,\bar{x}) \text{ for } 0 \leq \alpha < 1.$$

Using that $T(S,\bar{x})$ is closed, we get

$$v = \lim_{\alpha \to 1} r v_\alpha \in T(S,\bar{x}). \qquad \square$$

We give some examples.

Example 4.21. The set S is given as

$$g_1(x) = x_2 - x_1^3 \leq 0, \quad g_2(x) = x_2 \leq 0.$$

For $\bar{x} = 0$ we have $I(\bar{x}) = \{1,2\}$.

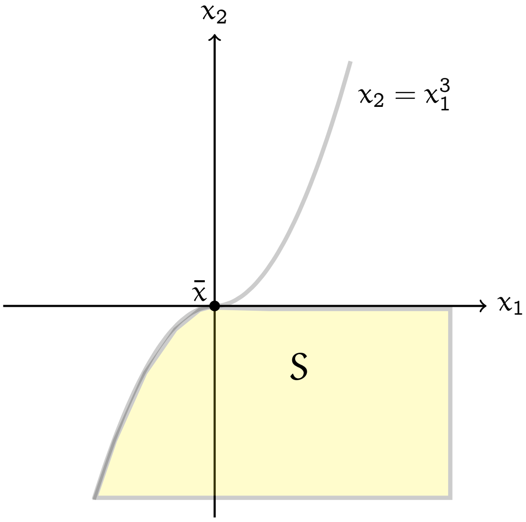


Figure 4.8: Set S.

The derivatives

$$g_1'(\bar{x}) = g_2'(\bar{x}) = (0,1)$$

are not linearly independent. Yet, the point $\bar{x} = 0$ is regular in the sense of Definition 4.18, because for every $v \in \mathbb{R}^2$ with $v_2 < 0$ we get

$$g_i'(\bar{x})v = v_2 < 0, \; i = 1,2.$$

We obtain

$$T(S,\bar{x}) = L(S,\bar{x}) = \{v \in \mathbb{R}^2 | v_2 \leq 0\}$$

according to Thm. 4.19, (2.).

Let us slightly change that example.

Example 4.22. S is given as

$$g_1(x) = x_2 - x_1^3 \leq 0, \quad g_2(x) = -x_2 \leq 0.$$

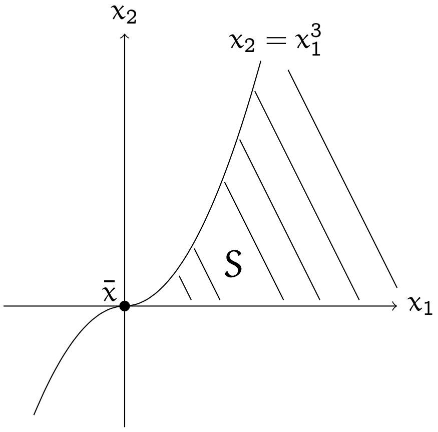


Figure 4.9: Set S.

For $\bar{x} = 0$ we get $I(\bar{x}) = \{1, 2\}$. The derivatives

$$g_1'(\bar{x}) = (0, 1) \text{ and } g_2'(\bar{x}) = (0, -1).$$

are linearly dependent. $\bar{x} = 0$ is not regular in the sense of Definition 4.18, since for no $v \in \mathbb{R}^2$, the conditions

$$g_1'(0)v = v_2 < 0, \text{ und } g_2'(0)v = -v_2 < 0$$

are satisfiable. We compute

$$T(S, \bar{x}) = \{v \in \mathbb{R}^2 | v_1 \geq 0, v_2 = 0\}$$

and with (4.6) we get

$$L(S, \bar{x}) = \{v \in \mathbb{R}^2 | v_2 = 0\}.$$

Here we have $T(S, \bar{x}) \subsetneq L(S, \bar{x})$.

Chapter 5

First Order Necessary Optimality Conditions

$$\min\{f(x)|\ g(x) \in K\} \tag{5.1}$$

We assume that $f, g \in C^1$.
Let $\bar{x}$ be a local minimum of (5.1). Then, there is a neighbourhood U of $\bar{x}$ such that the following subsets of $\mathbb{R} \times \mathbb{R}^m$ defined as

$$\{f(x) - f(\bar{x}), g(x)|x \in U\} \cap \{(r, y)|r < 0, y \in K\}$$

have an empty intersection. Thus, $(0, 0_m)$ lies on the boundary of

$$B = \{(f(x) - f(\bar{x}) + r, g(x) - y)|x \in U, r \geq 0, y \in K\} \subset \mathbb{R} \times \mathbb{R}^m. \tag{5.2}$$

Let us linearize the set B, i.e., we replace $f(x)$ and $g(x)$ with their first order Taylor expansion (without rest term) in $\bar{x}$, hence,

$$f(\bar{x}) + f'(\bar{x})v \text{ and } g(\bar{x}) + g'(\bar{x})v \text{ with } v = x - \bar{x},$$

and we obtain the convex set

$$\tilde{B} = \{(f'(\bar{x})v + r, g'(\bar{x})v + g(\bar{x}) - y|v \in \mathbb{R}^n, r \geq 0, y \in K\} \subset \mathbb{R} \times \mathbb{R}^m.$$

The conical hull of this set wrt. $(0, 0_m)$ is the following convex cone:

$$A = \{(f'(\bar{x})v + r, g'(\bar{x})v - y|v \in \mathbb{R}^n, r \geq 0, y \in K(g(\bar{x}))\} \subset \mathbb{R} \times \mathbb{R}^m. \tag{5.3}$$

Exercise 5.1. Show that indeed A is the conical hull of $\tilde{B}$ wrt. $(0, 0_m)$.

The set A can be interpreted as a convex approximation of the non-convex set B. The statement of the following theorem says that $(0, 0_m)$ lies on the boundary of A.

Theorem 5.2. Let $\bar{x}$ be a local minimum of (5.1). Then, the following statements are true.

1. Necessary Optimality Conditions of Fritz John:
   There is $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m$, $(\lambda_0, \lambda) \neq (0, 0_m)$ with
   $$\lambda_0 f'(\bar{x}) + \lambda g'(\bar{x}) = 0 \tag{5.4}$$
   $$\lambda_0 \geq 0,\ \lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})). \tag{5.5}$$

2. Necessary Optimality Conditions of Karush-Kuhn-Tucker:
   If $\bar{x}$ is regular, we get $\lambda_0 > 0$ in (1) and w.l.o.g. $\lambda_0 = 1$ holds. Thus, there is $\lambda \in \mathbb{R}^m$ with
   $$f'(\bar{x}) + \lambda g'(\bar{x}) = 0 \tag{5.6}$$
   $$\lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})). \tag{5.7}$$

Proof. Both statements are proven together.
1. Case: $\bar{x}$ is not regular. With Corollary 4.16 there is $\lambda \in \mathbb{R}^m, \lambda \neq 0$ with

$$\lambda g'(\bar{x}) = 0,\ \lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})).$$

The statement of the first part of the theorem follows with $\lambda_0 = 0$.
2. Case: Let $\bar{x}$ be regular. The variational inequality of Thm. 4.6 reads as:

$$f'(\bar{x})v \geq 0 \text{ for all } v \in T(S, \bar{x}),\ \text{ where } S := \{x \mid g(x) \in K\}.$$

With the regularity of $\bar{x}$, we get with Thm. 4.19 that

$$T(S, \bar{x}) \supset L(S, \bar{x}) = \{v \in \mathbb{R}^n | g'(\bar{x})v \in K(g(\bar{x}))\},$$

and hence

$$f'(\bar{x})v \geq 0 \text{ for all } v \in \mathbb{R}^n \text{ with } g'(\bar{x})v \in K(g(\bar{x})). \tag{5.8}$$

We consider the convex cone in (5.3):

$$A = \{(f'(\bar{x})v + r, g'(\bar{x})v - y | v \in \mathbb{R}^n, r \geq 0, y \in K(g(\bar{x}))\} \subset \mathbb{R} \times \mathbb{R}^m.$$

Because of (5.8) we have that $(0, 0_m)$ lies on the boundary of A (set $v = 0, r = 0, y = 0$) and with the separating hyperplane theorem there is $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m$, $(\lambda_0, \lambda) \neq 0$, with

$$\lambda_0(f'(\bar{x})v + r) + \lambda(g'(\bar{x})v - y) \geq 0 \text{ for all } v \in \mathbb{R}^n, r \geq 0, y \in K(g(\bar{x})). \tag{5.9}$$

For $r = 0$ and $y = 0$ we get

$$\lambda_0(f'(\bar{x})v) + \lambda g'(\bar{x})v \geq 0 \text{ for all } v \in \mathbb{R}^n$$

and therefore

$$\lambda_0(f'(\bar{x})) + \lambda g'(\bar{x}) = 0.$$

For $v = 0$ and $y = 0$ we get $\lambda_0 r \geq 0$ for all $r \geq 0$ and hence $\lambda_0 \geq 0$. Finally, (5.9) implies for $v = 0$ and $r = 0$

$$\lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})).$$

The case $\lambda_0 = 0$ contradicts the statement of Cor. 4.16 for regular points $\bar{x}$. Thus, $\lambda_0 > 0$ and w.l.o.g. $\lambda_0 = 1$.

□

5.1 Lagrange-Function and Multipliers

The last theorem is known as the Lagrange-Multiplier rule. The vector $\lambda_i, i = 0, \ldots, m$ is called Lagrange-Multiplier. The Lagrange-Function is defined as

$$L(x, \lambda_0, \lambda) := \lambda_0 f(x) + \lambda g(x), \ (\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m$$

and for regular points

$$L(x, \lambda) := f(x) + \lambda g(x), \ \lambda \in \mathbb{R}^m.$$

The last theorem then reads as:

1. John: $L_x(\bar{x}, \lambda_0, \lambda) = 0, \lambda_0 \geq 0, \lambda(-y) \geq 0$ for all $y \in K(g(\bar{x}))$
2. KKT: $L_x(\bar{x}, \lambda) = 0, \lambda(-y) \geq 0$ for all $y \in K(g(\bar{x}))$,

where L_x denote the partial derivative of L wrt. x.

Exercise 5.3. Show that the following sets

$$\Lambda(\bar{x}) := \{\lambda \in \mathbb{R}^m | L_x(\bar{x}, \lambda) = 0, \lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x}))\} \tag{5.10}$$

are convex and closed.
If $\bar{x}$ is regular, then the above set is the set of Lagrange-Multipliers at $\bar{x}$.

Theorem 5.4. Let $\bar{x}$ be a local minimum of (5.1). Then, the following statements are equivalent.

1. $\bar{x}$ is regular.
2. $\Lambda(\bar{x}) \neq \emptyset$ and $\Lambda(\bar{x})$ are bounded.

Proof. 1.⇒2.: $\Lambda(\bar{x}) \neq \emptyset$ follows from Theorem 5.2, (2.).
Assumption: $\Lambda(\bar{x})$ is unbounded. Then, there is a sequence $\lambda_i \in \Lambda(\bar{x}), i \in \mathbb{N}$, with $\|\lambda_i\| \to \infty$ for $i \to \infty$. Per definition of $\Lambda(\bar{x})$ we have

$$f'(\bar{x}) + \lambda_i g'(\bar{x}) = 0, \lambda_i(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})).$$

With $\lambda_i \neq 0$ for i large enough, we get

$$\frac{1}{\|\lambda_i\|} f'(\bar{x}) + \frac{\lambda_i}{\|\lambda_i\|} g'(\bar{x}) = 0, \quad \frac{\lambda_i}{\|\lambda_i\|}(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})). \tag{5.11}$$

As the boundary of $B_1(0)$ is compact in $\mathbb{R}^m$ we may assume that $\frac{\lambda_i}{\|\lambda_i\|} \to \lambda$ with $\|\lambda\| = 1$. Equation (5.11) yields for $i \to \infty$

$$\lambda g'(\bar{x}) = 0, \lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})).$$

With Corollary 4.16 we get a contradiction to the regularity at $\bar{x}$.
1.$\Leftarrow$2.: Let $\lambda_1 \in \Lambda(\bar{x})$.
Assumption: $\bar{x}$ is not regular.
With Corollary 4.16 there is $\lambda_2 \in \mathbb{R}^m, \lambda_2 \neq 0$ with

$$\lambda_2 g'(\bar{x}) = 0, \quad \lambda_2(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})).$$

This implies $\lambda_1 + r\lambda_2 \in \Lambda(\bar{x})$ for all $r \geq 0$ in contradiction to the boundedness of $\Lambda(\bar{x})$. ☐

We get the following implication.

Corollary 5.5. If $\bar{x}$ is a regular local minimum of (5.1), then $\Lambda(\bar{x})$ is a nonempty, compact and convex subset of $\mathbb{R}^m$.

Now we discuss uniqueness of the Lagrange-Multipliers.

Definition 5.6. $\bar{x}$ is called normal, if

$$\operatorname{Im} g'(\bar{x}) - V = \mathbb{R}^m, \quad V := K(g(\bar{x})) \cap (-K(g(\bar{x}))). \tag{5.12}$$

Recall that $V = K(g(\bar{x})) \cap (-K(g(\bar{x})))$ is the largest linear subspace contained in $K(g(\bar{x}))$. With $K(g(\bar{x})) \supset V$ we get: $\bar{x}$ is normal $\Rightarrow$ $\bar{x}$ is regular.

Theorem 5.7. If $\bar{x}$ is a normal local minimum of (5.1), then $\Lambda(\bar{x})$ is a singleton.

Proof. We get $\Lambda(\bar{x}) \neq \emptyset$, as $\bar{x}$ is regular. For $\lambda_1, \lambda_2 \in \Lambda(\bar{x})$ we show $\lambda_1 = \lambda_2$. Per definition of $\Lambda(\bar{x})$ we have

$$f'(\bar{x}) + \lambda_i g'(\bar{x}) = 0, \quad \lambda_i(-y) \geq 0 \text{ for all } y \in K(g(\bar{x})) \quad (i = 1, 2).$$

The vector $\lambda := \lambda_1 - \lambda_2$ satisfies

$$\lambda g'(\bar{x}) = 0, \quad \lambda y = 0 \text{ for all } y \in V.$$

With condition (5.12) in Definition 5.6 (note $\operatorname{Im} g'(\bar{x}) - V = \mathbb{R}^m$) we get $\lambda = 0$ and therefore $\lambda_1 = \lambda_2$. ☐

5.2 Specialization for the Standard Cone

Exercise 5.8. Show that if the set K in formulation 5.1 is a convex cone, we get $K(g(\bar{x})) = K + \mathbb{R}\, g(\bar{x})$.

$$\lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x}))$$

is in this case equivalent to

$$\lambda(-y) \geq 0 \text{ for all } y \in K,\ \lambda g(\bar{x}) = 0. \tag{5.13}$$

The equation $\lambda g(\bar{x}) = 0$ is called complementarity conditions.

Exercise 5.9. Show that for $K(g(\bar{x})) = K + \mathbb{R}\, g(\bar{x})$ we get that

$$\lambda(-y) \geq 0 \text{ for all } y \in K(g(\bar{x}))$$

is equivalent to

$$\lambda(-y) \geq 0 \text{ for all } y \in K,\ \lambda g(\bar{x}) = 0.$$

Now we consider the standard problem of nonlinear optimization.

$$\begin{aligned} \min \{f(x)|\ & g_i(x) \leq 0,\ i = 1, \dots, k, \\ & g_i(x) = 0,\ i = k+1, \dots, m\} \end{aligned} \tag{5.14}$$

This problem is obtained as special case of 5.1 by setting $K = \mathbb{R}^k_- \times \{0_{m-k}\}$. Recall:

$$\begin{aligned} I(\bar{x}) &:= \{i \in \{1, \dots, k\}|\ g_i(\bar{x}) = 0\} \\ J(\bar{x}) &:= I(\bar{x}) \cup \{k+1, \dots, m\} \end{aligned}$$

For the Lagrange-Multipliers $\lambda = (\lambda_1, \dots, \lambda_m)$ we get from (5.13)

$$\lambda_i \geq 0 \text{ for all } i \in I(\bar{x}),\ \lambda_i = 0 \text{ for all } i \notin J(\bar{x}).$$

We compute

$$K(g(\bar{x})) = \{y \in \mathbb{R}^m | y_i \leq 0,\ i \in I(\bar{x}),\ y_i = 0,\ i = k+1, \dots, m\}$$

and the linear subspace V in (5.12) is given by

$$V = \{y \in \mathbb{R}^m | y_i = 0 \text{ for all } i \in J(\bar{x})\}.$$

Corollary 5.10. $\bar{x}$ is normal for 5.14 if and only if the gradients

$$g_i'(\bar{x})\ i \in J(\bar{x}) \text{ are linearly independent.} \tag{5.15}$$

Theorem 5.11. Let $\bar{x}$ be a local minimum of (5.14). Then, there is $\lambda_0 \geq 0$ and $\lambda \in \mathbb{R}^m$ with $(\lambda_0, \lambda) \neq (0, 0_m)$, such that:

1. $L_x(\bar{x}, \lambda_0, \lambda) = \lambda_0 f'(\bar{x}) + \lambda g'(\bar{x}) = \lambda_0 f'(\bar{x}) + \sum_{i=1}^m \lambda_i g_i'(\bar{x}) = 0 \in \mathbb{R}^n$.
2. $\lambda_i = 0$ for $i \notin J(\bar{x})$, i.e., $i \in \{1, \dots, k\}$ with $g_i(\bar{x}) < 0$.
3. $\lambda_i \geq 0$ for $i \in I(\bar{x})$.

We have $\lambda_0 > 0$, if $\bar{x}$ regular. Then, w.l.o.g. $\lambda_0 = 1$ (divide Lagrange-Function by $\lambda_0 > 0$). If $\bar{x}$ is normal, then $\lambda \in \mathbb{R}^m$ is unique.

Every feasible $\bar{x}$ with multipliers (λ_0, λ) satisfying the conditions of Thm. 5.11are called critical point. It turns out that not every critical point is a local minimum of (5.14). In particular, for problems 5.11 with equations, the required conditions for the minimization

$$\min\{f(x) | g(x) = 0\}$$

and maximization variant

$$\max\{f(x) | g(x) = 0\}$$

coincide.

Example 5.12.

$$\min \{f(x) = x_1 + x_2 \\ g(x) = x_1^2 + x_2^2 - 2 = 0.\}$$

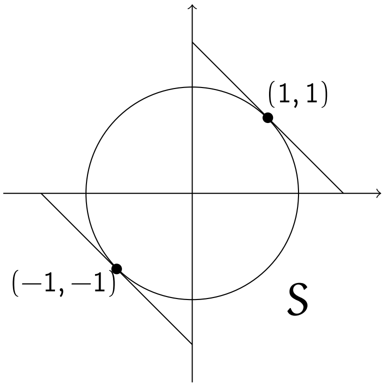


Figure 5.1: Set S and critical points.

Every point of S defined as

$$S := \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 - 2 = 0\}$$

is regular. The necessary KKT-conditions are

$$f'(\bar{x}) + \lambda g'(\bar{x}) = (1 + 2\lambda\bar{x}_1, 1 + 2\lambda\bar{x}_2) = 0.$$

This implies $\lambda \neq 0$. Together with $g(\bar{x}) = 0$ we get

$$\begin{aligned} \bar{y} &= (1, 1)^T,\ \lambda = -1/2 \\ \bar{x} &= (-1, -1)^T,\ \lambda = 1/2 \end{aligned}$$

Obviously $\bar{y}$ is a local maximum, while $\bar{x}$ is a local minimum. With the sufficient optimality conditions that come up in the next section, we can show this formally.

