Demand Allocation Games: Integrating Discrete and Continuous Strategy Spaces

Tobias Harks and Max Klimm*

Institut für Mathematik, Technische Universität Berlin
{harks,klimm}@math.tu-berlin.de

Abstract. In this paper, we introduce a class of games which we term *demand allocation games* that combines the characteristics of finite games such as congestion games and continuous games such as Cournot oligopolies. In a strategy profile each player may choose both an action out of a finite set and a non-negative demand out of a convex and compact interval. The utility of each player is assumed to depend solely on the action, the chosen demand, and the aggregated demand on the action chosen. We show that this general class of games possess a pure Nash equilibrium whenever the players' utility functions satisfy the assumptions *negative externality*, *decreasing marginal returns* and *homogeneity*. If one of the assumptions is violated, then a pure Nash equilibrium may fail to exist. We demonstrate the applicability of our results by giving several concrete examples of games that fit into our model.

1 Introduction

The problem of allocating scarce resources to satisfy demands is a central topic in the operations research and optimization literature. While a central planer may compute and implement an optimal allocation, in many applications this may be impossible as the allocation of resources is determined by selfish players. A prominent example for this scenario are congestion games. In a congestion game, there is a set of resources and a pure strategy of a player consists of a subset of resources. The profit of a resource depends only on the number of players choosing the resource, and the utility of a player is the sum of the profits of the chosen resources. Under these assumptions, Rosenthal proved the existence of a pure Nash equilibrium (PNE for short) [25]. Another well-known variant of congestion games arises if players can *fractionally* demand the resources, see Beckmann [2] and Haurie and Marcotte [10] for related models. For this *continuous* variant, the quite general result of Rosen [24] implies the existence of a PNE provided the strategy space is convex and compact and utility functions are concave. In the context of such discrete and continuous classes of games, there are mainly two types of existence theorems for PNE. The first type applies to discrete games (such as classical congestion games and many variants thereof)

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where each player has a *finite* strategy space. For this type, the existence of PNE is proved by either potential function arguments (as in [1,4,5,6,9,22,25]), or by using the combinatorial structure of the finite strategy space (as in [13,16,26]). On the other hand, for continuous games, existence of PNE is usually established via fixed-point theorems of Kakutani (as in [24] for general concave games) and Brouwer (as in [19] for mixed extensions of finite games), or by a monotonicity property of the best reply functions (as in [21,23] for Cournot oligopolies).

While existence of PNE for both extremes is well understood, much less is known for strategic games that exhibit continuous and discrete elements at the same time. To motivate this point we give an example. Consider the classical Cournot oligopoly (cf. [3,30]). In a Cournot oligopoly game, there is a set of firms each producing quantities so as to satisfy an elastic demand. The production cost for every player is modeled by a cost function and the interaction of firms comes from the market price function which is dependent on the total supply on the market. In this form, a Cournot oligopoly game belongs to the class of continuous games and under mild assumptions on the market aggregation function, the existence of a pure Cournot equilibrium follows from Rosen [24]. The situation changes, if there are several (parallel) markets, and each Cournot player can select exactly one market to offer its quantity. The restriction of choosing only one market arises if the market is regulated, e.g., if each firm may only purchase one market license, see for instance Stähler and Upmann [29] for related models. In this case, the strategy of a player is now discrete in the sense that exactly one market can be chosen, and it is continuous in the sense that the production quantity is still continuously variable on the chosen market. Yet, we give another example related to models of population behavior in biology. Suppose there is a set of exhaustible food patches distributed on an area shared by different populations of animals (e.g., sticklebacks as in the experiment of Milinsky [18] or herds of zebras and elephants sharing water locations). Analyzing the equilibrium behavior of such systems belongs to the field of population games, see the book by Sandholm [27] and further references therein. Here, every population is represented by a fixed-sized continuum of infinitesimal small individuals each choosing a food patch. By definition (cf. [27, Chapter 2, condition (v)]) such games are continuous in the sense that the individuals are sufficiently small and may be assigned to different locations even if they belong to the same population. If the populations of animals correspond to swarms or herds the continuity assumption breaks down as swarms or herds move as a whole. Moreover, in reality the size of every population is not fixed but it correlates with the available amount of food supply. For systems having the above described characteristics, a new model is needed that integrates continuous and discrete action spaces.