We slightly modify the example.

Example 5.13.

$$\begin{aligned} \min \{f(x) &= x_1 + x_2 \\ g(x) &= x_1^2 + x_2^2 - 2 \leq 0.\} \end{aligned}$$

Every $\bar{x} \neq 0$ of the set S defined as

$$S := \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 - 2 \leq 0\}$$

is regular. Th necessary KKT-conditions read as

$$f'(\bar{x}) + \lambda g'(\bar{x}) = (1 + 2\lambda\bar{x}_1, 1 + 2\lambda\bar{x}_2) = 0,\ \lambda \geq 0 \text{ for } g(\bar{x}) = 0.$$

We get the unique solution

$$\bar{x} = (-1, -1)^T,\ \lambda = 1/2$$

Hence, the sign constraint $\lambda \geq 0$ sorts out the solution $\bar{y}$.

We give another example.

Example 5.14.

$$\begin{aligned} \min \{f(x) &= -x_2 \\ g_1(x) &= x_1^2 + x_2 \leq 0 \\ g_2(x) &= -x_1^2 + x_2 \leq 0.\} \end{aligned}$$

$\bar{x} = (0, 0)^T$ is the global minimum. We get $I(\bar{x}) = \{1, 2\}$ and

$$g_1'(0, 0) = (0, 1), g_2'(0, 0) = (0, 1)$$

are linearly dependent, thus, $\bar{x}$ is not normal. But $\bar{x}$ is regular, as it satisfies the Mangasarian-

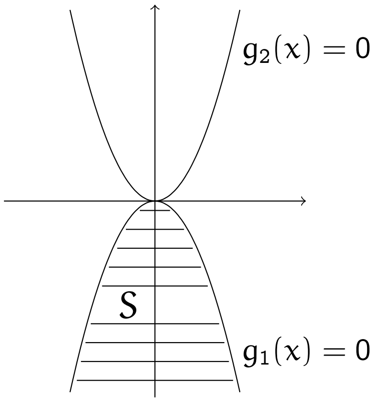


Figure 5.2: Set S and critical points.

Fromowitz-conditions for $v = (0, -1)^\top$.

$$g_1'(0,0)v = g_2'(0,0)v = (0,1) \cdot (0,-1)^\top = -1 < 0.$$

The necessary KKT-conditions read as

$$f'(\bar{x}) + \lambda g'(\bar{x}) = (0, -1 + \lambda_1 + \lambda_2) = 0,\ \lambda_1 \geq 0, \lambda_2 \geq 0.$$

The set of multipliers is given by

$$\Lambda(\bar{x}) = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2_+ |\ \lambda_1 + \lambda_2 = 1\}.$$

This set is convex and compact in compliance with Corollary 5.5.

Chapter 6

Second-Order Necessary and Sufficient Optimality Conditions

$$\min\{f(x)|\ \ g(x)\in K\} \tag{6.1}$$

where we assume $f, g \in C^2$.
The Hesse-matrix of f in $\bar{x}$ is denoted by

$$f''(\bar{x}) = \left(\frac{\partial^2 f(\bar{x})}{\partial x_i \partial x_j}\right)_{i,j=1,\ldots,n}.$$

The Hesse-matrix of the Lagrange function reads as

$$L_{xx}(\bar{x}, \lambda_0, \lambda) = \lambda_0 f''(\bar{x}) + \lambda g''(\bar{x}) = \lambda_0 f''(\bar{x}) + \sum_{i=1}^{m} \lambda_i g_i''(\bar{x}),$$

where the Hessian tensor of g in x is defined as

$$g''(x) := (g_1'', \ldots, g_m'') \text{ with } \lambda g''(x) := \sum_{i=1}^{m} \lambda_i g_i''(x).$$

The linearized cone of S in $\bar{x}$ is per definition in (4.5) given as

$$L(S, \bar{x}) = \{v \in \mathbb{R}^n | g'(\bar{x})v \in K(g(\bar{x}))\}.$$

We now derive second order conditions using the Hesse-matrix of the Lagrange-function

$$\lambda_0 f''(\bar{x}) + \lambda g''(\bar{x})$$

wrt. the convex cone

$$\begin{aligned} C &:= \{v \in \mathbb{R}^n | f'(\bar{x})v \le 0,\ g'(\bar{x})v \in K(g(\bar{x}))\} \\ &= \{v \in \mathbb{R}^n | f'(\bar{x})v \le 0\} \cap L(S, \bar{x}). \end{aligned} \tag{6.2}$$

The set C represents the set of vectors v of the linearized cone $L(S, \bar{x})$ that constitute descent directions of f in $\bar{x}$.

Theorem 6.1. Let $\bar{x} \in S$ and $K(g(\bar{x}))$ closed.

1. Second Order Necessary Conditions:
   If $\bar{x}$ is a local minimum of (6.1), then, for every $v \in C$ there is $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m$, $(\lambda_0, \lambda) \neq 0$ with
   (a) $\lambda_0 \geq 0$, $\lambda(-y) \geq 0$ for all $y \in K(g(\bar{x}))$.
   (b) $\lambda_0 f'(\bar{x}) + \lambda g'(\bar{x}) = 0$
   (c) $v^T(\lambda_0 f''(\bar{x}) + \lambda g''(\bar{x}))v \geq 0$
2. Second Order Sufficient Conditions:
   Suppose for every $v \in C \setminus \{0\}$ there exists $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m$, $(\lambda_0, \lambda) \neq 0$ with
   (a) $\lambda_0 \geq 0$, $\lambda(-y) \geq 0$ for all $y \in K(g(\bar{x}))$.
   (b) $\lambda_0 f'(\bar{x}) + \lambda g'(\bar{x}) = 0$
   (c) $v^T(\lambda_0 f''(\bar{x}) + \lambda g''(\bar{x}))v > 0$.

   Then, there is $\epsilon > 0$ and a constant $c > 0$ with
   $$f(x) \geq f(\bar{x}) + c\,\|x - \bar{x}\|^2 \text{ for all } x \in S \text{ with } \|x - \bar{x}\| \leq \epsilon. \tag{6.3}$$
   In particular, $\bar{x}$ is a strong local minimum of (6.1).
3. If $\bar{x}$ is regular, then $\lambda_0 > 0$, i.e., w.l.o.g. $\lambda_0 = 1$ can be chosen in (1.) and (2.). If $\bar{x}$ is normal, then $\lambda_0 = 1$ and λ in (1.) and (2.) is unique and independent of $v \in C$.

Proof. For 1: The proof is similar to that for the first order conditions, see Lempio and Zowe (1981) for a complete proof.

For 2: We assume that (6.3) is false. Then, there is a sequence $\{x_i\} \subset S$ with
$$x_i \neq \bar{x}, \quad \lim_{i \to \infty} x_i = \bar{x}.$$
and
$$f(x_i) < f(\bar{x}) + \frac{1}{i}\|x_i - \bar{x}\|^2. \tag{6.4}$$
We set $v_i := x_i - \bar{x} \neq 0$. The boundary of the unit ball is compact and hence we can assume w.l.o.g. that
$$v := \lim_{i \to \infty} \frac{v_i}{\|v_i\|} \quad (\text{with } \|v\| = 1).$$
With (6.4) we get
$$f'(\bar{x})v = \lim_{i \to \infty} \frac{f(x_i) - f(\bar{x})}{\|x_i - \bar{x}\|} \leq 0,$$

and with the closedness of $K(g(\bar{x}))$ we get

$$g'(\bar{x})v = \lim_{i\to\infty} \frac{g(x_i) - g(\bar{x})}{\|x_i - \bar{x}\|} \in K(g(\bar{x})).$$

Per definition of the cone (6.2) we get $v \in C \setminus \{0\}$. For such v there is per assumption $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m$, $(\lambda_0, \lambda) \neq 0$, satisfying $(a), (b), (c)$. The Taylorexpansion in $\bar{x}$ of second order of f and g yields together with (6.4):

$$f'(\bar{x})v_i + \frac{1}{2}v_i^T f''(\bar{x})v_i + o(\|v_i\|^2) = f(x_i) - f(\bar{x}) \leq \frac{1}{i}\|v_i\|^2 \tag{6.5}$$

$$g'(\bar{x})v_i + \frac{1}{2}v_i^T g''(\bar{x})v_i + o(\|v_i\|^2) = g(x_i) - g(\bar{x}) \in K(g(\bar{x})). \tag{6.6}$$

Multiplying (6.5) with λ_0 and (6.6) with λ, we get via addition of both equalities and considering $(a), (b)$ the inequality

$$\frac{1}{2}v_i^T(\lambda_0 f''(\bar{x}) + \lambda g''(\bar{x}))v_i + o(\|v_i\|^2) \leq \frac{\lambda_0}{i}\|v_i\|^2.$$

(We use in particular (a), i.e., $g(x_i) - g(\bar{x}) \in K(g(\bar{x}))$ and hence $\lambda(g(x_i) - g(\bar{x})) \leq 0$.) The division by $\|v_i\|^2$ yields in the limit $i \to \infty$ the inequality

$$\frac{1}{2}v^T(\lambda_0 f''(\bar{x}) + \lambda g''(\bar{x}))v \leq 0$$

in contradiction to (c).

For 3: If $\bar{x}$ is regular, we get $\lambda_0 > 0$ from Thm. 5.2(2.). If $\bar{x}$ is normal, we get the statement from Thm. 5.7. $\square$

The cone C in (6.2) can also described without using f. Suppose that $\bar{x}$ is normal. If $\bar{x}$ is a local minimum, then there is a unique $\lambda \in \mathbb{R}^m$ with

$$f'(\bar{x}) + \lambda g'(\bar{x}) = 0, \quad \lambda(-y) \geq 0 \text{ for } y \in K(g(\bar{x})). \tag{6.7}$$

For $v \in C$ we get per definition

$$f'(\bar{x})v \leq 0, \quad \lambda(-y) \geq 0 \text{ for } y \in K(g(\bar{x})).$$

In conjunction with (6.7) we get

$$f'(\bar{x})v = -\lambda g'(\bar{x})v \leq 0, \lambda(-g'(\bar{x})v) \geq 0.$$

Hence,

$$f'(\bar{x})v = 0 \text{ and } \lambda g'(\bar{x})v = 0 \text{ for all } v \in C.$$

Thus, C has the form

$$C = \{v \in \mathbb{R}^n | \lambda g'(\bar{x})v = 0, \quad g'(\bar{x})v \in K(g(\bar{x}))\}. \tag{6.8}$$

6.1 Specialization to the Standard Cone

For $K = \{0\}$ we get

$$C = L(S, \bar{x}) = \{v \in \mathbb{R}^n | g'(\bar{x})v = 0\}.$$

For the standard problem we get using $I(\bar{x})$ and $J(\bar{x}) = I(\bar{x}) \cup \{k+1, \dots, m\}$ (see Section 5) for the multiplier $\lambda \in \mathbb{R}^m$:

$$\lambda_i \geq 0 \text{ for all } i \in I(\bar{x}),\ \lambda_i = 0 \text{ for all } i \notin J(\bar{x}).$$

Using

$$K(g(\bar{x})) = \{y \in \mathbb{R}^m | y_i \leq 0, i \in I(\bar{x}),\ y_i = 0, i = k+1, \dots, m\}$$

C has the form

$$\begin{aligned} C = \{v \in \mathbb{R}^n | g_i'(\bar{x})v \leq 0, i \in I(\bar{x}),\ \lambda_i = 0,\\ g_i'(\bar{x})v = 0, i \in I(\bar{x}),\ \lambda_i > 0,\\ g_i'(\bar{x})v = 0, i = k+1, \dots, m\}. \end{aligned} \tag{6.9}$$

Altogether we obtain the following conditions for problem (6.1).

Theorem 6.2. Let $\bar{x} \in S$ be normal, i.e., the gradients $g_i'(\bar{x})^T, i \in J(\bar{x})$ are linearly independent.

1. Second-order necessary conditions:
   If $\bar{x}$ is a local minimum (6.1), then there is a unique $\lambda_i \in \mathbb{R}, i \in J(\bar{x})$, with
   (a) $\lambda_i \geq 0$ for $i \in I(\bar{x})$,
   (b) $f'(\bar{x}) + \sum_{i \in J(\bar{x})} \lambda_i g_i'(\bar{x}) = 0$
   (c) $v^T(f''(\bar{x}) + \sum_{i \in J(\bar{x})} \lambda_i g_i''(\bar{x}))v \geq 0$ for all $v \in C$ mit C as in (6.9).
2. Second-order sufficient conditions:
   Suppose there are $\lambda_i \in \mathbb{R}, i \in J(\bar{x})$ with
   (a) $\lambda_i \geq 0$ for $i \in I(\bar{x})$,
   (b) $f'(\bar{x}) + \sum_{i \in J(\bar{x})} \lambda_i g_i'(\bar{x}) = 0$
   (c) $v^T(f''(\bar{x}) + \sum_{i \in J(\bar{x})} \lambda_i g_i''(\bar{x}))v > 0$ for all $v \in C \setminus \{0\}$ with C as in (6.9).

   Then, there is $\epsilon > 0$ and $c > 0$ with

   $$f(x) \geq f(\bar{x}) + c\,\|x - \bar{x}\|^2 \text{ for all } x \in S \text{ with } \|x - \bar{x}\| \leq \epsilon. \tag{6.10}$$

   In particular, $\bar{x}$ is a strong local minimum of (6.1).

6.2 Examples

We revisit the example 5.12

$$\begin{aligned} \min\{f(x) &= x_1 + x_2 \\ g(x) &= x_1^2 + x_2^2 - 2 = 0.\} \end{aligned}$$

which has the two critical points

$$\begin{aligned} \bar{x}_1 &= (1,1)^\top \text{ with } \lambda_1 = -1/2 \\ \bar{x}_2 &= (-1,-1)^\top \text{ with } \lambda_2 = 1/2. \end{aligned}$$

The Hesse-matrix of the Lagrange-function in those points is

$$f''(\bar{x}) + \lambda g''(\bar{x}) = \lambda \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

For $\bar{x}_2$ and λ_2 we get that the matrix is pos. def. on $\mathbb{R}^2$, hence, the conditions of Thm. 6.2(2.) are satisfied with the subspace

$$C = \{v \in \mathbb{R}^2 | v_2 = -v_1\}.$$

Thus, $\bar{x}_2$ is a strong local minimum.

For $\bar{x}_1$ and λ_1 the matrix is negativ definite, and hence $\bar{x}_1$ is a strong local maximum.

We give another example.

$$\begin{aligned} \min\{f(x) &= x_1^2 + x_2 \\ g_1(x) &= x_1^2 + x_2^2 - 9 \le 0, \\ g_2(x) &= x_1 + x_2 - 1 \le 0.\} \end{aligned}$$

As candidate for a local minimnum consider $\bar{x} = (0,-3)^\top$, for which $g_1(\bar{x}) = 0$ and $g_2(\bar{x}) < 0$. Hence, $I(\bar{x}) = \{1\}$. The point $\bar{x}$ is normal and the KKT-conditions in Thm. 5.11 has with $\lambda_2 = 0$ the solution $\lambda_1 = 1/6$. The cone C in (6.9) is the subspace

$$\begin{aligned} C &= \{v \in \mathbb{R}^2 | 2\bar{x}_1 v_1 + 2\bar{x}_2 v_2 = 0\} \\ &= \{v \in \mathbb{R}^2 | v_2 = 0\} \end{aligned}$$

The Hesse-matrix of the Lagrange-function

$$f''(\bar{x}) + \lambda_1 g_1''(\bar{x}) = \begin{pmatrix} 2(1+\lambda_1) & 0 \\ 0 & 2\lambda_1 \end{pmatrix}.$$

is positiv definite on $\mathbb{R}^2$ and in particular

$$v^\mathsf{T}(f''(\bar{x}) + \lambda_1 g_1'(\bar{x}))v = 2v_1^2(1 + \lambda_1) > 0$$

for all $v \in C \setminus \{0\}$. With Thm. 6.2(2.), we get that $\bar{x} = (0, -3)^\mathsf{T}$ is a strong local minimum.

Chapter 7

Sensitivity Analysis

We consider the standard problem:

$$\begin{aligned} \min\,\{f(x)|\ \ & g_i(x) \le 0,\ i = 1,\dots,k, \\ & g_i(x) = 0,\ i = k+1,\dots,m\} \end{aligned} \tag{7.1}$$

Suppose that the right-hand side (7.1) is perturbed:

$$\begin{aligned} \min\,\{f(x)|\ \ & g_i(x) \le \epsilon_i,\ i = 1,\dots,k, \\ & g_i(x) = \epsilon_i,\ i = k+1,\dots,m\} \end{aligned} \tag{7.2}$$

We obtain a parameterized family of optimization problems depending on $\epsilon := (\epsilon_1,\dots,\epsilon_m) \in \mathbb{R}^m$ denoted by (7.1). For $\epsilon = 0$ the perturbed problem (7.2) becomes (7.1) which we denote as the unperturbed problem.
More generally for $\epsilon \in \mathbb{R}^p, p \ge 1$, we obtain:

$$\begin{aligned} \min\,\{f(x,\epsilon)|\ \ & g_i(x,\epsilon) \le 0,\ i = 1,\dots,k, \\ & g_i(x,\epsilon) = 0,\ i = k+1,\dots,m\} \end{aligned} \tag{7.3}$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ are mappings with certain differentiability assumptions to be specified later and we allow even that f and g depend in a nonlinear way on ϵ.
For $K = \mathbb{R}^k_- \times \{0_{m-k}\}$. we obtain:

$$\min_{x\in\mathbb{R}^n}\{f(x,\epsilon)|g(x,\epsilon) \in K\}. \tag{7.4}$$

The feasible set (7.4) is given as

$$S(\epsilon) := \{x \in \mathbb{R}^n|g(x,\epsilon) \in K\},\ \ \epsilon \in \mathbb{R}^p. \tag{7.5}$$

The optimal value function of (7.4) is denoted as:

$$\begin{aligned} w : \mathbb{R}^p \to \bar{\mathbb{R}} &:= \mathbb{R} \cup \{-\infty, +\infty\} \text{ defined as} \\ w(\epsilon) &:= \inf_{x \in \mathbb{R}^n} \{f(x,\epsilon) | g(x,\epsilon) \in K\}, \epsilon \in \mathbb{R}^p. \end{aligned} \tag{7.6}$$

7.1 Local Sensitivity Analysis

Under which conditions can we embed a local minimum $x(0)$ of the unperturbed problem into a continuously differentiable family of perturbed local minima $x(\epsilon)$. W.l.o.g., we use the following notation:

$$\begin{aligned} I(\bar{x}) &= \{i \in \{1,\dots,k\} | g_i(\bar{x},0) = 0\} = \{1,\dots,k_0\}, \\ J(\bar{x}) &= \{1,\dots,k_0, k+1,\dots,m\}, \\ m_0 : &= |J(\bar{x})| = m + k_0 - k. \end{aligned} \tag{7.7}$$

Theorem 7.1. Let $\bar{x} \in S(0)$ be a local minimum of (7.4). Let $\bar{x}$ be normal, i.e., the gradients $g_i'(\bar{x})^T$ are linearly independent for $i \in J(\bar{x})$. Assume there are uniquely determined $\bar{\lambda}_i, i \in J(\bar{x})$ such that the second order sufficient optimality conditions of Thm. 6.1, (2) are satisfied with

1. $\bar{\lambda}_i > 0$ for $i = 1,\dots,k_0$,
2. $f_x(\bar{x},0) + \sum_{i \in J(\bar{x})} \bar{\lambda}_i g_{ix}(\bar{x},0) = 0$,
3. $v^T \Big(f_{xx}(\bar{x},0) + \sum_{i \in J(\bar{x})} \bar{\lambda}_i g_{ixx}(\bar{x},0) \Big) v > 0$ for all $v \neq 0$ with
$$v \in C = \{v \in \mathbb{R}^n | g_{ix}(\bar{x},0)v = 0,\ i \in J(\bar{x})\}.$$

Then there is a neighbourhood $E \subset \mathbb{R}^p$ of $\epsilon = 0$ and continuously differentiable functions $x : E \to \mathbb{R}^n, \lambda_i : E \to \mathbb{R}, i \in J(\bar{x})$, with:

1. $x(0) = \bar{x},\ \lambda_i(0) = \bar{\lambda}_i,\ i \in J(\bar{x})$,
2. for all $\epsilon \in E$: $x(\epsilon), \lambda_i(\epsilon), i \in J(\bar{x})$ satisfy the conditions of Thm. 6.1(2.) for the perturbed problem (7.4). In particular, $x(\epsilon)$ is a local minimum of (7.4).

Proof. We define

$$G : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{m_0},\ m_0 = m + k_0 - k,$$

via

$$G(x,\epsilon) = (g_1(x,\epsilon),\dots,g_{k_0}(x,\epsilon), g_{k+1}(x,\epsilon),\dots,g_m(x,\epsilon))^T.$$

Per construction, $\bar{x}$ is a strong local minimum of

$$\min\{f(x,0) |\ G(x,0) = 0\}.$$

The wanted function $x(\epsilon)$ of local minima to (7.4) should be C^1, thus, they should satisfy

$$g_i(x(\epsilon), \epsilon) < 0 \text{ for } i \notin J(\bar{x}), \|\epsilon\| \text{ small enough.}$$

Hence, $x(\epsilon)$ should be a local minimum of the perturbed problem with equality constraints

$$\min\{f(x, \epsilon) | \, G(x, \epsilon) = 0\}. \tag{7.8}$$

The Lagrange-function therefore is constructed as

$$L(x, \nu, \epsilon) = f(x, \epsilon) + \nu G(x, \epsilon), \;\; \nu \in \mathbb{R}^{m_0}. \tag{7.9}$$

$x(\epsilon)$ and the multiplier $\nu(\epsilon) = (\lambda_i(\epsilon))_{i \in J(\bar{x})}$ needs to solve

$$F(x, \nu^T, \epsilon) := \begin{pmatrix} L_x(x, \nu, \epsilon)^T \\ G(x, \epsilon) \end{pmatrix} = 0, \tag{7.10}$$

where

$$F : \mathbb{R}^n \times \mathbb{R}^{m_0} \times \mathbb{R}^p \to \mathbb{R}^n \times \mathbb{R}^{m_0}.$$

Note that ν^T as the argument of F appears as a column vector. For $\epsilon = 0$ we get per assumption $F(\bar{x}, \bar{\nu}^T, 0) = 0$ with $\bar{\nu} := (\bar{\lambda}_i)_{i \in J(\bar{x})}$ and F is C^1 in a neighbourhood of $(\bar{x}, \bar{\nu}^T, 0)$. The Jacobi-matrix of F wrt. (x, ν^T) in $(\bar{x}, \bar{\nu}^T, 0)$ is given by the $(n+m_0) \times (n+m_0)$ matrix

$$A_0 := \frac{\partial F}{\partial(x, \nu^T)}(\bar{x}, \bar{\nu}^T, 0) = \begin{pmatrix} L_{xx}(\bar{x}, \bar{\nu}, 0) & G_x(\bar{x}, 0)^T \\ G_x(\bar{x}, 0) & 0 \end{pmatrix}. \tag{7.11}$$

In order to apply the implicit function theorem, we will show that A_0 is non-singular. Let $(v, w) \in \mathbb{R}^n \times \mathbb{R}^{m_0}$ with

$$A_0 \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} L_{xx}(\bar{x}, \bar{\nu}, 0)\, v + G_x(\bar{x}, 0)^T\, w \\ G_x(\bar{x}, 0)\, v \end{pmatrix} = 0. \tag{7.12}$$

Multiplying the equation with $(v, 0)^T$ from the left and using $G_x(\bar{x}, 0)v = 0$, we get

$$v^T L_{xx}(\bar{x}, \bar{\nu}, 0) v = 0.$$

The assumption of the Thm. yield $v = 0$. The equation (7.12) reduces to

$$G_x(\bar{x}, 0)^T\, w = 0.$$

Using that $\bar{x}$ is normal, the matrix $G_x(\bar{x}, 0)^T$ has rank m_0 and therefore $w = 0$. Thus, A_0 is non-singular.

We can apply the implicit function theorem on the system of equations (7.10) and get the existence of a neighbourhood $E \subset \mathbb{R}^p$ of $\epsilon = 0$ and C^1 functions $x : E \to \mathbb{R}^n, \nu(\epsilon) = (\lambda_i(\epsilon))_{i \in J(\bar{x})} : E \to \mathbb{R}^{m_0}$, with:

$$\begin{aligned} F(x(\epsilon), \nu(\epsilon)^T, \epsilon) = 0 \text{ for all } \epsilon \in E, \\ x(0) = \bar{x}, \quad \nu(0) = \bar{\nu}. \end{aligned} \tag{7.13}$$

For completing the proof, we need to verify that $x(\epsilon)$ and $(\lambda_i(\epsilon))_{i \in J(\bar{x})}$ satisfy the second order sufficient optimality conditions. Because of

$$\begin{aligned} \lambda_i(0) = \bar{\lambda}_i > 0, \quad i = 1, \dots, k_0, \\ g_i(x(0), 0) = g_i(\bar{x}, 0) < 0, \text{ for } i = k_0, \dots, k \end{aligned}$$

and the continuity of the functions, we can choose E small enough, such that for all $\epsilon \in E$ we get

$$\begin{aligned} \lambda_i(\epsilon) > 0, \quad i = 1, \dots, k_0, \\ g_i(x(\epsilon), \epsilon) < 0, \text{ for } i = k_0, \dots, k \end{aligned}$$

and hence $J(x(\epsilon)) = J(\bar{x})$ for all $\epsilon \in E$. With (7.10) and (7.13) we get that $x(\epsilon) \in S(\epsilon)$ for all $\epsilon \in E$. Moreover, the KKT-conditions are satisfied:

$$L_x(x(\epsilon), \nu(\epsilon), \epsilon) = 0 \text{ for all } \epsilon \in E.$$

Again using continuity, we can choose E small enough such that for all $\epsilon \in E$:

$$v^T \Big(\lambda_0 f_{xx}(x(\epsilon), \epsilon) + \sum_{i \in J(\bar{x})} \bar{\lambda}_i(\epsilon) g_{ixx}(x(\epsilon), \epsilon) \Big) v > 0$$

for all $v \neq 0$ with

$$v \in C = \{v \in \mathbb{R}^n | g_{ix}(x(\epsilon), \epsilon) v = 0, \ i \in J(\bar{x})\}.$$

The matrix $G_x(x(\epsilon), \epsilon)$ has rank m_0 implying that $x(\epsilon)$ is normal. Altogether, $x(\epsilon)$ is a strong local minimum of the perturbed problem (7.4). $\square$

Corollary 7.2. For the functions $x : E \to \mathbb{R}^n, \nu(\epsilon) = (\lambda_i(\epsilon))_{i \in J(\bar{x})} : E \to \mathbb{R}^{m_0}$ appearing in Thm. 7.1, the following statements are true:

1. With the non-singular $(n + m_0) \times (n + m_0)$ matrix A_0 and the $(n + m_0) \times p$ matrix

$$B_0 = \begin{pmatrix} L_{x\epsilon}(\bar{x}, \bar{\nu}, 0) \\ G_\epsilon(\bar{x}, 0) \end{pmatrix}$$

we can compute $\dot{x}(0)$ and $\dot{\nu}(0)$ as

$$\begin{pmatrix} \dot{x}(0) \\ \dot{\nu}(0)^T \end{pmatrix} = -A_0^{-1} B_0.$$

2. Generalized Shadow-Price:

$$\frac{d}{d\epsilon} f(x(\epsilon),\epsilon)\Big|_{\epsilon=0} = L_\epsilon(\bar{x},\bar{\nu},0) = f_\epsilon(\bar{x},0) + \bar{\nu} G_\epsilon(\bar{x},0).$$

Proof. For (1): The differentiation of (7.13) yieds with (7.10) and (7.11):

$$\frac{d}{d\epsilon} F(x(\epsilon),\nu(\epsilon)^T,\epsilon)\Big|_{\epsilon=0} = A_0 \begin{pmatrix} \dot{x}(0) \\ \dot{\nu}(0)^T \end{pmatrix} + B_0 = 0.$$

For (2): From $G(x(\epsilon),\epsilon) = 0$ we get

$$0 = \frac{d}{d\epsilon} G(x(\epsilon),\epsilon)\Big|_{\epsilon=0} = G_x(\bar{x},0)\,\dot{x}(0) + G_\epsilon(\bar{x},0).$$

Together with

$$f_x(\bar{x},0) = -\bar{\nu} G_x(\bar{x},0)$$

we get

$$\begin{aligned} \frac{d}{d\epsilon} f(x(\epsilon),\epsilon)\Big|_{\epsilon=0} &= f_x(\bar{x},0)\,\dot{x}(0) + f_\epsilon(\bar{x},0) \\ &= -\bar{\nu} G_x(\bar{x},0)\,\dot{x}(0) + f_\epsilon(\bar{x},0) \\ &= \bar{\nu} G_\epsilon(\bar{x},0) + f_\epsilon(\bar{x},0) \\ &= L_\epsilon(\bar{x},\bar{\nu},0). \end{aligned}$$

□

7.2 Application to Real-Time Optimization

Part (1) of the Corollary allows to represent the solution $x(\epsilon)$ of the perturbed problem via a Taylor-expansion at the unperturbed solution :

$$\begin{pmatrix} x(\epsilon) \\ \nu(\epsilon)^T \end{pmatrix} \approx \begin{pmatrix} \bar{x} \\ \bar{\nu}^T \end{pmatrix} + \begin{pmatrix} \dot{x}(0) \\ \dot{\nu}(0)^T \end{pmatrix} \epsilon. \tag{7.14}$$

The formula 2. becomes for (7.2):

$$\frac{d}{d\epsilon_i} f(x(\epsilon),\epsilon)\Big|_{\epsilon=0} = \begin{cases} -\bar{\lambda}_i & \text{for } i \in J(\bar{x}) \\ 0, & \text{for } i \notin J(\bar{x}). \end{cases} \tag{7.15}$$

This formula shows how to interpret the Lagrange multipliers as shadow prices.

The formula (7.14) allows for an application in the area of real-time optimization. Suppose we compute offline a solution of the unperturbed problem. If the system data changes, a new solution can be computed in real-time (without resolving the equation system) via the approximation (7.14).

Example 7.3.

$$\begin{aligned} \min\ f(x,\epsilon) = -(0.5+\epsilon)\sqrt{x_1} - (0.5-\epsilon)x_2 \\ x_1 + x_2 \leq 1, \\ x_1 \geq 0.1, \\ x_2 \geq 0. \end{aligned}$$

for $\epsilon = 0$ the assumptions of Thm. 7.1 are satisfied with

$$\begin{aligned} \bar{x} &= (0.25, 0.75),\ \bar{x} \text{ is normal} \\ J(\bar{x}) &= \{1\},\ G_1(x_1,x_2) = x_1 + x_2 - 1 \\ \bar{\nu} &= \bar{\lambda}_1 = 0.5 > 0. \end{aligned}$$

The Lagrange-function (7.9) is given by

$$L(\bar{x},\bar{\nu},\epsilon) = -(0.5+\epsilon)\sqrt{\bar{x}_1} - (0.5-\epsilon)\bar{x}_2 + \bar{\nu}(\bar{x}_1 + \bar{x}_2 - 1).$$

The sufficient conditions of 2. order are valid, because

$$\begin{aligned} L_{xx}(\bar{x},\bar{\nu},\epsilon) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ C &= \{v \in \mathbb{R}^2 | G_x(\bar{x})v = v_1 + v_2 = 0\} \\ v^T L_{xx}(\bar{x},\bar{\nu},\epsilon)v &= v_1^2 > 0 \text{ for all } v \in C \setminus \{0\}. \end{aligned}$$

The formula gives

$$\begin{aligned} A_0 &= \begin{pmatrix} L_{xx} & G_x^T \\ G_x & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\ B_0 &= \begin{pmatrix} L_{x\epsilon} \\ G_\epsilon \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{pmatrix} &= -A_0^{-1}B_0 = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}. \end{aligned}$$

Hence, the approximation of first order yields

$$\begin{pmatrix} x_1(\epsilon) \\ x_2(\epsilon) \\ x_3(\epsilon) \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.75 \\ 0.5 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \epsilon.$$

For $\epsilon = 0.05$ we get

$$\begin{aligned} approximation: &\quad (0.35, 0.65, 0.45) \\ exact\ \ value: &\quad (0.373, 0.627, 0.45). \end{aligned}$$

Chapter 8

Duality

$$\begin{aligned} \min \{f(x) |\ & g_i(x) \le 0,\ i = 1, \dots, k, \\ & g_i(x) = 0,\ i = k+1, \dots, m\} \end{aligned} \tag{8.1}$$

with equivalent representation $\min\{f(x) | x \in S\}$, where $S = \{x \in \mathbb{R}^n | g(x) \in K\}$ and $K = \mathbb{R}^k_- \times \{0_{m-k}\}$.

Consider the Lagrange-Function:

$$L(x, \lambda) := f(x) + \lambda g(x),\ \lambda \in \mathbb{R}^k_+ \times \mathbb{R}^{m-k}.$$

We define the Lagrangian-Dual:

$$\begin{aligned} \mu &: \mathbb{R}^m \to \mathbb{R} \\ \mu(\lambda) &= \inf_{x \in \mathbb{R}^n} L(x, \lambda) = \inf_{x \in \mathbb{R}^n} \{f(x) + \lambda g(x)\}. \end{aligned}$$

We assume $\mu(\lambda) = -\infty$, if $L(x, \lambda)$ is not bounded from below on $\mathbb{R}^n$.

Theorem 8.1. If μ is finite on $R \subset \mathbb{R}^m$, them μ is concave on R.

Proof. Let $\lambda_1, \lambda_2 \in R$ and let $\alpha \in [0, 1]$. We obtain:

$$\begin{aligned} \mu(\alpha\lambda_1 + (1-\alpha)\lambda_2) &= \inf_{x \in \mathbb{R}^n} \{f(x) + (\alpha\lambda_1 + (1-\alpha)\lambda_2) g(x)\} \\ &\ge \inf_{y \in \mathbb{R}^n} \{\alpha f(y) + \alpha\lambda_1 g(y)\} + \inf_{z \in \mathbb{R}^n} \{(1-\alpha) f(z) + ((1-\alpha)\lambda_2) g(z)\} \\ &= \alpha\mu(\lambda_1) + (1-\alpha)\mu(\lambda_2). \end{aligned}$$

□

8.1 Dual Problem and Weak Duality

Let $p^* = \min\{f(x) | x \in S\}$ be the optimal value of (8.1). We show that for every Lagrange multiplier, i.e., $\lambda_i \ge 0, i = 1, \dots, k$, every value of $\mu(\lambda)$ yields a lower bound on p^*.

Theorem 8.2. For $\lambda \in \mathbb{R}^m$ with $\lambda_i \geq 0, i = 1, \dots, k$ and $\bar{x} \in S$, we have:

$$\mu(\lambda) \leq f(\bar{x}). \tag{8.2}$$

In particular, $\mu(\lambda) \leq p^*$.

Proof. Let $\bar{x} \in S$, i.e. $g_i(\bar{x}) \leq 0, i = 1, \dots, k$ and $g_i(\bar{x}) = 0$ for $i = k+1, \dots, m$. Then,

$$\sum_{i=1}^{k} \lambda_i \, g_i(\bar{x}) + \sum_{i=k+1}^{m} \lambda_i \, g_i(\bar{x}) \leq 0.$$

We get

$$L(\bar{x}, \lambda) = f(\bar{x}) + \sum_{i=1}^{k} \lambda_i \, g_i(\bar{x}) + \sum_{i=k+1}^{m} \lambda_i \, g_i(\bar{x}) \leq f(\bar{x})$$

and obtain

$$\mu(\lambda) = \inf_{x \in \mathbb{R}^n} L(x, \lambda) \leq L(\bar{x}, \lambda) \leq f(\bar{x}).$$

□

For every Lagrange multiplier λ, the corresponding value $\mu(\lambda)$ yields a lower bound on p^*. The dual problem maximizes this lower bound:

$$\boxed{\max\{\mu(\lambda) | \, \lambda \in \mathbb{R}^m, \lambda_i \geq 0, i = 1, \dots, k\} \qquad (8.3)}$$

This problem is termed dual problem while the original problem (8.1) is the primal problem.

We say that λ with $\lambda_i \geq 0, i = 1, \dots, k$ is dual feasible, if $\mu(\lambda) > -\infty$. We denote with λ^* the optimal Lagrange multiplier.

Remark 8.3. Problem (8.3) is a convex optimization problem, because the objective is concave and the feasible region is convex. This property is independent on whether or not the primal is convex.

Let d^* be an optimal solution to the dual problem. Thm. 8.2 implies weak duality:

$$d^* \leq p^*.$$

Note that weak duality also holds, if d^* and p^* are infinite. If $p^* = -\infty$, then $d^* = -\infty$ and the dual is infeasible. Otherwise, if $d^* = \infty$, we get $p^* = \infty$, hence the primal problem is infeasible. The difference $p^* - d^*$ is known as duality gap.

8.2 Strong Duality and Saddle Points

If

$$d^* = p^*,$$

we say that strong duality holds.

Theorem 8.4. Let x^* be feasible for (8.1) and λ^* feasible for (8.3). Suppose that strong duality holds, i.e.

$$f(x^*) = \mu(\lambda^*).$$

Then:

1. x^* and λ^* are globally optimal for (8.1) and (8.3).
2. $\lambda_i^* g_i(x^*) = 0$ for all $i = 1, \ldots, m$.

Proof. For 1. let $\bar{x}$ be a global optimal solution of (8.1).

$$f(\bar{x}) \geq \mu(\lambda^*) = f(x^*),$$

where the first inequality follows from (8.2).
For 2. we observe that for all $i > k$, the feasibility of x^* yields $g_i(x^*) = 0$ and therefore $\lambda_i^* g_i(x^*) = 0$ is implied. For $i \leq k$ we consider the inequality:

$$f(x^*) = \mu(\lambda^*) \leq L(x^*, \lambda^*) = f(x^*) + \lambda^* g(x^*).$$

We get $\lambda^* g(x^*) \geq 0$. On the other hand $\lambda^* g(x^*) \leq 0$, and hence we get $\lambda^* g(x^*) = 0$. As for every $i \leq k$ we have $\lambda_i^* \geq 0$ and $g_i(x^*) \leq 0$, the claim follows for all $i \leq k$. ☐

We define a saddle point.