In this paper, we introduce a class of games which we term *demand allocation games* that comprises the characteristics of the examples above. Suppose we are given a finite set A of actions and a finite set N of players. Each player is associated with a subset $A_i \subseteq A$ of actions allowable to her and a convex and compact interval of non-negative demands. In a strategy profile, a player chooses both a feasible action and a feasible demand for her. We additionally require the following assumptions on the player's utility functions. We assume that the utility of each player is not affected by the strategic choices of players on other actions. This assumption is often referred to as "Independence of Irrelevant Choices", see for instance Konishi et al. [13] and Voorneveld et al. [31] for a similar model with fixed demands. Moreover, we require that the game is anonymous in the sense that the utility of each player depends solely on the aggregate demand of all players playing the same action, which is a common assumption, see e.g. Konishi et al. [13]. It is a useful observation that under these basic assumptions the utility of each player i, when choosing action a_i together with demand d_i can be represented by an indirect utility function $v_i^{a_i} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ so that $u_i(a,d) = v_i^{a_i}(d_i, \ell_{-i}^{a_i}(a,d))$, where $\ell_{-i}^{a_i}(a,d) = \sum_{j \in N \setminus \{i\}: a_j = a} d_j$ denotes the aggregated demand (or load) of other players on action a_i. Clearly, in this general form nothing can be said about the existence of pure Nash equilibrium. Therefore, we require more structure about the player's utility functions. We define the following three assumptions on the player's utility functions that capture the properties of the above examples. The first assumption is called "Negative Externality" (EXT for short) and requires that the utility of a player using an action a_i decreases if the aggregate demand of other players playing the same action increases. Informally, the second assumption "Decreasing Marginal Returns" (DMR for short) requires that for every player the marginal return function exists and decreases when both that player's demand and the total demand of the chosen action increase. The last assumption is called "Homogeneity" (HOM for short) and requires that for all $i \in N$, we have $v_i^{a_i} = v_i^{b_i}$ for all $a_i, b_i \in A_i$. This last assumption is clearly the most restrictive and controversial one. We will show, however, that if it is dropped, there are instances without PNE.

Our Results. As our main result, we prove that every demand allocation game satisfying EXT, DMR and HOM possesses a PNE. This result is tight in the sense that if one of the assumptions is dropped, there is a demand allocation game without PNE. For proving this existence result we provide an algorithm that computes a PNE. Our algorithm relies on iteratively computing a (partial) equilibrium on every action separately using Rosen's theorem. Here, a partial equilibrium is a strategy profile that is resilient against unilateral demand deviations. Given a partial equilibrium, the algorithm selects a player that can play a better and best response. After such a best response it recomputes the partial equilibrium and proceeds in the same fashion. We prove that a player-specific load vector of the partial equilibria lexicographically decreases in every iteration and thus, the algorithm terminates. A perhaps surprising property of our proof is that even though we iteratively recompute a partial equilibrium by using Rosen's theorem as a black box, there is enough structure of such a partial equilibrium to prove that the algorithm terminates. We also show that demand allocation games do not have the finite improvement property even if EXT, DMR and HOM are satisfied, thus, they are not potential games. For demand allocation games with only two players, we prove that already EXTand DMR are enough to yield a PNE. In the final section of the paper, we give a series of concrete examples that fit into our model: Cournot games on parallel markets, singleton congestion games with player-specific payoff functions and variable demands, and games in biology.