Definition 8.5. We are given a problem of the form (8.1). Let $(\bar{x}, \bar{\lambda})$ satisfy:

1. $\bar{\lambda}_i \geq 0, i = 1, \ldots, k$
2. $L(\bar{x}, \lambda) \leq L(\bar{x}, \bar{\lambda})$ for all $\lambda \in \mathbb{R}^m$ with $\lambda_i \geq 0, i = 1, \ldots, k$.
3. $L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda})$ for all $x \in \mathbb{R}^n$.

Then, $(\bar{x}, \bar{\lambda})$ is called saddle point for (8.1). The conditions are called saddle point conditions.

Theorem 8.6 (saddle point theorem). Let $(\bar{x}, \bar{\lambda})$ be a saddle point for (8.1). Then strong duality holds for $\bar{x}$ and $\bar{\lambda}$.

Proof. Condition 2. implies

$$f(\bar{x}) + \nu g(\bar{x}) \leq f(\bar{x}) + \bar{\lambda} g(\bar{x}) \text{ for all } \nu \in \mathbb{R}^m \text{ with } \nu_i \geq 0, i = 1, \ldots, k.$$

After basic manipulations, we get

$$(\nu - \bar{\lambda})g(\bar{x}) \leq 0 \text{ for all } \nu \in \mathbb{R}^m \text{ with } \nu_i \geq 0, i = 1, \dots, k. \tag{8.4}$$

Inserting $\nu_i = \bar{\lambda}_i$ for $i = 1, \dots, k$, we get

$$\sum_{i=k+1}^{m} (\nu_i - \bar{\lambda}_i)g_i(\bar{x}) \leq 0 \text{ for all } \nu_i \in \mathbb{R}, i = k+1, \dots, m. \tag{8.5}$$

This implies $g_i(\bar{x}) = 0$ for all $i = k+1, \dots, m$. Hence, inequality (8.4) reduces to

$$\sum_{i=1}^{k} (\nu_i - \bar{\lambda}_i)g_i(\bar{x}) \leq 0 \text{ for all } \nu_i \geq 0, i = 1, \dots, k. \tag{8.6}$$

This condition implies $g_i(\bar{x}) \leq 0$ for all $i = 1, \dots, k$. Thus, $\bar{x} \in S$. Moreover, (8.6) yields

$$\bar{\lambda}_i g_i(\bar{x}) = 0, \text{ for all } i = 1, \dots, k.$$

Exercise 8.7. Show that (8.6) implies the following:

1. $g_i(\bar{x}) \leq 0$ for all $i = 1, \dots, k$.
2. $\bar{\lambda}_i g_i(\bar{x}) = 0$, for all $i = 1, \dots, k$.

Altogether, we obtain

$$f(\bar{x}) = f(\bar{x}) + \sum_{i=1}^{k} \bar{\lambda}_i g_i(\bar{x}) = f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) = L(\bar{x}, \bar{\lambda}).$$

The condition 3. yields

$$L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}) \text{ for all } x \in \mathbb{R}^n.$$

Hence,

$$f(\bar{x}) = L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}) \text{ for all } x \in \mathbb{R}^n,$$

which in turn implies

$$f(\bar{x}) \leq \inf_{x \in \mathbb{R}^n} L(x, \bar{\lambda}) = \mu(\bar{\lambda}).$$

With weak duality, we get $f(\bar{x}) = \mu(\bar{\lambda})$. ☐

Exercise 8.8. Show that for any saddle point $(\bar{x}, \bar{\lambda})$ of (8.1) the KKT-conditions of Thm. 5.2 are satisfied.

8.3 Strong Duality for Convex Problems

We consider a convex optimization problem:

$$\min \; f(x)$$

$$\begin{aligned} g_i(x) &\le 0,\ i = 1, \ldots, k \\ Ax &= b, \\ x &\in S, \end{aligned} \tag{8.7}$$

where $S \subset \mathbb{R}^n$ convex, A being a $(m-k) \times n$ matrix of full rank and f and g_i convex for $i = 1, \ldots, k$. We assume that the following regularity conditions are satisfied (Slater): there is $\bar{x} \in \operatorname{int}(S)$ with

$$g_i(\bar{x}) < 0,\ i = 1, \ldots, k,\ A\,\bar{x} = b.$$

> **Theorem 8.9** (Strong Duality). Consider (8.7) and assume that the Slater-regularity conditions are satisfied. Then, $d^* = p^*$.

Proof. Suppose that p^* is finite. As the primal problem is feasible, only the case $p^* = -\infty$ can occur which implies $d^* = -\infty$ by weak duality. We define

$$\begin{aligned} &A_1 \subseteq \mathbb{R}^k \times \mathbb{R}^{m-k} \times \mathbb{R} \\ &A_1 = \{(u, v, t) | \exists x \in S,\ g_i(x) \le u_i,\ i = 1, \ldots, k,\ A_j\, x - b_j = v_j,\ j = 1, \ldots, m-k,\ f(x) \le t\} \end{aligned}$$

Note that A_1 is convex. We define the convex set:

$$A_2 = \{(0, 0, s) \in \mathbb{R}^k \times \mathbb{R}^{m-k} \times \mathbb{R} | s < p^*\}.$$

We get that A_1 and A_2 do not intersect: Assume by contradiction $(u, v, t) \in A_1 \cap A_2$. With $(u, v, t) \in A_2$ we get $u = 0, v = 0$, and $t < p^*$. Because $(u, v, t) \in A_1$, we get x with $g_i(x) \le 0, i = 1, \ldots, k$, $Ax - b = 0$, and $f(x) \le t < p^*$, in contradiction to the optimality of p^*.
With the separating hyperplane theorem we get $(w_1, w_2, w_3) \neq 0$ and $\alpha \in \mathbb{R}$ with

$$w_1^T u + w_2^T v + w_3\, t \ge \alpha \text{ for all } (u, v, t) \in A_1, \tag{8.8}$$

and

$$w_1^T u + w_2^T v + w_3\, t \le \alpha \text{ for all } (u, v, t) \in A_2. \tag{8.9}$$

From (8.8) follows $w_1 \ge 0$ and $w_3 \ge 0$, as otherwise $w_1^T u + w_3\, t$ is unbounded from below on A_1, in contradiction to (8.8). Condition (8.9) implies $w_3\, t \le \alpha$ for all $t \le p^*$ and hence $w_3\, p^* \le \alpha$. Together with (8.8) we get for all $x \in S$ (choose $g(x) = u$ and $Ax - b = v$) that

$$\sum_{i=1}^{k} (w_1)_i\, g_i(x) + w_2^T (Ax - b) + w_3\, f(x) \ge \alpha \ge w_3\, p^*. \tag{8.10}$$

We argue now by a case distinction: If $w_3 > 0$, we divide (8.10) by w_3 and obtain

$$L(x, w_1/w_3, w_2/w_3) \ge p^*,$$

for all $x \in S$. We get $\mu(\lambda, \nu) \ge p^*$, where $\lambda = w_1/w_3$ and $\nu = w_2/w_3$. (Here $\lambda \in \mathbb{R}^k$ and

$\nu \in R^{m-k}$ are the corresponding multipliers for $g(x) \le 0$ and $Ax = b$). With weak duality $\mu(\lambda, \nu) \le p^*$ we get equality.

Now, we consider the case $w_3 = 0$ and lead this to a contradiction. From (8.10), we get for all $x \in S$:

$$\sum_{i=1}^{m} (w_1)_i\, g_i(x) + w_2^T(Ax - b) \ge 0. \tag{8.11}$$

We apply this to the point $\bar{x}$ satisfying the Slater-conditions and get

$$\sum_{i=1}^{k} (w_1)_i\, g_i(\bar{x}) \ge 0.$$

As $g_i(\bar{x}) < 0$ and $w_1 \ge 0$ we get $w_1 = 0$.

From $(w_1, w_2, w_3) \ne 0$ and $w_1 = 0, w_3 = 0$, we get $w_2 \ne 0$. Hence, (8.11) implies $w_2^T(Ax - b) \ge 0$ for all $x \in S$. The point $\bar{x}$ satisfies $w_2^T(A\bar{x} - b) = 0$. With $\bar{x} \in \operatorname{int}(S)$ we get that $\bar{x} \pm \epsilon \in \operatorname{int}(S)$ for $\epsilon \in \mathbb{R}^n$ with $\|\epsilon\|$ small enough. We get $A^T w_2 = 0$, as otherwise there are points in $\operatorname{int}(S)$ with $w_2^T(Ax - b) < 0$, contradiction. Conditions $A^T w_2 = 0$ and $w_2 \ne 0$ contradict the assumption of A having full rank. □

Chapter 9

Pricing in Resource Allocation Games

In this chapter, we give an application of Lagrangian duality theory within a self-contained game-theoretical resource allocation model.

9.1 Model

A resource allocation model is compactly described by a tuple $I = (N, R, X, \pi)$, where $N = \{1, \ldots, n\}$ describes a nonempty finite set of players and $R = \{1, \ldots, m\}$ denotes a nonempty finite set of resources. The set $X := \times_{i \in N} X_i$ describes the combined strategy space of the players where $X_i \subseteq \mathbb{R}^m$ is the nonempty and compact strategy space of player $i \in N$. For $x_i = (x_{ij})_{j \in R} \in X_i$ the entry $x_{ij} \in \mathbb{R}$ can be interpreted as the level of resource usage of player i for resource j.

We call the vector of resource usage $x = (x_{ij}) \in \mathbb{R}^{n \cdot m}$ a strategy profile. Given $x \in X$, we can define the load on resource $j \in R$ as $\ell_j(x) := \sum_{i \in N} x_{ij}$, where x_{ij} is the j-th component of x_i. We denote the set of feasible loads, or the load space, by the Minkowski sum $\ell(X) := \sum_{i \in N} X_i$. We assume that cost functions are parameterized by an exogenously given vector $u \in \mathbb{R}^m$ and depend on the own strategy vector only. In this regard, for $u \in \mathbb{R}^m$, the costs of a player $i \in N$ at the strategy profile $x_i \in X_i$ are given by $\pi_i(u, x_i)$ for some function $\pi_i : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ wich is is assumed to be continuous.

For a model I, a vector $u \in \mathbb{R}^m$ leads to a strategic game $G(u) = (N, X, (\pi_i(u, x_i))_{i \in N})$ and the vector u can be interpreted as the induced load of an equilibrium.

We are concerned with the problem of defining prices $\lambda_j \geq 0, j \in R$ on the resources in order to incentivize an efficient usage of the resources. If player i uses resource j at consumption level x_{ij}, she needs to pay $\lambda_j x_{ij}$. For any vector $u \in \mathbb{R}^m$, the quantities $\pi_i(u, x_i)$ and $\lambda^\intercal x_i$ are assumed to be normalized to represent the same unit (say money in Euro) and we assume that the private cost functions are quasi-linear: $\pi_i(u, x_i) + \lambda^\intercal x_i$. We write $G(u, \lambda) = (N, X, (\pi_i(u, x_i) + \lambda^\intercal x_i)_{i \in N})$ as the resulting strategic game augmented with prices λ. If the parameter $u = (u_j)_{j \in R} \in \mathbb{R}^m$ represents a targeted load vector, then, the task is to find prices $\lambda \in \mathbb{R}^m_{\geq 0}$ so that an equilibrium of the game with prices $G(u, \lambda)$ realizes this load.

Definition 9.1 (Implementability). Let I be a resource allocation model.

1. A vector $u \in \mathbb{R}^m$ is implementable for I, if there is a tuple $(x^*, \lambda^*) \in X \times \mathbb{R}^m_{\geq 0}$ such that the following two conditions are satisfied:
    (a) $\ell_j(x^*) = u_j$ for all $j \in R$.
    (b) $x_i^* \in \arg\min_{x_i \in X_i} \{\pi_i(u, x_i) + (\lambda^*)^\intercal x_i\}$ for all $i \in N$.
2. A vector $u \in \mathbb{R}^m$ is weakly implementable for I, if there is a tuple (x^*, λ^*) that satisfies (1b) but (1a) is replaced with $\ell(x^*) \leq u$ and λ^* satisfies $\ell_j(x^*) < u_j \Rightarrow \lambda_j^* = 0$ for all $j \in R$.

If the model I is clear from the context, we say that $u \in \mathbb{R}^m$ is (weakly) implementable.

Condition (1a) requires that x^* realizes the targeted load $\ell(x^*) = u$ while Condition (1b) implements x^* as a pure Nash equilibrium of the game augmented with prices $G(u, \lambda^*)$. Let us consider the following example:

Example 9.2 (Market Equilibria). Suppose there are items $R = \{1, \dots, m\}$ for sale and there is a set $N = \{1, \dots, n\}$ of potential buyers. For every subset $S \subseteq R$ of items, player i experiences value $w_i(S) \in \mathbb{R}$ giving rise to a valuation function $w_i : 2^m \to \mathbb{R}, i \in N$. The market manager wants to determine a price vector $\lambda^* \in \mathbb{R}^m_{\geq 0}$ for selling the items such that every player receives a subset $S_i^* \subseteq R$ maximizing her quasi-linear utility $S_i^* \in \arg\max_{S_i \subseteq R} \{w_i(S_i) - \sum_{j \in S_i} \lambda_j^*\}$ and unsold items have prices equal to zero. The tuple $((S_i^*)_{i \in N}, \lambda^*)$ is known as a Walrasian competitive equilibrium.
This model fits into the framework by defining $I = (N, R, X, \pi)$ with $X_i = \{0,1\}^m, i \in N$ representing the set of incidence vectors of R. The cost function $\pi_i := -v_i$ is given by the negative valuation function $v_i : \mathbb{R}^m \times X_i \to \mathbb{R}, (u, x_i) \mapsto w_i(R(x_i))$ for any $u \in \mathbb{R}^m$. One can easily verify that any strategy profile which weakly enforces the load $1 \in \mathbb{R}^m$ corresponds to a Walrasian competitive equilibrium and vice versa.

9.2 Connection to Lagrangean Duality in Optimization

Let us fix a model I. For $u \in \mathbb{R}^m$, we define the following minimization problem that we call master problem:

$$\begin{aligned} \min_x \; & \pi(u, x) && \text{(P(u))} \\ \text{s.t.: } & \ell(x) \leq u, && (9.1) \\ & x \in X, \end{aligned}$$

where the objective function is defined as $\pi(u, x) := \sum_{i \in N} \pi_i(u, x_i)$.

The Lagrangian function for problem $P(u)$ becomes $L(x, \lambda) := \pi(u, x) + \lambda^\intercal(\ell(x) - u)$, $\lambda \in \mathbb{R}^m_{\geq 0}$, and we can define the Lagrangian-dual as: $\mu : \mathbb{R}^m_{\geq 0} \to \mathbb{R}$, $\mu(\lambda) = \min_{x \in X} L(x, \lambda) =$

$\min_{x \in X}\{\pi(u,x) + \lambda^\intercal(\ell(x) - u)\}$. The dual problem is defined as:

$$\max_{\lambda \geq 0} \mu(\lambda) \tag{9.2}$$

Definition 9.3. Problem $P(u)$ has zero-duality gap, if there is $\lambda^* \in \mathbb{R}^m_{\geq 0}$ and $x^* \in X$ with $\pi(u,x^*) = \mu(\lambda^*)$. In this case, we say that the pair (x^*, λ^*) is primal-dual optimal.

If problem $P(u)$ has zero-duality gap, the two solutions $\lambda^* \in \mathbb{R}^m_{\geq 0}$ and $x^* \in X$ are optimal for their respective problems (9.2) and $P(u)$ and infima/suprema in the definition of μ become a minimum/maximum.

In the following, we prove a key structure, namely that $\min_{x \in X} L(x, \lambda)$ decomposes into independent problems, one for each player.

Lemma 9.4. Let $\lambda \in \mathbb{R}^m_{\geq 0}$. For a problem of type $P(u)$, the following holds true:

$$x^* \in \arg\min_{x \in X} L(x,\lambda) \Leftrightarrow x_i^* \in \arg\min_{x_i \in X_i}\{\pi_i(u,x_i) + \lambda^\intercal x_i\} \text{ for all } i \in N. \tag{9.3}$$

Proof. We calculate:

$$\begin{aligned} \min_{x \in X} L(x,\lambda) &= \min_{x_i \in X_i, i \in N} \left\{ \left[\sum_{i \in N} (\pi_i(u,x_i) + \lambda^\intercal x_i \right] - \lambda^\intercal u \right\} \\ &= \sum_{i \in N} \min_{x_i \in X_i} \{\pi_i(u,x_i) + \lambda^\intercal x_i\} - \lambda^\intercal u, \end{aligned}$$

where the first equality follows by the linearity of $\ell(x)$ w.r.t. $x_i, i \in N$ and the last equality by the assumption that $\pi_i(u,x_i)$ only depends on $x_i \in X_i$ and the fact that $\lambda^\intercal u$ does not depend at all on x. ☐

We obtain the following main result concerning the Implementability of a load u.

Theorem 9.5. The following equivalences hold for I.

1. A vector $u \in \mathbb{R}^m$ is implementable via (x^*, λ^*) if and only if (x^*, λ^*) has zero duality gap for $P(u)$ and x^* satisfies (9.1) with equality.
2. A vector $u \in \mathbb{R}^m$ is weakly implementable via (x^*, λ^*) if and only if (x^*, λ^*) has zero duality gap for $P(u)$.

Proof. For the proof we only show 2., since 1. follows from 2. as the additional condition $\ell(x^*) = u$ holds true for both statements of 1.

For 2.: $\Leftarrow$: Assume there are $\lambda^* \in \mathbb{R}^m_{\geq 0}, x^* \in X$ with $\ell(x^*) \leq u$ so that $\mu(\lambda^*) = \pi(u,x^*)$. We obtain

$$\mu(\lambda^*) = \min_{x \in X}\{\pi(u,x) + (\lambda^*)^\intercal(\ell(x) - u)\} \leq \pi(u,x^*) + (\lambda^*)^\intercal(\ell(x^*) - u)$$

$$\leq \pi(u,x^*) = \mu(\lambda^*).$$

Hence, all inequalities must be tight leading to $(\lambda^*)^\intercal(\ell(x^*)-u)=0$ as claimed. It remains to prove Condition (1b). With $x^* \in \arg\min_{x\in X} L(x,\lambda^*)$ we get

$$x^* \in \arg\min_{x\in X} L(x,\lambda^*) \underset{\text{Lem. 9.4}}{\Leftrightarrow} x_i^* \in \arg\min_{x_i\in X_i}\{\pi_i(u,x_i) + \sum_{j\in R}\lambda_j^* x_{ij}\} \text{ for all } i\in N.$$

$\Rightarrow$: Let $u \in \mathbb{R}^m$ be weakly implementable via $(x^*,\lambda^*) \in X \times \mathbb{R}^m_{\geq 0}$, that is, $\ell(x^*) \leq u$, $(\lambda^*)^\intercal(\ell(x^*)-u) = 0$ and $x_i^* \in \arg\min_{x_i\in X_i}\{\pi_i(u,x_i) + (\lambda^*)^\intercal x_i\}$ for all $i \in N$. We calculate

$$\begin{aligned}
\mu(\lambda^*) &= \min_{x\in X}\{\pi(u,x) + (\lambda^*)^\intercal(\ell(x)-u)\} \\
&= \pi(u,x^*) + (\lambda^*)^\intercal(\ell(x^*)-u) && (9.4)\\
&= \pi(u,x^*), && (9.5)
\end{aligned}$$

where (9.4) follows from Lemma 9.4 and (9.5) uses the condition $(\lambda^*)^\intercal(\ell(x^*)-u) = 0$. Hence, strong duality holds for the pair (x^*,λ^*). ☐

The theorem does not rely on any further assumption on the functions $\pi_i(u,x_i), i \in N$ nor on the feasible sets X_i but only on the duality gap of $P(u)$. This is in particular interesting as several classes of optimization problems are known to have zero duality gap even without convexity of feasible sets and objective functions.