2 The Model

Let A be a finite set of actions and let N be a finite set of players. For each player $i \in N$ we are given a convex and closed interval $D_i = [\alpha_i, \omega_i] \subseteq \mathbb{R}_{\geq 0}$ of allowable demands and a subset $A_i \subseteq A$ of allowable actions. A *strategy* of player i is a tuple (a_i, d_i) where $a_i \in A_i$ is an allowable action and $d_i \in D_i$ is an allowable demand for player i. A *strategy profile* of the game is a tuple (a, d) where $a = (a_i)_{i \in N}$ is the action vector and $d = (d_i)_{i \in N}$ is the demand vector. We assume that the *utility* of player i under strategy profile (a, d) depends solely on the action a_i and the demand d_i chosen by player i, and the total demand of other players with the same action $\ell_{-i}^{a_i}(a, d) = \sum_{j \in N \setminus \{i\} : a_i = a_j} d_j$. To measure this utility, we introduce for each player i and each of her allowable actions $a_i \in A_i$ an *indirect utility function* $v_i^{a_i} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$. The utility of player i under strategy profile (a, d) is then defined as $u_i(a, d) = v_i^{a_i}(d_i, \ell_{-i}^{a_i}(a, d))$. We are interested in establishing conditions on the indirect utility functions that ensure the existence of at least one pure Nash equilibrium. Formally, a strategy profile (a, d) is a *pure Nash equilibrium*, PNE for short, if $u_i(a, d) \geq u_i(a_i', a_{-i}, d_i', d_{-i})$ for all players $i \in N$ and all strategies $(a_i', d_i') \in A_i \times D_i$. We make the following three assumptions on the indirect utility functions $v_i^{a_i}$ of player i and action $a_i \in A_i$. The first assumption is called "Negative Externality" and requires that the utility of every player increases as the total demand of other players with the same action decreases.

Assumption 1 (Negative Externality (EXT)). *For all $i \in N$, $a_i \in A_i$ and $d_i \in D_i$, the indirect utility function $v_i^{a_i}(d_i, \cdot)$ is non-increasing in the second entry, that is $v_i^{a_i}(d_i, \ell_{-i}) \geq v_i^{a_i}(d_i, \ell'_{-i})$ for all $\ell_{-i}, \ell'_{-i} \in \mathbb{R}_{\geq 0}$ with $\ell_{-i} \leq \ell'_{-i}$.*

This assumption is natural when players compete over scare resources to satisfy their demand and has been made explicitly or implicitly in various contexts ranging from traffic and communication networks (i.e. [2,10,12]) to biology (i.e. [18]) and economics (i.e. in Cournot oligopolies [3,30] and Cournot oligopsonies [11,20]).

The second assumption is called "Decreasing marginal returns" and requires that for players with a non-trivial interval of allowable demands, the marginal utility function exists, is continuously differentiable, and decreases if the player's demand and the total demand of the chosen action increase.

Assumption 2 (Decreasing Marginal Returns (DMR)). *For all $i \in N$ with $\alpha_i < \omega_i$, $d_i \in D_i$, $a_i \in A_i$, and $\ell_{-i} \in \mathbb{R}_{\geq 0}$, the marginal return function $\partial\, v_i^{a_i}(d_i, \ell_{-i}) \,/\, \partial\, d_i$ exists and is continuously differentiable in d_i. Moreover, $\partial\, v_i^{a_i}(d_i, \ell_{-i}) \,/\, \partial\, d_i > \partial\, v_i^{a_i}(d_i', \ell'_{-i}) \,/\, \partial\, d_i'$ for all $d_i, d_i' \in [\alpha, \omega]$ and $\ell_{-i}, \ell'_{-i} \in \mathbb{R}_{\geq 0}$ with $d_i \leq d_i'$ and $d_i + \ell_{-i} \leq d_i' + \ell'_{-i}$, where at least one of these two inequalities is strict.*

The assumption that the utility of player i is concave in her demand often appears in the literature on Cournot oligopolies (cf. [21,23]) in order to get the existence of an equilibrium. Also many works in telecommunications (cf. [12,28]) justify concavity of the utility function in the demand variable by application-specific characteristics such as the rate-control algorithm used in the TCP protocol. Note that if for some $i \in N$ it holds that $\alpha_i = \omega_i$, then DMR is trivially satisfied.