In cost minimization games, the feasible sets X_i usually contain some sort of covering conditions on the resource consumption. For example in network routing, one needs to send some prescribed amount of flow.

In this regard, we introduce a natural candidate set of vectors u for which we know that any feasible solution satisfying (9.1) does so with equality.

Definition 9.6. A vector $u \in \mathbb{R}^m$ is called minimal w.r.t. $\ell(X)$, if

$$\{\tilde{u} \in \mathbb{R}^m : \tilde{u} \leq u\} \cap \ell(X) = \{u\}$$

Exercise 9.7. A vector $u \in \mathbb{R}^m$ is minimal w.r.t. $\ell(X)$, if and only if there are strictly increasing functions $h_j : \mathbb{R} \to \mathbb{R}, j \in R$ such that

$$u \in \arg\min_{\tilde{u}\in\ell(X)} \sum_{j\in R} h_j(\tilde{u}_j).$$

Corollary 9.8. Let $u \in \mathbb{R}^m$ be minimal w.r.t. $\ell(X)$. Then, the following two statements are equivalent:

1. u is implementable via price vector $\lambda^* \in \mathbb{R}^m_{\geq 0}$ and $x^* \in X$.
2. (x^*,λ^*) satisfies $\pi(u,x^*) = \mu(\lambda^*)$.

9.3 Convexified Games

So far, the strategy spaces $X_i, i \in N$ and the cost functions $\pi_i(u, \cdot), i \in N$ of a model I were not restricted and are allowed to be non-convex. For instance integrality restrictions in $X_i \subseteq \mathbb{Z}^m, i \in N$ are allowed. In what follows, we connect I with a related convexified resource allocation model I^{co}.

For $Z \subseteq \mathbb{R}^m$ denote $co(Z) := \cap\{K \subseteq \mathbb{R}^m | Z \subseteq K, K \text{ convex}\}$ the convex hull of Z.

Definition 9.9. For a resource allocation model $I = (N, R, X, \pi)$ and load u, the associated convexified model I^{co} is defined as

$$I^{co} = (N, R, X^{co}, (\phi_i)_{i \in N}),$$

where $X^{co} := \times_{i \in N} co(X_i)$ and for all $u \in \mathbb{R}^m$:

$$\begin{aligned} \phi_i(u, \cdot) : co(X_i) &\to \mathbb{R} \cup \{-\infty\} \\ \bar{x}_i &\mapsto \min_{\alpha_{ik}, x_i^k} \left\{ \sum_{k=1}^{m+1} \alpha_{ik} \pi_i(u, x_i^k) \;\middle|\; \begin{array}{l} \sum_{k=1}^{m+1} \alpha_{ik} x_i^k = \bar{x}_i, \alpha_i \in \Lambda, \\ x_i^k \in X_i, 1 \le k \le m+1 \end{array} \right\} \end{aligned} \tag{9.6}$$

in which $\Lambda := \{\alpha \in \mathbb{R}^{m+1}_{\ge 0} | 1^\intercal \alpha = 1\}$. Note that ϕ_i is constant in $\tilde{u} \in \mathbb{R}^m$.

Theorem 9.10. Let $x^* \in X$, $\lambda^* \in \mathbb{R}^m_{\ge 0}$. The following statements are equivalent.

1. u is implementable for I via (x^*, λ^*).
2. $\phi_i, i \in N$ are real-valued functions, u is implementable for I^{co} via (x^*, λ^*) and $\phi(u, x^*) = \pi(u, x^*)$ holds.

The equivalence remains true by replacing the term “implementable” with “weakly implementable”.

Proof. We first derive another description of the dual for the master problem $P(u)$ of I. We get for all $\lambda \in \mathbb{R}^m_{\ge 0}$:

$$\begin{aligned} \mu(\lambda) &= \min_{x_i \in X_i, i \in N} \sum_{i \in N} \pi_i(u, x_i) + \lambda^\intercal(\ell(x) - u) \\ &= \sum_{i \in N} \min_{\alpha_{ik}, x_i^k} \left\{ \sum_{k=1}^{m+1} \alpha_{ik} \left[\pi_i(u, x_i^k) + \lambda^\intercal x_i^k\right] \;\middle|\; \begin{array}{l} \alpha_i \in \Lambda, x_i^k \in X_i, \\ 1 \le k \le m+1 \end{array} \right\} - \lambda^\intercal u \qquad (9.7) \\ &= \min_{\bar{x} \in X^{co}} \sum_{i \in N} \phi_i(u, \bar{x}_i) + \lambda^\intercal(\ell(\bar{x}) - u) \qquad (9.8) \end{aligned}$$

where (9.7) follows by the linearity of $\alpha_i \mapsto \sum_{k=1}^{m+1} \alpha_{ik} \left[\pi_i(u, x_i^k) + \lambda^\intercal x_i^k\right]$. Equation (9.8) follows as $\sum_{k=1}^{m+1} \alpha_{ik} \lambda^\intercal x_i^k = \lambda^\intercal \sum_{k=1}^{m+1} \alpha_{ik} x_i^k$ holds.

Now we are ready to prove 1.⇔2..

1.⇒2.: By Theorem 9.5, we have $\mu(\lambda^*) = \pi(u, x^*) > -\infty$ and thus, $\phi_i, i \in N$ need to be real-valued by the above description of μ. Subsequently I^{co} belongs to the resource allocation model defined in Section 9.1. The dual $\mu^{co}(\lambda)$ of the master problem $P(u)$ of I^{co} is then given by the expression in (9.8). We get $\pi(u, x^*) = \mu(\lambda^*) = \mu^{co}(\lambda^*) \leq \phi(u, x^*)$, where the last inequality follows from weak duality. Since $\phi(u, x) \leq \pi(u, x)$ for all $x \in X$, we get $\mu^{co}(\lambda^*) = \phi(u, x^*)$ and $\phi(u, x^*) = \pi(u, x^*)$. Thus the result follows by Theorem 9.5 and the fact that the load of x^* in I equals the load of x^* in I^{co}.

2.⇒1.: As $\phi_i, i \in N$ are real-valued, I^{co} belongs to the resource allocation model defined in Section 9.1. Thus, the result follows by Theorem 9.5 together with $x^* \in X$, $\pi(u, x^*) = \phi(u, x^*)$, $\mu^{co}(\lambda^*) = \mu(\lambda^*)$ and using that the respective induced loads of x^* in I and x^* in I^{co} coincide. ☐

9.4 LP-based characterizations of Implementability

We now discuss a special class of models I, which allow for an LP-based characterization of Implementability. The main property needed is a special structure of the Lagrangian-dual function of the master problem $P(u)$.

Assumption 9.11. For every $i \in N$, there exist $\{x_i^1, \dots, x_i^{k_i}\} \subseteq X_i$ for some $k_i \in \mathbb{N}$ such that the dual of $P(u)$ may be represented as follows:

$$\mu(\lambda) = \min_{x_i \in \{x_i^1, \dots, x_i^{k_i}\}, i \in N} \sum_{i \in N} \pi_i(u, x_i) + \lambda^\intercal (\ell(x) - u)$$

Clearly, this assumption is fulfilled in the important case of finite models where the strategy sets are finite point sets (see Fig. 9.1 left). We will show in Corollary 9.14 that this assumption also holds for concave models, that are, models where the convex hull of each $X_i, i \in N$ is finitely generated and the functions $\pi_i(u, x_i), i \in N$ are concave on $co(X_i)$ (see Fig. 9.1 right).

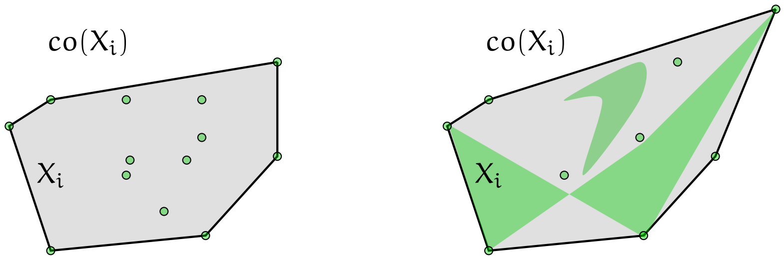


Figure 9.1: Left is the scenario of X_i consisting of a finite point set. Right, X_i may consist of connected components (in green) and isolated points but the convex hull is assumed to be finitely generated and additionally $\pi_i(u, x_i)$ are assumed to be concave on $co(X_i)$.

For a model I fulfilling Assumption 9.11, we define the following LP in the variable $\alpha =$

$(\alpha_i)_{i\in N}$.

$$\begin{aligned} \min_{\alpha} \sum_{i\in N} \pi_i^\intercal \alpha_i & & \text{(LP(u))} \\ \text{s.t.: } \ell(\alpha) \leq u, & & (9.9) \\ \alpha_i \in \Lambda_i, i \in N, & & \end{aligned}$$

where $\pi_i := (\pi_i(u, x_i^k))_{k\in\{1,\dots,k_i\}}$, $\ell(\alpha) := \sum_{i\in N}\sum_{k\in\{1,\dots,k_i\}} \alpha_{ik} x_i^k$ and $\Lambda_i := \{\alpha_i \in \mathbb{R}^{k_i}_{\geq 0} | 1^\intercal \alpha_i = 1\}, i \in N$.

Theorem 9.12. Let I be a model for which Assumptions 9.11 holds. Then, the following statements are equivalent.

1. The vector $u \in \mathbb{R}^m$ is implementable for I.
2. There exists $x^* \in X$ with $\ell(x^*) = u$ and an optimal solution α^* of LP(u) such that $\pi(u, x^*) = \sum_{i\in N} \pi_i^\intercal \alpha_i^*$.

The equivalence remains true by replacing the term “implementable” with “weakly implementable” and replacing the condition $\ell(x^*) = u$ in Statement 2. by $\ell(x^*) \leq u$.

Proof. We first show that the duals of LP(u) and P(u) coincide. We get

$$\begin{aligned} \mu^{\mathrm{LP}}(\lambda) &:= \min_{\alpha_i\in\Lambda_i, i\in N} \sum_{i\in N} \pi_i^\intercal \alpha_i + \lambda^\intercal(\ell(\alpha) - u) \\ &= \min_{x_i\in\{x_i^1,\dots,x_i^{k_i}\}, i\in N} \sum_{i\in N} \pi_i(u, x_i) + \lambda^\intercal(\ell(x) - u) && (9.10) \\ &= \mu(\lambda), && (9.11) \end{aligned}$$

where (9.10) follows by the linearity of the objective function w.r.t. α_i, $i \in N$ and equation (9.11) by Assumption 9.11.
Let us denote by $\pi^*, (\pi^{\mathrm{LP}})^*, \mu^*, (\mu^{\mathrm{LP}})^*$ the respective optimal values of P(u), LP(u) and their duals. By weak duality of P(u) and strong duality of LP(u), we get

$$\pi^* \geq \mu^* = (\mu^{\mathrm{LP}})^* = (\pi^{\mathrm{LP}})^*.$$

Now for 1.⇒ 2., we have $\pi^* = \pi(u, x^*) = \mu(\lambda^*) = \mu^*$ for one $(x^*, \lambda^*) \in X \times \mathbb{R}^m_{\geq 0}$ with $\ell(x^*) = u$ by Theorem 9.5. Thus, $\pi^* = (\pi^{\mathrm{LP}})^*$ and there is an optimal solution α^* of LP(u) such that $\pi(u, x^*) = \sum_{i\in N} \pi_i^\intercal \alpha_i^*$. For the converse 2.⇒ 1., we observe that by $\pi(u, x^*) = \sum_{i\in N} \pi_i^\intercal \alpha_i^* = (\mu^{\mathrm{LP}})^* = \mu^*$, there exists one $\lambda^* \in \mathbb{R}^m_{\geq 0}$ with $\pi(u, x^*) = \mu^* = \mu(\lambda^*)$. Therefore, the statement follows by Theorem 9.5 and the assumption that $\ell(x^*) = u$ holds. □

In the following, we describe the consequences of Theorem 9.12 for the important cases of finite and concave models:

Definition 9.13. We call a model I

1. finite, if $X_i = \{x_i^1, \dots, x_i^{k_i}\}$ for some $k_i \in \mathbb{N}$ and all $i \in N$.
2. concave, if for all $i \in N$, there exist $\{x_i^1, \dots, x_i^{k_i}\} \subseteq X_i$ for some $k_i \in \mathbb{N}$ such that $\operatorname{co}(X_i) = \operatorname{co}\big(\{x_i^1, \dots, x_i^{k_i}\}\big)$. Furthermore, $\pi_i(u, \cdot), i \in N$ can be extended to the domain $\operatorname{co}(X_i)$ so that they are concave on $\operatorname{co}(X_i)$.

Corollary 9.14. For a finite model I, the following statements are equivalent.

1. The vector $u \in \mathbb{R}^m$ is implementable for I.
2. $LP(u)$ admits an integral optimal solution $\tilde{\alpha}$ for which (9.9) is tight.

For a concave model I, the following statements are equivalent.

3. The vector $u \in \mathbb{R}^m$ is implementable for I.
4. $LP(u)$ admits an optimal solution $\tilde{\alpha}$ with $x_i^{\tilde{\alpha}} := \sum_{j=1}^{k_i} \tilde{\alpha}_{ij} x_i^j \in X_i, i \in N$ so that $\ell(x^{\tilde{\alpha}}) = u$ and $\pi(u, x^{\tilde{\alpha}}) = \sum_{i \in N} \pi_i^T \tilde{\alpha}_i$.

The equivalences remain true by replacing the term “implementable” with “weakly implementable” and removing the condition that (9.9) needs to be tight in Statement 2. and replacing $\ell(x^{\tilde{\alpha}}) = u$ in Statement 4. by $\ell(x^{\tilde{\alpha}}) \leq u$.

The proof and further complexity theoretic consequences can be found in Harks and Schwarz (SIAM J. Opt. 2023).

Chapter 10

Numerical Methods

We first recap the definition of the matrix spectral norm:

$$\|A\| := \max\left\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\right\}$$

This norm can be characterized by

$$\|A\| = \sqrt{\lambda_{\max}(A^T A)},$$

where $\lambda_{\max}(A^T A)$ denotes the largest eigenvalue of the symmetric and positive semidefinite matrix $A^T A$. In particular, for a symmetric matrix $A \in \mathbb{R}^{n\times n}$ we obtain

$$\|A\| = |\lambda_{\max}(A)|.$$

10.1 Unconstrained Optimization Problems

We first consider

$$\min\{f(x) | \, x \in \mathbb{R}^n\} \tag{10.1}$$

without constraints, where $f \in C^2$ on $\mathbb{R}^n$.

Definition 10.1. $v \in \mathbb{R}^n$ is called descent direction of f in $\bar{x} \in \mathbb{R}^n$, if there is $\bar{t} > 0$ with

$$f(\bar{x} + tv) < f(\bar{x}) \text{ for all } t \in (0, \bar{t}).$$

We obtain the following Lemma.

Lemma 10.2. If $f'(\bar{x})v < 0$, then v is a descent direction of f in $\bar{x}$.

Proof. Definie $\Phi(t) := f(\bar{x} + tv)$. We get $\dot{\phi}(0) = f'(\bar{x})v < 0$ and the claim follows. □

Remark 10.3.

- The condition $f'(\bar{x})v < 0$ means that the angle ϕ between v and $-f'(\bar{x})$ in $\bar{x}$ is less than $\pi/2$ (or 90^o). Consider:

$$0 > f'(\bar{x})v \Rightarrow 0 < -f'(\bar{x})v = \cos(\phi) \left\|-f'(\bar{x})\right\| \|v\|.$$

We get $\cos(\phi) > 0$ and hence $\phi \in [0, \pi/2)$.

- The criterion $f'(\bar{x})v < 0$ is not necessary. If $\bar{x}$ is a strict local maximum, then all $v \in \mathbb{R}^n$ are directions of descent for f in $\bar{x}$, but $f'(\bar{x})v < 0$ need not be satisfied.

Exercise 10.4. Let $B \in \mathbb{R}^{n\times n}$ be symmetric and positive definit. Then, $v = -B\, f'(\bar{x})$ is a descent direction of f in $\bar{x}$, if $f'(\bar{x}) \neq 0$.

We describe a descent method to compute $\bar{x}$ with $f'(\bar{x}) = 0$.

1. Choose $x^0 \in \mathbb{R}^n$, $k = 0$ and fix $\epsilon_0 > 0$;
2. If $\left\|f'(x^k)\right\| \leq \epsilon_0$:STOP (Termination);
3. Compute a descent direction v^k with $f'(x^k)v^k < 0$;
4. Compute a step-length t_k with $f(x^k + t_k v^k) < f(x^k)$;
5. Set $x^{k+1} \leftarrow x^k + t_k v^k$; $k \leftarrow k + 1$ and go to step 2.

Figure 10.1: Generic Method of Descent.

Definition 10.5 (Gradient-Method, Newton-Method, Quasi-Newton-Method).

1. For
$$v^k := -f'(x^k)^T$$
we obtain the Gradient-Method.
2. For the Newton-Method, we choose:
$$v^k := -f''(x^k)^{-1} f'(x^k)^T.$$
In every point x^k in which the Hesse-matrix $f''(x^k)$ is positive definite, the vector v^k is a descent direction (assuming $f'(x^k) \neq 0$).
3. The Quasi-Newton-Method chooses
$$v^k := -H_k^{-1} f'(x^k)^T,$$
for a suitable positive definite matrix H_k.

Theorem 10.6. If $H \in \mathbb{R}^{n\times n}$ is symmetric, positive definite and $f'(x) \neq 0$, then the gradient direction

$$v := \frac{H^{-1}f'(x)^T}{\|H^{-1}f'(x)^T\|_H}$$

maximizes the descent of $f'(x)v$ over all $v \in \mathbb{R}^n$ with $\|v\|_H = 1$, where $\|x\|_H := \sqrt{x^T Hx}$.

Proof. We prove the theorem only for $H = I$. With Cauchy-Schwarz-inequality we get using $\|v\| = 1$:

$$|f'(x)v| \leq \left\|f'(x)\right\| \|v\| = \left\|f'(x)\right\|.$$

This bound is attained for $v := \pm\frac{f'(x)^T}{\|f'(x)^T\|}$. ☐

We go back to the angle-condition discussed before.

Definition 10.7 (Angle-Condition). A generic descent method of the form (10.1) satisfies the angle-condition, if:

$$\text{there is } c > 0, \text{ such that for all } k \in \mathbb{N} \text{ we have: } c_k := -\frac{f'(x^k)v^k}{\|f'(x^k)\| \, \|v^k\|} \geq c. \tag{10.2}$$

A weaker condition is the Zoutendijk-Condition, which only requires $\sum_{k=0}^{\infty} c_k = \infty$.

Choice of the Step-Length

We discuss the degree of freedom regarding the step-length choice in 10.1. Let x^0 be the initial point and assume that $N(f, f(x^0)) = \{x \in \mathbb{R}^n | f(x) \leq f(x^0)\}$ is compact. With the continuity of $f''(x)$ on $N(f, f(x^0))$, there is $C > 0$ with

$$\left\|f''(x)\right\| \leq C \text{ for all } x \in N(f, f(x^0)).$$

We apply Taylor expansion of f in x in direction of $tv, v \in \mathbb{R}^n$:

$$\begin{aligned} f(x + tv) &= f(x) + tf'(x)v + \frac{t^2}{2}v^T f''(z)v \\ &\leq f(x) + tf'(x)v + \frac{t^2}{2}C\|v\|^2. \end{aligned} \tag{10.3}$$

Here $z = x + \xi tv$ is an intermediate point with $0 < \xi < 1$. The bound (10.3) is valid for all $t > 0$ with $x + [0, t]v \subset N(f, f(x^0))$.
The last term of (10.3) is a polynomial $p(t)$ of degree two in t and attains at

$$t^* = -\frac{f'(x)v}{C\|v\|^2} > 0$$

its strict global minimum. Let $\bar{t}$ be the unique maximal step-length with

$$x + tv \in N(f, f(x^0)) \text{ for all } t \in [0, \bar{t}].$$

We get

$$p(\bar{t}) \geq f(x+\bar{t}v) \geq f(x) = p(0).$$

Using $p'(0) = f'(x)v < 0$ we get that t^* lies in $(0,\bar{t})$ and thus $x + t^*v \in N(f, f(x^0))$. With (10.3) we get

$$f(x+t^*v) \leq p(t^*) = f(x) - \frac{1}{2C}\left(\frac{f'(x)v}{\|v\|}\right)^2.$$

This bounds the minimum descent. Since C is not known a priori, we define the following.

Definition 10.8. A step-length strategy $t(x,v)$ is called efficient, if for every $x^0 \in \mathbb{R}^n$, there is $\xi > 0$ with

$$f(x + t(x,v)v) \leq f(x) - \xi\left(\frac{f'(x)v}{\|v\|}\right)^2 \quad \text{for all } x \in N(f, f(x^0)),$$

and v is a descent direction of f in x with $f'(x)v < 0$.

Under the assumptions we get that $t := \arg\min\{f(x+tv) | t > 0\}$ is efficient.

We obtain the following theorem on the general descent method (10.1) with efficient step-length strategies.

Theorem 10.9. Let $f \in C^2$ and let (10.1) with $\epsilon_0 = 0$ satisfy the condition (10.2). Suppose we choose an efficient step-length strategy. Then, one of the following statements is true:

1. After finitely many iterations we have $f'(x^k) = 0$.
2. $\lim_{k\to\infty} f(x^k) = -\infty$
3. $\lim_{k\to\infty} f'(x^k) = 0$, i.e., every accumulation point of $x^k, k \in \mathbb{N}$ is a zero of $f'(x)$.

Proof. If 10.1 terminates after finitely many iterations, we get using $\epsilon_0 = 0$ the condition $f'(x^k) = 0$.

So suppose that this does not hold. Using the angle- and efficiency condition we get for iteration k:

$$f(x^{k+1}) - f(x^k) \leq -\xi\left(\frac{f'(x^k)v^k}{\|v^k\|}\right)^2 = -\xi c_k^2 \left\|f'(x^k)\right\|^2.$$

After $N \in \mathbb{N}$ iterations we get

$$f(x^N) - f(x^0) = \sum_{k=0}^{N-1} f(x^{k+1}) - f(x^k) \leq -\xi \sum_{k=0}^{N-1} c_k^2 \left\|f'(x^k)\right\|^2.$$

We divide by $-\xi < 0$ and get

$$-\frac{f(x^N) - f(x^0)}{\xi} \geq \sum_{k=0}^{N-1} c_k^2 \left\| f'(x^k) \right\|^2.$$

If f is bounded from below we get

$$\lim_{N \to \infty} f(x^N) > -\infty$$

and therefore

$$\lim_{N \to \infty} -\frac{f(x^N) - f(x^0)}{\xi} < \infty.$$

Hence we get

$$\sum_{k=0}^{\infty} c_k^2 \left\| f'(x^k) \right\|^2 < \infty.$$

Using the angle- and efficiency condition, we get $\left\| f'(x^k) \right\| \to 0$.

□

We provide two additional step-length methods, the Armijo-Rule and the Goldstein-Rule.

Definition 10.10 (Armijo-Rule). For $\sigma \in (0,1), \alpha \in (0,1)$ we choose $t := \alpha^\ell$ with

$$\ell := \min\{j \in \mathbb{N}_0 | f(x + \alpha^j v) \leq f(x) + \sigma \alpha^j (f'(x)v)\}.$$

As for the interpretation of the Armijo-Rule, we define for

$$\Phi(t) = f(x + tv), t \geq 0$$

the auxiliary function

$$\Psi(t) = \Phi(t) - (f(x) + \sigma t(f'(x)v)).$$

Per construction, we get $\Psi(0) = 0$ and

$$\begin{aligned} \Psi'(0) &= \Phi'(0) - \sigma f'(x)v \\ &= f'(x)v - \sigma f'(x)v = (1-\sigma) f'(x)v < 0. \end{aligned}$$

Assuming $N(f, f(x^0))$ to be compact, we know that $\Phi(t)$ grows for t large enough. Hence, there is a unique $\ell \in \mathbb{N}_0$ and thus a maximal $t = \alpha^\ell > 0$ satisfying the Armijo-Condition. This t is usually computed via enumeration $\ell = 1, 2, \dots$.

Remark 10.11.

- The Armijo-Method may not be efficient in general.
- The scaled Armijo-step-length works with a scaling factor $s > 0$ and is defined as

$$\ell := \min\{j \in \mathbb{N}_0 | f(x + s\alpha_j v) \leq f(x) + \sigma s \alpha_j (f'(x)v)\}.$$

For large enough s, the Armijo-Variant is efficient.

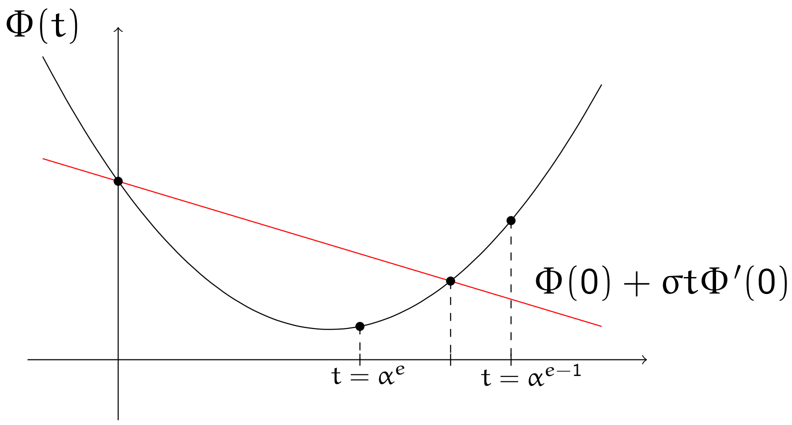


Figure 10.2: Armijo-step-length strategy.

Definition 10.12 (Goldstein-Rule). In order to avoid small step-length, we bound the feasible space from below. The step-length $t > 0$ satisfies the Goldstein-Condition, if for fixed $\sigma \in (0, 0.5)$, we have

$$\Phi_u(t) \le \Phi(t) \le \Phi_o(t), \tag{10.4}$$

where $\Phi_u(t)$ and $\Phi_o(t)$ are defined as follows:

$$\Phi_o(t) := \Phi(0) + \sigma t \Phi'(0), \Phi_u(t) := \Phi(0) + (1-\sigma) t \Phi'(0). \tag{10.5}$$

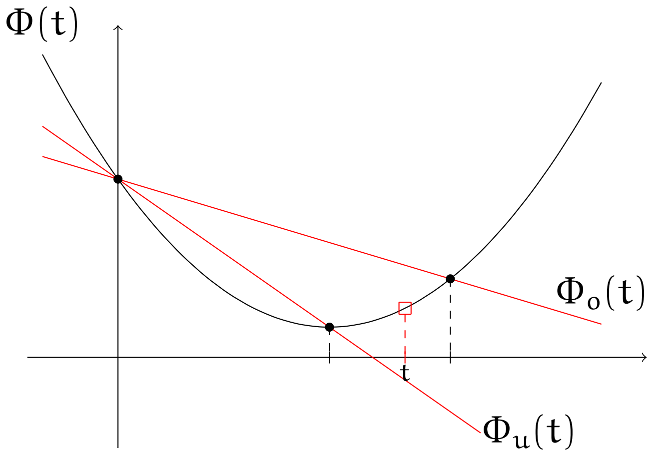


Figure 10.3: Goldstein-step-length strategy.

Any step-length method satisfying the Goldstein-Rule is efficient. We provide a concrete implementation.

```
1. Set $t_u := 0$, $t_o > 0$ (arbitrary);
2. If $\Phi(t_o) < \Phi_u(t_o)$, set $t_u := t_o, t_o := 2t_o$.
   Repeat until $\Phi(t_o) \geq \Phi_u(t_o)$;
3. If $\Phi(t_o) \leq \Phi_o(t_o)$, set $t := t_o$, Stop.;
4. Repeat: set $t := (t_u + t_o)/2$
   If $\Phi(t) < \Phi_u(t)$, set $t_u := t$;
   If $\Phi(t) > \Phi_o(t)$, set $t_o := t$;
   Until: $\Phi_u(t) \leq \Phi(t) \leq \Phi_o(t)$, Stop.
```

Figure 10.4: Step-length choice after Goldstein.

Theorem 10.13. Let $f \in C^2$ and $x \in \mathbb{R}^n$ with $N(f, (f(x))$ compact. Let $v \in \mathbb{R}^n$ be a descent direction with $f'(x)v < 0$ and

$$C := \max \left\{ \left\| f''(y) \right\|^2 \mid y \in N(f, (f(x)) \right\}.$$

Then, every step-length t which satisfies the Goldstein-Rule also satisfies the following efficiency condition:

$$f(x + tv) \leq f(x) - \frac{\sigma}{C} \left(\frac{f'(x)v}{\|v\|} \right)^2 .$$

Proof. Let t^* be the infimum of the positive local minimizers of Φ. The value t^* is then a stationary point and Φ is strictly decreasing in $[0, t^*]$.

Case 1: $t \leq t^*$
With $\Phi_u(t) := \Phi(0) + (1 - \sigma)t\Phi'(0) \leq \Phi(t)$ we get using Taylor expansion

$$(1 - \sigma)t\Phi'(0) \leq \Phi(t) - \Phi(0) = t\Phi'(0) + \frac{t^2}{2}\Phi''(\tilde{t}) \leq t\Phi'(0) + C \|v\|^2$$

and therefore

$$-\sigma t\Phi'(0) \leq \frac{t^2}{2} C \|v\|^2 ,$$

or equivalently

$$t \geq -\frac{2\sigma}{C} \frac{\Phi'(0)}{\|v\|^2} := \hat{t} > 0.$$

Using monotonicity of Φ on $0 < \hat{t} \leq t \leq t^*$, we get

$$\begin{aligned}\Phi(t) \leq \Phi(\hat{t}) &\leq \Phi(0) + \hat{t}\Phi'(0) + \frac{\hat{t}^2}{2} C \|v\|^2 \\ &= \Phi(0) - \frac{2\sigma(1-\sigma)}{C}\left(\frac{\Phi'(0)}{\|v\|^2}\right)^2 \\ &\leq \Phi(0) - \frac{\sigma}{C}\left(\frac{\Phi'(0)}{\|v\|^2}\right)^2\end{aligned}$$

The last inequality uses $\sigma \in (0, 0.5)$.
Case 2: $t > t^*$

We get

$$0 = \Phi'(t^*) = \Phi'(0) + t^*\Phi''(\tilde{t}) \leq \Phi'(0) + C\|v\|^2$$

and thus $t^* \geq -\frac{\Phi'(0)}{C\|v\|^2}$. We obtain

$$\begin{aligned}\Phi(t) &\leq \Phi_o(t) = \Phi(0) + \sigma t \Phi'(0) \\ &\leq \Phi(0) + \sigma t^* \Phi'(0) \leq \Phi(0) - \frac{\sigma}{C}\left(\frac{\Phi'(0)}{\|v\|^2}\right)^2.\end{aligned}$$

☐

10.2 Penalty Methods for Constrained Problems

We consider first the equality constrained problem:

$$\min\{f(x) \mid h(x) = 0,\ x \in \mathbb{R}^n\} \tag{10.6}$$

where $f \in C^1$ on $\mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ with $h \in C^1$. The key idea of penalty approaches is to reduce (10.6) to an unconstrained problem with a changed objective that penalizes the violation of the constraint $h(x) = 0$. An example is the following penalty function

$$P(x; \alpha) := f(x) + \frac{\alpha}{2}\|h(x)\|^2.$$

Theorem 10.14. Let f, h be continuous functions, let $\{\alpha_k\}$ be a strictly increasing sequence with $\alpha_k \to \infty$, let the feasible set

$$S := \{x \in \mathbb{R}^n \mid h(x) = 0\}$$

be nonempty, and let $\{x^k\}$ be a sequence generated by Algorithm 10.5 (in particular, this sequence is assumed to exist). Then the following statements hold:

1. The sequence $\{P(x^k; \alpha_k)\}$ is monotonically increasing.

```
1. Choose $\alpha_0 > 0$ and $k := 0$;
2. Compute optimal solution $x^k$ to
       $\min P(x; \alpha)$ s.t. $x \in \mathbb{R}^n$.
3. If $\left\|h(x^k)\right\| = 0$ :STOP (Termination);
4. Compute $\alpha_{k+1} > \alpha_k$; $k \leftarrow k + 1$ and go to step 2.
```

Figure 10.5: Penalty Method.

> 2. The sequence $\{\|h(x^k)\|\}$ is monotonically decreasing.
> 3. The sequence $\{f(x^k)\}$ is monotonically increasing.
> 4. $\lim_{k\to\infty} h(x^k) = 0$.
> 5. Every accumulation point of the sequence $\{x^k\}$ is a solution of problem (5.10).

Proof. 1: Since $\alpha_k < \alpha_{k+1}$, we have

$$P(x^k; \alpha_k) \leq P(x^{k+1}; \alpha_k) \leq P(x^{k+1}; \alpha_{k+1}),$$

which already proves statement 1:.

2: From

$$P(x^k; \alpha_k) \leq P(x^{k+1}; \alpha_k) \quad \text{and} \quad P(x^{k+1}; \alpha_{k+1}) \leq P(x^k; \alpha_{k+1}),$$

addition yields

$$P(x^k; \alpha_k) + P(x^{k+1}; \alpha_{k+1}) \leq P(x^{k+1}; \alpha_k) + P(x^k; \alpha_{k+1}).$$

This implies

$$\alpha_k\|h(x^k)\|^2 + \alpha_{k+1}\|h(x^{k+1})\|^2 \leq \alpha_k\|h(x^{k+1})\|^2 + \alpha_{k+1}\|h(x^k)\|^2,$$

and hence

$$(\alpha_k - \alpha_{k+1})\left(\|h(x^k)\|^2 - \|h(x^{k+1})\|^2\right) \leq 0.$$

Since $\alpha_k < \alpha_{k+1}$, it follows that

$$\|h(x^k)\| \geq \|h(x^{k+1})\| \quad \text{for all } k \in \mathbb{N}.$$

3.: From $P(x^k; \alpha_k) \leq P(x^{k+1}; \alpha_k)$ and part 2.:, it follows that

$$f(x^k) \leq f(x^{k+1}),$$

which proves statement 3.

4.: Since $S \neq \emptyset$, we obtain

$$P(x^k; \alpha_k) \leq \inf_{x \in S} P(x; \alpha_k) = \inf_{x \in S} f(x) < +\infty. \tag{10.7}$$

Because $\alpha_k \to \infty$ and $f(x^k) \geq f(x^0)$ by 3., it already follows that

$$\lim_{k \to \infty} \|h(x^k)\| = 0,$$

using that $\lim_{k \to \infty} P(x^k; \alpha_k) < +\infty$.

5.: Let x^* be an accumulation point of the sequence $\{x^k\}$ and let $\{x^k\}_{k \in L}$ be a subsequence converging to x^*. By 4. we have $h(x^*) = 0$, so x^* is at least a feasible point of (10.6). From (10.7) we further obtain

$$f(x^*) = \lim_{k \in L} f(x^k) \leq \lim_{k \in L} P(x^k; \alpha_k) \leq \inf_{x \in S} f(x).$$

Thus statement 5. is also proven. ☐

We note that Theorem 10.14 does not require any differentiability assumptions on the functions f and h, and the feasible set S does not need to satisfy any regularity condition. Only the continuity of f and h is required. Therefore, Theorem 10.14 also remains valid for the more general optimization problem:

$$\min f(x) \quad \text{s.t.} \quad h(x) = 0,\ g(x) \leq 0$$

since this can obviously be reformulated in the form

$$\min f(x) \quad \text{s.t.} \quad h(x) = 0,\ \max\{0, g(x)\} = 0$$

with

$$\max\{0, g(x)\} := (\max\{0, g_1(x)\}, \ldots, \max\{0, g_k(x)\})^T \in \mathbb{R}^m.$$

The corresponding penalty function is then

$$P(x; \alpha) := f(x) + \frac{\alpha}{2}\|h(x)\|^2 + \frac{\alpha}{2} \sum_{i=1}^{k} \Big(\max\{0, g_i(x)\}\Big)^2.$$

In the following, we return to the equality-constrained problem (10.6) and investigate how we can construct an associated sequence of Lagrange multipliers $\{\mu^k\}$ from the sequence $\{x^k\}$ generated by the penalty method, so that the pair $\{(x^k, \mu^k)\}$ converges to a KKT point

(x^*, μ^*) of (10.6). For this purpose, we assume f and h to be continuously differentiable. Since x^k is by construction a global minimum of $P(\cdot; \alpha_k)$, we have in particular

$$0 = P_x(x^k; \alpha_k) = f'(x^k) + \alpha_k \sum_{j=1}^{m} h_j(x^k) h_j'(x^k). \tag{10.8}$$

On the other hand, at a KKT point we have

$$0 = f'(x^*) + \sum_{j=1}^{m} \mu_j^* h_j'(x^*).$$

This suggests interpreting the values

$$\mu_j^k := \alpha_k h_j(x^k) \quad (j = 1, \dots, m) \tag{10.9}$$

as approximations for the Lagrange multipliers μ_j^* $(j = 1, \dots, m)$. Indeed, the following result holds true.

Theorem 10.15. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ be C^1, $\{x^k\}$ a sequence generated by the penalty method with $\lim_{k\to\infty} x^k = x^*$, the gradients $h_1'(x^*), \dots, h_m'(x^*)$ linearly independent, and $\{\mu^k\}$ the sequence defined according to (10.9). Then:

1. The sequence $\{\mu^k\}$ converges to a vector $\mu^* \in \mathbb{R}^m$.
2. The pair (x^*, μ^*) with μ^* from 1. is a KKT point of (10.6), i.e., μ^* is the uniquely determined Lagrange multiplier corresponding to the solution x^* of (10.6).