The next assumption "Homogeneity" imposes that players have no a priori preferences over actions, that is, each player's utility is solely defined by her own demand and the total demand of the chosen action and *not* by the identity of the action itself.

Assumption 3 (Homogeneity (HOM)). *For all $i \in N$, we have $v_i^{a_i} = v_i^{b_i}$ for all $a_i, b_i \in A_i$.*

In games that satisfy HOM, we may write $v_i = v_i^{a_i} = v_i^{b_i}$ for all $a_i, b_i \in A_i$. Note that HOM does not require symmetry among players, i.e., we still allow $v_i \neq v_j$ for $i \neq j$. We only require that every player is indifferent between any two allowable actions as long as their own demand and the total demand on these actions is equal. Clearly, HOM is the most restrictive and controversial assumption. We show, however, that homogeneity is necessary in the sense that if it is dropped, there are games without PNE.

3 Existence of Pure Nash Equilibria

In this section, we will give an existence result for demand allocation games. Specifically, we will show that demand allocation games satisfying the assumptions Negative Externality (EXT), Decreasing Marginal Return (DMR) and Homogeneity (HOM) always possess a PNE. Our results are "tight" in the sense that if any of the three assumptions is dropped, then there are instances without a PNE. To prove our main result, we first introduce the concept of a *partial equilibrium*. Intuitively, a partial equilibrium is a strategy profile that is resilient against unilateral demand deviations. Formally, a strategy profile (a, d) is a partial equilibrium if $u_i(a, d) \geq u_i(a, d_i', d_{-i})$ for all $i \in N$ and $d_i' \in D_i$. Using the result of Rosen [24], we will prove that under assumption DMR for every strategy profile (a, d), there is a partial equilibrium of the form $(a, \tilde{d})$. We say that $(a, \tilde{d})$ is an *associated* partial equilibrium to (a, d).

Proposition 1. *Let G be a demand allocation game. Under assumption DMR, for every strategy profile (a, d), there is an associated partial equilibrium $(a, \tilde{d})$.*

Proof. Pick an arbitrary strategy profile (a, d) of G. Consider the restricted demand allocation game $\tilde{G}$ with $\tilde{A}_i = \{a_i\}$. In $\tilde{G}$ the strategy space of each player reduces to the convex and closed interval $D_i \subseteq \mathbb{R}$. Using DMR, the utility function of each player is continuous and concave in d_i. By Rosen's existence theorem [24, Theorem 1], a pure Nash equilibrium of $\tilde{G}$ exists. Hence, each PNE of $\tilde{G}$ is an associated partial equilibrium to (a, d). $\square$

The following lemma will be important throughout this paper. It expresses the first-order optimality conditions of a partial equilibrium. The proof is straightforward and left to the reader.

Lemma 2. *Let (a, d) be a partial equilibrium. Then, for all $i \in N$ with $\alpha_i < \omega_i$ the following conditions hold: $\partial\, u_i(a, d) \,/\, \partial\, d_i \leq 0$ if $d_i = \alpha_i$, $\partial\, u_i(a, d) \,/\, \partial\, d_i = 0$ if $d_i \in (\alpha_i, \omega_i)$, and $\partial\, u_i(a, d) \,/\, \partial\, d_i \geq 0$ if $d_i = \omega_i$.*

For an action $b \in A$, we define the *active set* on action b under strategy profile (a, d) as $N^b(a, d) = \{i \in N : a_i = b\}$. We need the following lemma.

Lemma 3 (Uniqueness Lemma). *Let (a, d) and (a', d') be two partial equilibria of a demand allocation game satisfying* DMR*. Then,*

1. *$\ell^b(a, d) = \ell^b(a', d')$ for all $b \in A$ with $N^b(a, d) = N^b(a', d')$,*
2. *$\ell^b(a, d) \leq \ell^b(a', d')$ for all $b \in A$ with $N^b(a, d) \subseteq N^b(a', d')$.*

Proof. Obviously, it suffices to prove 2. Assume by contradiction that there is $b \in A$ with $\ell^b(a, d) > \ell^b(a', d')$ and $N^b(a, d) \subseteq N^b(a', d')$. This implies the existence of a player $i \in N^b(a, d)$ with $d_i > d'_i$. In particular, we have $\omega_i \geq d_i > d'_i \geq \alpha_i$. The conditions of Lemma 2 for a partial equilibrium give $\partial\, u_i(a, d) \,/\, \partial\, d_i \geq 0$ and $\partial\, u_i(a', d') \,/\, \partial\, d'_i \leq 0$. We get

$$0 \geq \frac{\partial\, u_i(a', d')}{\partial\, d'_i} = \frac{\partial\, v_i^b\big(d'_i, \ell_{-i}^b(a', d')\big)}{\partial\, d'_i} \overset{\mathsf{DMR}}{>} \frac{\partial\, v_i^b\big(d_i, \ell_{-i}^b(a, d)\big)}{\partial\, d_i} = \frac{\partial\, u_i(a, d)}{\partial\, d_i} \geq 0,$$

a contradiction. □

We are now ready to present a procedure for proving the existence of PNE. We claim that the following iterative contraction-switching procedure converges to a PNE.

1. Start with arbitrary strategy profile (a, d)
2. **Contraction phase:** Let $(a, \tilde{d})$ be an associated partial equilibrium
3. **Switching phase:** If there is a player i who can improve unilaterally, pick a best reply $(a'_i, d'_i) \in \arg\max_{(a''_i, d''_i) \in A_i \times D_i} u_i(a''_i, a_{-i}, d''_i, \tilde{d}_{-i})$, set $(a, d) = (a''_i, a_{-i}, d''_i, \tilde{d}_{-i})$ and proceed with 2. Else, return $(a, \tilde{d})$.

Note that in Step 2, we actually call an oracle that gives us an associated partial equilibrium. The oracle takes as input a restricted demand allocation game $\widetilde{G}$ and outputs an associated partial equilibrium. By Proposition 1 this is always possible.

In the following, we will show that this procedure ends after finitely many steps (involving finitely many calls of the oracle) and outputs a PNE. The following properties are the key to prove that the contraction-switching procedure terminates.

Lemma 4. *Let G be a demand allocation game satisfying* DMR, EXT *and* HOM, *let (a, d) be a partial equilibrium, let (a'_i, d'_i) be a best and better reply of player i and let $(a'_i, a_{-i}, \tilde{d})$ be an associated partial equilibrium. Then, the following properties hold.*

1. $\ell^{a_i'}(a_i', a_{-i}, d_i', d_{-i}) < \ell^{a_i}(a, d)$ *(Switching Property)*
2. $\ell^{a_i'}(a_i', a_{-i}, \tilde{d}) \leq \ell^{a_i'}(a_i', a_{-i}, d_i', d_{-i})$ *(Contraction Property)*
3. $\ell^{a_i}(a_i', a_{-i}, \tilde{d}) \leq \ell^{a_i}(a, d)$ *(Monotonicity Property)*

Proof. We begin proving the switching property. For the sake of a contradiction, assume $\ell^{a_i'}(a_i', a_{-i}, d_i', d_{-i}) \geq \ell^{a_i}(a, d)$. We consider the following three cases:

First case $d_i' > d_i$: As (a, d) is a partial equilibrium and $d_i < d_i' \leq \omega_i$, by Lemma 2 we have $0 \geq \partial\, u_i(a, d) \,/\, \partial\, d_i$. We calculate

$$0 \geq \frac{\partial\, u_i(a,d)}{\partial\, d_i} = \frac{\partial\, v_i(d_i, \ell_{-i}^{a_i}(a,d))}{\partial\, d_i} \stackrel{\mathsf{DMR}}{>} \frac{\partial\, v_i(d_i', \ell_{-i}^{a_i'}(a_i', a_{-i}, d_i', d_{-i}))}{\partial\, d_i'} = \frac{\partial\, u_i(a_i', a_{-i}, d_i', d_{-i}')}{\partial\, d_i'} \geq 0,$$

a contradiction. The equalities use the assumption HOM. The last inequality stem from the facts that (a_i', d_i') is a best reply of player i and that $d_i' > d_i \geq \alpha_i$.