Proof. (1): Let $A_k := h'(x^k) \in \mathbb{R}^{m\times n}$ and $A_* := h'(x^*) \in \mathbb{R}^{m\times n}$ be the Jacobian matrices of h at x^k and x^*, respectively. By continuity, $A_k \to A_*$. By assumption, the matrix $A_* A_*^T \in \mathbb{R}^{m\times m}$ is non-singular. For sufficiently large k, $A_k A_k^T$ is therefore also non-singular, and we have

$$(A_k A_k^T)^{-1} \to (A_* A_*^T)^{-1}.$$

With (10.8) and (10.9), we have

$$f'(x^k) + \sum_{j=1}^{m} \mu_j^k h_j'(x^k) = 0,$$

thus

$$A_k^T \mu^k = -f'(x^k).$$

Multiplying from the left by A_k and solving for μ^k yields

$$\mu^k = -(A_k A_k^T)^{-1} A_k f'(x^k).$$

Based on our preliminary considerations, the right-hand side converges, i.e., the sequence $\{\mu^k\}$ converges to some μ^* with

$$\mu^* = -(A_* A_*^T)^{-1} A_* f'(x^*).$$

(2): This assertion follows immediately from (10.8) and (10.9) by taking the limit $k \to \infty$, since from part (1) we already know that the sequence $\{\mu^k\}$ also converges. ☐

10.3 Lagrange-Newton Method

We recap the Newton-Method. Given is a C^1 function

$$F : \mathbb{R}^n \to \mathbb{R}^n$$

and we search for

$$F(x) = 0.$$

The linear approximation in x^k is defined as

$$F_k(x) := F(x^k) + F'(x^k)(x - x^k).$$

Hence,

$$x^{k+1} = x^k - F'(x^k)^{-1} F(x^k),$$

if the inverse F'^{-1} exists. In a concrete implementation, we don't compute the inverse but a direction vector $v = x - x^k$ solving the Newton-Equation:

$$F'(x^k)v = -F(x^k).$$

For a solution v^k we set

$$x^{k+1} := x^k + v^k.$$

1. Choose $x_0 \in \mathbb{R}^n$ and $k := 0$;
2. If $F(x^k) = 0$:STOP (Termination);
3. Compute v_k as a solution to the equation system
$$F'(x^k)v = -F(x^k). \tag{10.10}$$
4. set $x^{k+1} = x^k + v^k$, $k \leftarrow k + 1$ and go to step 2.

Figure 10.6: Newton's Method.

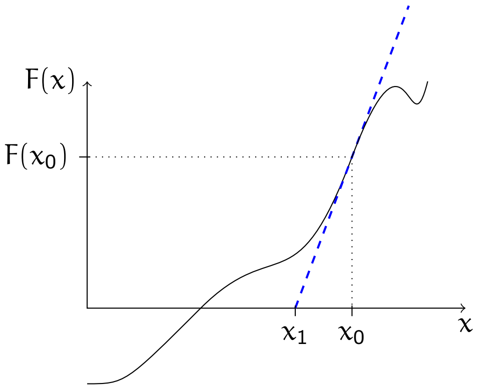


Figure 10.7: Illustration of the Newton-Method.

We now derive the so-called perturbation lemma which is heavily used in numerical linear algebra. We start with a preparatory lemma.

Lemma 10.16. Let $M \in M(n \times n, \mathbb{R})$ with $\|M\| < 1$. Then $I - M$ is regular, and the estimate

$$\|(I-M)^{-1}\| \leq \frac{1}{1-\|M\|}$$

holds.

Proof. For every $x \in \mathbb{R}^n$ we have

$$\|(I-M)x\| = \|x - Mx\| \geq \|x\| - \|Mx\| \geq (1 - \|M\|)\,\|x\|. \tag{10.11}$$

Since $\|M\| < 1$, we have $1 - \|M\| > 0$, and thus from (10.11) it follows immediately that $(I - M)x \neq 0$ for $x \neq 0$, i.e., the matrix $I - M$ is non-singular. In particular, for $x := (I - M)^{-1}y$ it follows from (10.11) that

$$\|y\| \geq (1 - \|M\|)\,\|(I-M)^{-1}y\| \quad \text{for all } y \in \mathbb{R}^n.$$

From the definition of a matrix norm we therefore obtain

$$\|(I-M)^{-1}\| = \max_{y \neq 0} \frac{\|(I-M)^{-1}y\|}{\|y\|} \leq \frac{1}{1-\|M\|},$$

which proves the claimed inequality. □

We now turn to the announced perturbation lemma.

Lemma 10.17. Let $A, B \in \mathbb{R}^{n \times n}$ satisfy $\|I - BA\| < 1$. Then both A and B are non-singular, and the estimate

$$\|B^{-1}\| \leq \frac{\|A\|}{1-\|I-BA\|}$$

holds. (A corresponding inequality also holds for A^{-1}.)

Proof. Let $M := I - BA$. By assumption, $\|M\| < 1$. By Lemma 10.16, it follows that

$$I - M = I - (I - BA) = BA$$

is non-singular. By the determinant multiplication theorem, both A and B are therefore non-singular. Moreover, Lemma 10.16 yields the estimate

$$\|(I-M)^{-1}\| \leq \frac{1}{1-\|M\|} = \frac{1}{1-\|I-BA\|}. \tag{10.12}$$

Since $I - M = BA$, we have $B^{-1} = A(I-M)^{-1}$, and hence, using (10.12),

$$\|B^{-1}\| \leq \|(I-M)^{-1}\| \, \|A\| \leq \frac{\|A\|}{1-\|I-BA\|}.$$

☐

Lemma 10.18. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable, $x^* \in \mathbb{R}^n$, and assume that $F'(x^*)$ is regular. Then there exists an $\varepsilon > 0$ such that $F'(x)$ is also regular for all $x \in B_\varepsilon(x^*)$. Moreover, there exists a constant $c > 0$ such that the estimate

$$\left\|F'(x)^{-1}\right\| \leq c$$

holds for all $x \in B_\varepsilon(x^*)$.

Proof. Since F' is continuous at x^* by assumption, there exists an $\varepsilon > 0$ such that

$$\left\|F'(x^*) - F'(x)\right\| \leq \frac{1}{2\left\|F'(x^*)^{-1}\right\|}$$

for all $x \in B_\varepsilon(x^*)$. Consequently,

$$\begin{aligned}\left\|I - F'(x^*)^{-1}F'(x)\right\| &\leq \left\|F'(x^*)^{-1}\right\| \left\|F'(x^*) - F'(x)\right\| \\ &\leq \frac{1}{2}\end{aligned}$$

for all $x \in B_\varepsilon(x^*)$. By the perturbation lemma (Lemma 10.17) it follows that for all $x \in B_\varepsilon(x^*)$ the matrix $F'(x)$ is regular with

$$\left\|F'(x)^{-1}\right\| \leq \frac{\left\|F'(x^*)^{-1}\right\|}{1-\left\|I - F'(x^*)^{-1}F'(x)\right\|} \leq 2\left\|F'(x^*)^{-1}\right\| =: c.$$

This proves the lemma. ☐

Definition 10.19 (Lipschitz-Continuity). A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous in $D \subset \mathbb{R}^n$ with Lipschitz constant $L > 0$, if the following holds true

$$\|F(x) - F(y)\| \leq L\,\|x - y\| \quad \text{for all } x, y \in D.$$

F is said to be locally Lipschitz at x, if there is $\varepsilon > 0$ so that F is Lipschitz on $B_\varepsilon(x)$

Lemma 10.20. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ and $\{x^k\} \subseteq \mathbb{R}^n$ be a sequence converging to some $x^* \in \mathbb{R}^n$. Then the following hold:

1. If F is continuously differentiable, then
$$\left\|F(x^k) - F(x^*) - F'(x^k)(x^k - x^*)\right\| = o(\left\|x^k - x^*\right\|).$$
2. If F is continuously differentiable and F' is locally Lipschitz continuous, then
$$\left\|F(x^k) - F(x^*) - F'(x^k)(x^k - x^*)\right\| = O(\left\|x^k - x^*\right\|^2).$$

Proof. 1.: Using the triangle inequality we obtain

$$\begin{aligned}&\left\|F(x^k) - F(x^*) - F'(x^k)(x^k - x^*)\right\| \\ &\quad\le \left\|F(x^k) - F(x^*) - F'(x^*)(x^k - x^*)\right\| + \left\|F'(x^*) - F'(x^k)\right\| \left\|x^k - x^*\right\|.\end{aligned}$$

Since F is differentiable at x^* by assumption, we have

$$\left\|F(x^k) - F(x^*) - F'(x^*)(x^k - x^*)\right\| = o(\left\|x^k - x^*\right\|).$$

Moreover, because F' is continuous at x^*, it follows that

$$\left\|F'(x^*) - F'(x^k)\right\| \to 0.$$

Combining the last three observations yields precisely

$$\left\|F(x^k) - F(x^*) - F'(x^k)(x^k - x^*)\right\| = o(\left\|x^k - x^*\right\|),$$

which proves claim 1.

2.: First, by the mean value theorem in integral form we have

$$\begin{aligned}&F(x^k) - F(x^*) - F'(x^k)(x^k - x^*) \\ &\quad= \int_0^1 F'\big(x^* + \tau(x^k - x^*)\big)(x^k - x^*)\,d\tau \;-\; F'(x^k)(x^k - x^*) \\ &\quad= \int_0^1 \Big[F'\big(x^* + \tau(x^k - x^*)\big) - F'(x^k)\Big](x^k - x^*)\,d\tau.\end{aligned}$$

Denoting by $L > 0$ the local Lipschitz constant of F' in a neighborhood of x^*, we obtain for all sufficiently large $k \in \mathbb{N}$:

$$\begin{aligned}&\left\|F(x^k) - F(x^*) - F'(x^k)(x^k - x^*)\right\| \\ &\quad\le \int_0^1 \left\|F'\big(x^* + \tau(x^k - x^*)\big) - F'(x^k)\right\|\,d\tau\,\left\|x^k - x^*\right\|\end{aligned}$$

$$
\begin{aligned}
&\leq L \left\| x^k - x^* \right\| \int_0^1 \left\| (\tau - 1)(x^k - x^*) \right\| \, d\tau \\
&= \frac{L}{2} \left\| x^k - x^* \right\|^2 \\
&= O(\left\| x^k - x^* \right\|^2).
\end{aligned}
$$

This is exactly claim 2. ☐

Definition 10.21 (Convergence).

1. A sequence $\{x_k\} \subset \mathbb{R}^n$ converges superlinearly towards x^*, if there is a sequence $\{\epsilon_k\} \subset \mathbb{R}_+$ with $\epsilon_k \to 0$ such that

   $$\left\| x^{k+1} - x^k \right\| \leq \epsilon_k \left\| x^k - x^* \right\| \quad \text{for all } k \in \mathbb{N}.$$

2. A sequence $\{x_k\} \subset \mathbb{R}^n$ converges quadratically towards x^*, if $\{x_k\}$ converges to x^* and there is a constant C with

   $$\left\| x^{k+1} - x^k \right\| \leq C \left\| x^k - x^* \right\|^2 \quad \text{for all } k \in \mathbb{N}.$$

Theorem 10.22 (Local Convergence of Newton's Method). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable, $x^* \in \mathbb{R}^n$ a zero of F, and suppose the Jacobian matrix $F'(x^*)$ is regular. Then there exists an $\varepsilon > 0$ such that for every $x^0 \in B_\varepsilon(x^*)$ the following hold:

1. The Newton method (Algorithm 10.6) is well-defined and generates a sequence $\{x^k\}$ converging to x^*.
2. The rate of convergence is superlinear.
3. If, in addition, F' is locally Lipschitz continuous, then the rate of convergence is even quadratic.

Proof. By Lemma 10.18 there exists an $\varepsilon_1 > 0$ such that $F'(x)$ is regular for all $x \in B_{\varepsilon_1}(x^*)$ with

$$\left\| F'(x)^{-1} \right\| \leq c$$

for some constant $c > 0$. Furthermore, by Lemma 10.20(a) there exists an $\varepsilon_2 > 0$ such that

$$\left\| F(x) - F(x^*) - F'(x)(x - x^*) \right\| \leq \frac{1}{2c} \left\| x - x^* \right\|$$

for all $x \in B_{\varepsilon_2}(x^*)$. Set $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ and choose $x^0 \in B_\varepsilon(x^*)$. Then x^1 is well-defined, and we have

$$
\begin{aligned}
\left\| x^1 - x^* \right\| &= \left\| x^0 - x^* - F'(x^0)^{-1} F(x^0) \right\| \\
&\leq \left\| F'(x^0)^{-1} \right\| \left\| F(x^0) - F(x^*) - F'(x^0)(x^0 - x^*) \right\| \\
&\leq c \, \frac{1}{2c} \left\| x^0 - x^* \right\|
\end{aligned}
$$

$$= \frac{1}{2} \left\| x^0 - x^* \right\| .$$

Hence, $x^1 \in B_\varepsilon(x^*)$ as well, and by induction we obtain

$$\left\| x^k - x^* \right\| \leq \left(\frac{1}{2} \right)^k \left\| x^0 - x^* \right\| \quad \text{for all } k \in \mathbb{N}.$$

Thus, the sequence $\{x^k\}$ is well-defined and converges to x^*, which proves part 1.
To show statements 2 and 3, we proceed similarly to the above estimate and obtain

$$\begin{aligned} \left\| x^{k+1} - x^* \right\| &= \left\| x^k - x^* - F'(x^k)^{-1} F(x^k) \right\| \\ &\leq \left\| F'(x^k)^{-1} \right\| \left\| F(x^k) - F(x^*) - F'(x^k)(x^k - x^*) \right\| \\ &\leq c \left\| F(x^k) - F(x^*) - F'(x^k)(x^k - x^*) \right\| . \end{aligned}$$

Applying Lemma 10.20 then yields claims 2 and 3. ☐

We consider now a constrained optimization problem:

$$\min \{ f(x) \mid \; g(x) = 0 \}, \; \text{where } g : \mathbb{R}^n \to \mathbb{R}^m. \tag{10.13}$$

Using the Lagrange-Function

$$L(x, \lambda) = f(x) + \lambda g(x) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

the KKT-conditions read as

$$F(x, \lambda) := \begin{pmatrix} L_x(x, \lambda) \\ L_\lambda(x, \lambda) \end{pmatrix} = \begin{pmatrix} f'(x) + \lambda g'(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{10.14}$$

The equation (10.14) is a nonlinear system of $n + m$ equations in x, λ. The corresponding Newton-Update reads as

$$\begin{pmatrix} x^{k+1} \\ \lambda^{k+1} \end{pmatrix} = \begin{pmatrix} x^k \\ \lambda^k \end{pmatrix} - F'(x^k, \lambda^k)^{-1} F(x^k, \lambda^k). \tag{10.15}$$

Formally, we get:
We obtain the following result regarding the Lagrange-Newton method.

> **Theorem 10.23.** Let $(\bar{x}, \bar{\lambda})$ be a normal KKT-point of 10.13, satisfying the second order sufficient optimality conditions of Thm. 6.1, Condition (2). Then, the Lagrange-Newton method converges quadratically against a KKT-point of 10.13, where $\epsilon_0 = 0$ is assumed.

1. Choose $x^0 \in \mathbb{R}^n, \lambda^0 \in \mathbb{R}^m$, $k = 0$ and fix $\epsilon_0 > 0$;
2. If $\left\|F(x^k, \lambda^k)\right\| \leq \epsilon_0$:STOP (Termination);
3. Compute $(\Delta x^k, \Delta\lambda^k)$ as solution of
$$F'(x^k, \lambda^k) \begin{pmatrix} \Delta x \\ \Delta\lambda \end{pmatrix} = -F(x^k, \lambda^k). \tag{10.16}$$
4. Set $x^{k+1} \leftarrow x^k + \Delta x^k; \lambda^{k+1} \leftarrow \lambda^k + \Delta\lambda^k;\ k \leftarrow k+1$ and go to step 2.

Figure 10.8: Lagrange-Newton Method.

Proof. We only need to show that the Jacobi-Matrix $F'(\bar{x}, \bar{\lambda})$ is regular. This was already shown in the proof of Thm. 7.1. ☐

Inequality Constraints

We now consider optimization problems with inequality constraints of the following form:

$$\min \{f(x) |\ g_i(x) \leq 0, i = 1, \dots, k,\ h_i(x) = 0, i = 1, \dots, m-k\}. \tag{10.17}$$

and we also write
$$\min \{f(x) |\ g(x) \leq 0, h(x) = 0\}.$$

With the Lagrangian
$$L(x, \lambda, \mu) = f(x) + \lambda h(x) + \mu g(x),$$
the KKT conditions are
$$\begin{aligned} L_x(x, \lambda, \mu) &= f'(x) + \lambda h'(x) + \mu g'(x) = 0 \\ g(x) &\leq 0 \\ h(x) &= 0 \\ \mu_i g_i(x) &= 0, i = 1, \dots, k \\ \mu_i &\geq 0, i = 1, \dots, k. \end{aligned} \tag{10.18}$$

We now introduce a so-called NCP function.

Definition 10.24. A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called an NCP function if
$$\phi(a, b) = 0 \Rightarrow a \geq 0, b \geq 0, ab = 0.$$

Example 10.25. The following mappings are NCP functions:

1. Minimum: $\phi(a, b) = 2\min\{a, b\}$.
2. Fischer–Burmeister: $\phi(a, b) = a + b - \sqrt{a^2 + b^2}$.

We obtain the following equivalent representation of the KKT conditions:

$$\begin{aligned} L_x(x, \lambda, \mu) = f'(x) + \mu g'(x) + \lambda h'(x) &= 0 \\ h(x) &= 0 \\ \phi(-g_i(x), \mu_i) &= 0, i = 1, \dots, k. \end{aligned}$$

This can again be formulated as a nonlinear system of equations

$$F(x, \lambda, \mu) = 0 \text{ with}$$

$$F(x, \lambda, \mu) := \begin{pmatrix} L_x(x, \lambda, \mu) \\ h(x) \\ \phi(-g(x), \mu) \end{pmatrix},$$

where

$$\phi(-g(x), \mu) := (\phi(-g_1(x), \mu_1), \dots, \phi(-g_k(x), \mu_k))^\mathsf{T} \in \mathbb{R}^k.$$

> **Remark 10.26.** In principle, one can apply a Newton method to the above system of equations. However, the function ϕ is not differentiable everywhere; therefore, the Newton method may need to be modified (using subdifferentials). A more detailed treatment of this topic can be found in Geiger and Kanzow (2011).