Second case $d_i' = d_i$: Using $\ell_{-i}^{a_i'}(a_i', a_{-i}, d_i', d_{-i}') \geq \ell_{-i}^{a_i}(a, d)$ and assumptions EXT and HOM, we obtain

$$u_i(a_i', a_{-i}, d_i', d_{-i}) = v_i\big(d_i', \ell_{-i}^{a_i'}(a_i', a_{-i}, d_i', d_{-i})\big) \leq v_i\big(d_i, \ell_{-i}^{a_i}(a, d)\big) = u_i(a, d).$$

We derive that player i does not improve, a contradiction to the fact that (a_i', d_i') is a better reply of player i.

Third case $d_i' < d_i$: Consider the strategy (a_i, d_i') of player i. Observe that $\ell_{-i}^{a_i}(a, d_i', d_{-i}) < \ell_{-i}^{a_i}(a, d)$ as $d_i' < d_i$. We obtain

$$u_i(a, d_i', d_{-i}) = v_i\big(d_i', \ell_{-i}^{a_i}(a, d_i', d_{-i})\big) \stackrel{\mathsf{EXT}}{\geq} v_i\big(d_i', \ell_{-i}^{a_i'}(a_i', a_{-i}, d_i', d_{-i})\big) = u_i(a_i', a_{-i}, d_i', d_{-i}) > u_i(a, d),$$

where the equalities use the assumption HOM and the first inequality uses the assumption EXT. Thus, (a, d) is not a partial equilibrium, contradiction!

We proceed by proving the contraction property. For a contradiction, suppose that $\ell^{a_i'}(a_i', a_{-i}, \tilde{d}) > \ell^{a_i'}(a_i', a_{-i}, d_i', d_{-i}')$. Then, at least one of the following two cases holds: Either $\tilde{d}_i > d_i'$ or there is a player $j \in N^{a_i'}(a_i', a_{-i}, \tilde{d}) \setminus \{i\}$ with $\tilde{d}_j > d_j$. If $\tilde{d}_i > d_i'$, we have $\partial\, u_i(a_i', a_{-i}, d_i', d_{-i}') \,/\, \partial\, d_i' \leq 0$ using the fact that (a_i', d_i') was a best reply of player i and that $d_i' < \tilde{d}_i \leq \omega_i$. By the assumptions of decreasing marginal values, we obtain

$$0 \geq \frac{\partial\, u_i(a_i', a_{-i}, d_i', d_{-i})}{\partial\, d_i'} = \frac{\partial\, v_i(d_i', \ell_{-i}^{a_i'}(a_i', a_{-i}, d_i', d_{-i}))}{\partial\, d_i'} \stackrel{\mathsf{DMR}}{>} \frac{\partial\, v_i\big(\tilde{d}_i, \ell_{-i}^{a_i'}(a_i', a_{-i}, \tilde{d})\big)}{\partial\, \tilde{d}_i} = \frac{\partial\, u_i(a_i', a_{-i}, \tilde{d})}{\partial\, \tilde{d}_i} \geq 0,$$

a contradiction.

If there is on the other hand $j \in N^{a'_i}(a'_i, a_{-i}, \tilde{d}) \setminus \{i\}$ with $\tilde{d}_j > d_j$, then we have $\partial u_j(a,d) \,/\, \partial d_j \leq 0$ as (a,d) was a partial equilibrium and $d_j < \tilde{d}_j \leq \omega_j$. We then get the same contradiction as for player i.

The monotonicity property follows directly from Lemma 3. □

We are now ready to state and prove our main result.

Theorem 5. *For demand allocation games, assumptions* DMR, EXT, HOM *yield the existence of a PNE.*

Proof. By using the previous lemmas, we show that the contraction-switching procedure terminates for any given starting profile (a,d). First notice that there are only finitely many action vectors $a = (a_i)_{i \in N}$ as both the number of players and the number of actions is finite. We will show that each possible action vector is visited at most once in the contraction-switching procedure.