10.4 SQP Methods

Consider again the equality-constrained optimization problem (10.13). In the Lagrange-Newton method, we set

$$x^{k+1} := x^k + \Delta x^k, \qquad \mu^{k+1} := \mu^k + \Delta\mu^k, \tag{10.19}$$

where $(\Delta x^k, \Delta\mu^k)$ denotes a solution of the linear system

$$F'(x^k, \mu^k) \begin{pmatrix} \Delta x \\ \Delta\mu \end{pmatrix} = -F(x^k, \mu^k). \tag{10.20}$$

Taking into account the special structure of $F'(x^k, \mu^k)$, this system can be written equivalently as

$$\begin{aligned} H_k\,\Delta x + h'(x^k)^\mathsf{T}\,\Delta\mu &= -L_x(x^k, \mu^k), \\ h_j'(x^k)^\mathsf{T}\,\Delta x &= -h_j(x^k), \qquad \forall j = 1, \dots, m, \end{aligned} \tag{10.21}$$

with $H_k = L_{xx}(x^k, \mu^k)$. In the following, we also allow approximations $H_k \approx L_{xx}(x^k, \mu^k)$. Defining $\mu^+ := \mu^k + \Delta\mu$, system (10.21) is equivalent to

$$\begin{aligned} H_k\,\Delta x + h'(x^k)^\top \mu^+ &= -f'(x^k), \\ h_j'(x^k)^\top \Delta x &= -h_j(x^k), \qquad \forall j = 1, \dots, m. \end{aligned} \tag{10.22}$$

System (10.22) admits another interpretation: it corresponds exactly to the Karush–Kuhn–Tucker (KKT) conditions of the quadratic optimization problem

$$\begin{aligned} \min_{\Delta x} \quad & f'(x^k)^\top \Delta x + \tfrac{1}{2}\Delta x^\top H_k\,\Delta x \\ \text{subject to} \quad & h_j(x^k) + h_j'(x^k)^\top \Delta x = 0, \qquad (j = 1, \dots, m). \end{aligned} \tag{10.23}$$

This observation motivates, in particular, the use of the quadratic subproblem

$$\begin{aligned} \min_{\Delta x} \quad & f'(x^k)^\top \Delta x + \tfrac{1}{2}\Delta x^\top H_k\,\Delta x \\ \text{subject to} \quad & g_i(x^k) + g_i'(x^k)^\top \Delta x \le 0, \qquad (i = 1, \dots, \ell), \\ & h_j(x^k) + h_j'(x^k)^\top \Delta x = 0, \qquad (j = \ell + 1, \dots, m). \end{aligned} \tag{10.24}$$

for the general problem

$$\min\,\{f(x) \mid\ g(x) \le 0, h(x) = 0\}. \tag{10.25}$$

with

$$g : \mathbb{R}^n \to \mathbb{R}^\ell \text{ and } h : \mathbb{R}^n \to \mathbb{R}^{m-\ell}.$$

The quadratic subproblem arising in step (3) of Algorithm 10.9 may admit multiple KKT points. The restriction imposed in step (3) on the choice of $(x^{k+1}, \lambda^{k+1}, \mu^{k+1})$ is hardly realizable in practice; however, it allows us to prove the following local convergence result.

> **Theorem 10.27** (Local Convergence of SQP). Let $(x^*, \lambda^*, \mu^*) \in \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ be a KKT point of (10.25) satisfying the following assumptions:
>
> (i) $g_i(x^*) + \lambda_i^* \neq 0$ for all $i = 1, \dots, \ell$ (strict complementarity).
>
> (ii) The gradients $h_j'(x^*)$ $(j = 1, \dots, m - \ell)$ and $g_i'(x^*)$ for $i \in I(x^*) := \{i \mid g_i(x^*) = 0\}$ are linearly independent (LICQ).
>
> (iii)
> $$d^\top L_{xx}(x^*, \lambda^*, \mu^*)d > 0$$
> for all $d \neq 0$ such that
> $$h_j'(x^*)^\top d = 0 \quad (j = 1, \dots, m - \ell), \qquad g_i'(x^*)^\top d = 0 \quad (i \in I(x^*)),$$
> (second-order sufficient condition).
>
> Then there exists $\varepsilon > 0$ such that for every initial vector $(x^0, \lambda^0, \mu^0) \in B_\varepsilon(x^*, \lambda^*, \mu^*)$ and every sequence $\{(x^k, \lambda^k, \mu^k)\}$ generated by Algorithm 10.9, the following statements hold:

(SQP method with $H_k = L_{xx}(x^k, \lambda^k, \mu^k)$)

1. Choose $(x^0, \lambda^0, \mu^0) \in \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ and set $k := 0$.
2. If (x^k, λ^k, μ^k) is a KKT point of (10.25): **STOP**.
3. Compute a KKT point $(x^{k+1}, \lambda^{k+1}, \mu^{k+1}) \in \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ of the quadratic subproblem

$$\begin{aligned} \min_x \quad & f'(x^k)^T(x - x^k) + \tfrac{1}{2}(x - x^k)^T L_{xx}(x^k, \lambda^k, \mu^k)(x - x^k) \\ \text{subject to} \quad & g_i(x^k) + g_i'(x^k)^T(x - x^k) \le 0, \quad i = 1, \dots, \ell, \\ & h_j(x^k) + h_j'(x^k)^T(x - x^k) = 0, \quad j = 1, \dots, m - \ell. \end{aligned} \tag{10.26}$$

   If this quadratic subproblem has several KKT points, choose the KKT point $(x^{k+1}, \lambda^{k+1}, \mu^{k+1})$ such that

$$\left\| (x^{k+1}, \lambda^{k+1}, \mu^{k+1}) - (x^k, \lambda^k, \mu^k) \right\|$$

   is minimal.
4. Set $k \leftarrow k + 1$ and return to (2).

Figure 10.9: SQP Method.

(a) Algorithm 10.9 is well defined, and $(x^k, \lambda^k, \mu^k) \to (x^*, \lambda^*, \mu^*)$.

(b) The rate of convergence is superlinear.

(c) If f'', g_i'' $(i = 1, \dots, \ell)$, and h_j'' $(j = 1, \dots, m - \ell)$ are locally Lipschitz continuous, then the convergence rate is quadratic.

Proof. We only sketch the proof here. The full proof can be found in Kanzow and Geiger. We define the mapping

$$F(x, \lambda, \mu) := \begin{pmatrix} \nabla_x L(x, \lambda, \mu) \\ \\ h(x) \\ \\ \min\{-g(x), \mu\} \end{pmatrix}$$

with

$$\min\{-g(x), \mu\} := \left(\min\{-g_1(x), \mu_1\}, \dots, \min\{-g_m(x), \mu_m\}\right)^\top \in \mathbb{R}^m,$$

which already appeared before (however, using an arbitrary NCP function instead of the minimum function).

Obviously, (x^*, λ^*, μ^*) is a KKT point of if and only if (x^*, λ^*, μ^*) satisfies the nonlinear system of equations

$$F(x, \lambda, \mu) = 0. \tag{10.27}$$

We first investigate the properties of Newton's method applied to this nonlinear system. Due to the assumed strict complementarity of the KKT point (x^*, λ^*, μ^*), see (i), the function F is continuously differentiable in an open neighborhood $B_\varepsilon(x^*, \lambda^*, \mu^*)$ (and thus Newton's method is applicable in this neighborhood). Furthermore, because of assumption (ii), the Jacobian matrix $F'(x^*, \lambda^*, \mu^*)$ is regular.

By analogy with the proof of Theorem 10.22, this implies the existence of some $\varepsilon_2 > 0$ (without loss of generality $\varepsilon_2 \leq \varepsilon_1$) such that the Newton method is well defined for every starting vector $(x^0, \lambda^0, \mu^0) \in B_{\varepsilon_2}(x^*, \lambda^*, \mu^*)$ and generates a sequence $\{(x^k, \lambda^k, \mu^k)\}$. The rest of the proof shows that (under the made assumptions) the Newton-Sequence coincides with the SQP-sequence. □

10.5 The Frank-Wolfe Algorithm

In this chapter, we consider the Frank-Wolfe (FW) algorithm for a problem of type

$$\min f(x), \ \text{s.t. } x \in K, \tag{10.28}$$

for some convex set $K \in \mathbb{R}^m$. FW relies on the following basic idea: we compute the gradient $f'(x_k)$ of f for $f \in C^1$ at some iterate $x_k \in K$. Then, we solve the simpler linear-objective problem

$$\min_{v \in K} f'(x_k)v \text{ s.t. } v \in K, \tag{10.29}$$

and for the optimal value v_k, we choose the new iterate as

$$x_{k+1} = (1 - \gamma_k)x_k + \gamma_k v_k. \tag{FW-Alg.}$$

A common choice of γ_k is $\gamma_k = \frac{2}{k+2}$. In the literature, the linear optimization problem is assumed to be easily solvable and we can abstract away the resulting running time by having access to a Linear Minimization Oracle (LMO).

10.5.1 Convex Objectives

In order to analyze the convergence of FW, we need some more concepts.

Definition 10.28 (Smootheness). Let $f : K \to \mathbb{R}$ with $f \in C^1$ is L-smooth, if

$$f(y) - f(x) \leq f'(x)(y - x) + \frac{L}{2} \|y - x\|^2 \ \text{ for all } x, y \in K. \tag{10.30}$$

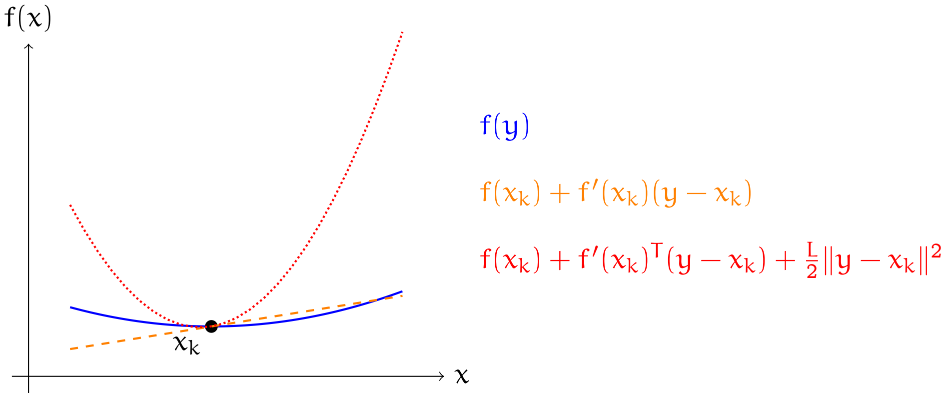


Figure 10.10: Smoothness property.

Lemma 10.29. Let $f : K \to \mathbb{R}$ with $f \in C^1$ be convex and let K be convex and compact. Then, the following statements hold.

1. x^* is optimal for (10.28) if and only if $f'(x^*)(x^* - v^*) \leq 0$, where $v^* \in K$ is optimal for (10.29) with $x_k = x^*$.
2. For all $x \in K$ it holds:

$$\underbrace{f(x) - f(x^*)}_{\text{primal gap}} \leq \underbrace{f'(x)(x - x^*)}_{\text{dual gap}} \leq \max_{v \in K} f'(x)(x - v) = f'(x)(x - v^*), \qquad \text{(FW-gap)}$$

where $v^* \in K$ is optimal for (10.29) with $x_k = x$.

The proof follows by convexity of f and K and the property of v^*. Now we measure the progress during the execution of (FW-Alg.).

Lemma 10.30. (Primal progress)
Consider (FW-Alg.) and fix $k \in \mathbb{N}$. Let $f : K \to \mathbb{R}$ with $f \in C^1$ and L-smooth. Let K be convex and compact. Then, we have

$$f(x_k) - f(x_{k+1}) \geq \gamma_k f'(x_k)(x_k - v_k) - (\gamma_k)^2 \frac{L}{2} \|x_k - v_k\|^2 . \tag{10.31}$$

Proof. We just insert the definition of x_{k+1} according to (FW-Alg.) and x_k into (10.30). ☐

The the two basic Lemmata 10.29 and 10.30 lead to the following convergence result.

Theorem 10.31. Let f be an L-smooth convex function and let K be a compact convex set of diameter $\operatorname{diam}(K) = D$, where $\operatorname{diam}(K) := \max_{x,y \in K} \|y - x\|$. Consider the iterates of (FW-Alg.) with $\gamma_k = \frac{2}{k+2}$. Then the following holds for all $k \in \mathbb{N}$:

$$f(x_k) - f(x^*) \leq \frac{2LD^2}{k+2} \tag{10.32}$$

and thus, for a given accuracy $\epsilon > 0$ we have $f(x_k) - f(x^*) \leq \epsilon$ for all $k \geq \frac{2LD^2}{\epsilon}$.

Proof. Using Lemma 10.30, we get

$$\begin{aligned} f(x_k) - f(x_{k+1}) &\geq \gamma_k f'(x_k)(x_k - v_k) - (\gamma_k)^2 \frac{L}{2} \|x_k - v_k\|^2 \\ &\geq \gamma_k (f(x_k) - f(x^*)) - (\gamma_k)^2 \frac{L}{2} \|x_k - v_k\|^2 . && \text{(By (FW-gap))} \end{aligned}$$

Multiplying with -1 and using $\|x_k - v_k\|^2 \leq D^2$, we obtain:

$$f(x_{k+1}) \leq (1-\gamma_k) f(x_k) + \gamma_k f(x^*) + (\gamma_k)^2 \frac{LD^2}{2}.$$

Subtracting $f(x^*)$ on both sides yields:

$$f(x_{k+1}) - f(x^*) \leq (1-\gamma_k)(f(x_k) - f(x^*)) + (\gamma_k)^2 \frac{LD^2}{2}.$$

Now, we proceed via induction over k. For $k = 0$, we get $\gamma_0 = 1$ and thus

$$f(x_1) - f(x^*) \leq \frac{LD^2}{2} \leq \frac{2LD^2}{2}.$$

For the induction step $k \to k+1$, we get:

$$\begin{aligned} f(x_{k+1}) - f(x^*) &\leq (1-\gamma_k)(f(x_k) - f(x^*)) + (\gamma_k)^2 \frac{LD^2}{2} \\ &\leq \frac{k}{k+2} (f(x_k) - f(x^*)) + \frac{4}{(k+2)^2} \frac{LD^2}{2} \\ &\leq \left(\frac{k}{k+2}\right) \left(\frac{2LD^2}{k+2}\right) + \frac{4}{(k+2)^2} \frac{LD^2}{2} && \text{(induction hypothesis)} \\ &= 2LD^2 \left(\frac{(1+k)}{(k+2)^2}\right) = \left(\frac{2LD^2}{k+3}\right) \left(\frac{(k+3)(1+k)}{(k+2)^2}\right) \\ &\leq \left(\frac{2LD^2}{k+3}\right), \end{aligned}$$

where the last inequality follows from $\left(\frac{(k+3)(1+k)}{(k+2)^2}\right) \leq 1$. ☐

Example 10.32 (Primal Gap Lower Bound). We provide an example where $\Omega(LD^2/\varepsilon)$ linear minimizations are needed to achieve a primal gap additive error at most ε for an L-smooth convex function over a feasible region with diameter D, for any positive numbers L, D, and ε.

We consider the problem

$$\min_{x \in \operatorname{co}(e_1,\ldots,e_n)} \|x\|_2^2,$$

that is, minimizing the quadratic objective function $f(x) = \|x\|_2^2$ over the probability simplex $P = \operatorname{co}(e_1, \ldots, e_n) \subset \mathbb{R}^n$, where the e_i are the coordinate vectors, i.e., the vectors in the standard basis of $\mathbb{R}^n$. The unique optimal solution to the problem is $x_i^* = (\frac{1}{n}), i = 1, \ldots, n$, the point whose coordinates are all equal to $1/n$.

If we now run the Frank–Wolfe algorithm from any extreme point x_0 of P, then after $k < n$ iterations, we have made k LMO calls, and hence have picked up at most $k+1$ of the n standard basis vectors. This is the only information available to us about the feasible region and by convexity the only feasible points the algorithm can create are convex combinations x_k of these picked up extreme points. Thus it holds

$$f(x_k) \geq \min_{\substack{x \in \operatorname{co}(S) \\ S \subseteq \{e_1,\ldots,e_n\} \\ |S| \leq k+1}} f(x) = \frac{1}{k+1}.$$

Therefore the primal gap after k iterations satisfies

$$f(x_k) - f(x^*) \geq \frac{1}{k+1} - \frac{1}{n},$$

and for $n \gg 1/\epsilon$, we need $\Omega(1/\epsilon)$ LMO calls to guarantee a primal gap of at most ϵ. Finally, observe that this example also provides an inherent sparsity vs. optimality tradeoff: if we aim for a solution with sparsity k, then the primal gap can be as large as

$$f(x_k) - f(x^*) \geq \frac{1}{k+1} - \frac{1}{n}.$$

10.5.2 Non-Convex Objectives

Now we turn to general non-convex functions $f : K \to \mathbb{R}$ with $f \in C^1$ and K convex.

Definition 10.33 (Frank-Wolfe Gap). The Frank-Wolfe gap of a function $f : K \to \mathbb{R}$ with $f \in C^1$ at $x \in K$ is defined as

$$g(x) := \max_{v \in K} f'(x)(x - v). \tag{10.33}$$

With $x \in K$, we get $g(x) \geq 0$.

Note that for convex functions f, the condition $g(x) = 0$ is by convexity of K equivalent to x being a global minimum. For non-convex functions f, the condition $g(x) = 0$ is by convexity of K only necessary but not sufficient. Yet, due to the general difficulty of non-convex problems, we search for points x^* with $\|g(x^*)\| = 0$ and the quantity $\|g(x)\|$ for the current iterate x will serve as a measure for termination, namely, for a prescribed precision $\epsilon > 0$, we will ask for convergence towards some point $x \in K$ with $\|g(x)\| \leq \epsilon$.

Theorem 10.34. Let f be an L-smooth function and let K be a compact convex set of diameter $\operatorname{diam}(K) = D$. Consider the iterates of (FW-Alg.) with $\gamma_k = \gamma = \frac{1}{\sqrt{T+1}}$ for some fixed $T \in \mathbb{N}$. Then after T steps of (FW-Alg.) the following holds:

$$G_T := \min_{0 \le k \le T} g(x_k) \le \frac{\max\{2h_0, LD^2\}}{\sqrt{T+1}}, \tag{10.34}$$

where $h_0 := f(x_0) - f(x^*)$ denotes the primal gap at $x_0 \in K$.

Proof. At iterate x_k we can bound the progress via Lemma 10.30 as

$$f(x_k) - f(x_{k+1}) \ge \gamma f'(x_k)(x_k - v_k) - \frac{\gamma^2 L}{2} \|x_k - v_k\|^2.$$

Summing up the above for $k = 0, \dots, T$ and rearranging gives

$$\begin{aligned}\gamma \sum_{k=0}^{T} f'(x_k)(x_k - v_k) &\le f(x_0) - f(x_{T+1}) + \gamma^2 \sum_{k=0}^{T} \frac{L}{2} \|x_k - v_k\|^2 \\ &\le f(x_0) - f(x^\star) + \gamma^2 \sum_{k=0}^{T} \frac{LD^2}{2} = h_0 + \gamma^2 (T+1) \frac{LD^2}{2}.\end{aligned}$$

We divide by $\gamma(T+1)$ on both sides to obtain

$$G_T \le \frac{1}{T+1} \sum_{k=0}^{T} f'(x_k)(x_k - v_k) \le \frac{h_0}{\gamma(T+1)} + \gamma \frac{LD^2}{2},$$

and for $\gamma = \frac{1}{\sqrt{T+1}}$ we get

$$G_T \le \frac{1}{T+1} \sum_{k=0}^{T} f'(x_k)(x_k - v_k) \le \frac{2h_0 + LD^2}{2\sqrt{T+1}} \le \frac{\max\{2h_0, LD^2\}}{\sqrt{T+1}},$$

completing the proof. □