To this end, we consider for a strategy profile (a,d), the vector $L(a,d) = (\ell^{a_i}(a,d))_{i \in N}$. We shall prove that $L(a,d)$ strictly decreases with respect to the sorted lexicographical order $\prec_{\text{lex}}$ that is defined as follows. For two vectors $u, v \in \mathbb{R}^n_{\geq 0}$ we say that u is *sorted lexicographically* smaller than v, written $u \prec_{\text{lex}} v$, if there is an index $k \in \{1, \dots, n\}$ such that $u_{\pi(i)} = v_{\psi(i)}$ for all $i < k$ and $u_{\pi(k)} < v_{\psi(k)}$ where π and ψ are permutations that sort the vectors u and v non-increasingly, that is, $u_{\pi(1)} \geq u_{\pi(2)} \geq \cdots \geq u_{\pi(n)}$ and $v_{\psi(1)} \geq v_{\psi(2)} \geq \cdots \geq v_{\psi(n)}$. To see that $L(a,d)$ lexicographically decreases, let (a,d) be a partial equilibrium and let (a'_i, d'_i) be a best and better reply of player i. Denote by $(a'_i, a_{-i}, \tilde{d})$ the partial equilibrium associated with strategy profile $(a'_i, a_{-i}, d'_i, d_{-i})$. Clearly, for every player $j \in N \setminus \big(N^{a_i}(a,d) \cup N^{a'_i}(a,d)\big)$ we have $L_j(a,d) = L_j(a'_i, a_{-i}, d'_i, d_{-i})$. The switching property proven in Lemma 4 ensures that the load on the new action a'_i stays strictly below that of the old action a_i, that is, $\ell^{a'_i}(a'_i, a_{-i}, d'_i, d_{-i}) < \ell^{a_i}(a,d)$. The contraction property ensures that, after the new set of players on the new action a'_i settles to an associated partial equilibrium, the total demand will not increase, that is, $\ell^{a'_i}(a'_i, a_{-i}, \tilde{d}) \leq \ell^{a'_i}(a'_i, a_{-i}, d'_i, d_{-i})$. It follows that $\ell^{a'_i}(a'_i, r_{-i}, \tilde{d}) < \ell^{a_i}(a,d)$. Also, by the monotonicity property we have $\ell^{a_i}(a'_i, a_{-i}, \tilde{d}) \leq \ell^{a_i}(a,d)$. Thus, we have shown that the entry $L_i(\cdot)$ of player i strictly decreases and that none of the changed entries becomes larger than $L_i(a,d)$, hence, the vector $L(\cdot)$ lexicographically decreases after one iteration of the contraction-switching procedure. This fact, together with the uniqueness of the load vector proven in Lemma 3, implies that the algorithm never visits the same action vector twice and, thus, terminates after finitely many steps. □

Note that the existence result of Theorem 5 is tight; if one of the assumption three assumptions DMR, EXT, and HOM is dropped then we can construct a game satisfying the other two assumptions that does not have a PNE. We can also provide an example of a game satisfying DMR, EXT, and HOM that has an improvement cycle. Thus, demand allocation games are not potential games, in general. Formal proofs of the above results appear in the full version of this paper.

3.1 Two Player Demand Allocation Games

In this section, we turn to the case of two players. We will show that any two-player demand allocation game that satisfies the assumptions EXT and DMR possesses a PNE.

Theorem 6. *For two-player demand allocation games, assumptions EXT and DMR yield the existence of a PNE.*

Proof. We shall prove that the following procedure computes a PNE. Start with the empty strategy profile and let player 1 choose a best reply (a_1, d_1). Then, let player 2 choose a best reply (a_2, d_2) to (a_1, d_1). If $a_1 \neq a_2$, we have reached a PNE as EXT implies that player 1 has no interest in switching to action a_2. The only interesting case is $a_1 = a_2$. Let $\tilde{x} = (a_1, a_2, \tilde{d}_1, \tilde{d}_2)$ be an associated partial equilibrium to $x = (a_1, a_2, d_1, d_2)$. We first show that $\tilde{d}_1 \leq d_1$. For a contradiction, suppose $\tilde{d}_1 > d_1$. Because $d_1 < \tilde{d}_1 \leq \omega_1$, we have $\partial v_1^{a_1}(d_1, 0) \,/\, \partial d_1 \leq 0$ as (a_1, d_1) was a best reply. On the other hand, we have $\partial v_1^{a_1}(\tilde{d}_1, \tilde{d}_2) \,/\, \partial \tilde{d}_1 \geq 0$ as $\tilde{d}_1 > d_1 \geq \alpha_1$ and $\tilde{x}$ is a partial equilibrium. We obtain $0 \leq \partial v_1^{a_1}(\tilde{d}_1, \tilde{d}_2) \,/\, \partial \tilde{d}_1 < \partial v_1^{a_1}(d_1, 0) \,/\, \partial d_1 \leq 0$, by the assumption DMR, a contraction.

Next, we show $u_2(\tilde{x}) \geq u_2(x)$. To see this, note that $u_2(\tilde{x}) = v_2^{a_1}(\tilde{d}_2, \tilde{d}_1) \geq v_2^{a_1}(d_2, \tilde{d}_1) \geq v_2^{a_1}(d_2, d_1) = u_2(x)$, where the first inequality uses the fact that $\tilde{x}$ is a partial equilibrium and the second inequality stems from the assumption EXT and the fact that $\tilde{d}_1 \leq d_1$.

Because $u_2(\tilde{x}) \geq u_2(x)$ and (a_2, d_2) was a best reply of player 2, there is no improvement move of player 2 from $\tilde{x}$. If player 1 does not want to deviate as well, $\tilde{x}$ is a PNE and we are done. If on the other hand (a_1', d_1') is a best reply of player 1, we let player 1 deviate and let player 2 play a best reply (a_2', d_2'). Note that player 2 will only adapt her demand, that is $a_2' = a_2 = a_1$. It is shown in the proof of Lemma 3 that the equilibrium demand of a player does not increase as the load increases, thus, $d_2' \geq \tilde{d}_2$. Then, player 1 will not want to switch again to action a_1. Also player 2 will not deviate as her payoff may only decrease when adapting her demand. Hence, we have reached a PNE. □

Note that the above result is tight in the sense that if either DMR or EXT are dropped, then there exist two-player games without a PNE.

4 Examples

We now give several examples of games that fall into the class of demand allocation games.

Cournot Competition on Parallel Markets. Cournot games (cf. Cournot [3], Mas-Colell et al. [15] and Tirole [30]) are among the most fundamental models of strategic interaction between firms. In a Cournot game, players correspond to firms that produce a homogeneous product. In each strategy, each firm chooses its production quantity d_i out of a compact and convex interval $[\alpha_i, \omega_i]$ of allowable production quantities. The price for which these quantities are sold is given by a non-increasing market reaction function $P : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that maps the total supply of the market $\ell = \sum_{i \in N} d_i$ to the market price for selling the produced quantity. Given a strategy profile $d = (d_i)_{i \in N}$, the utility of firm i is given as $u_i(d) = P(\ell)\, d_i - C_i(d_i)$, where $C_i : [\alpha_i, \omega_i] \to \mathbb{R}$ is a non-decreasing production cost function of player i.

Demand allocation games contain a natural generalization of Cournot games that we term *Cournot games on parallel markets*. In such games, there is a set A of markets each endowed with a non-increasing market reaction function $P_a, a \in A$. The markets are called *identical* if $P_a = P_b$ for all $a, b \in A$. In each strategy profile, each player chooses both a market a_i out of a player-specific set $A_i \subseteq A$ of allowable markets and a production quantity $d_i \in [\alpha_i, \omega_i]$. Given a strategy profile (a, d), the utility of player i is then defined as $u_i(a, d) = P_{a_i}\big(\ell^{a_i}(a, d)\big)\, d_i - C_i(d_i)$. Cournot games on identical parallel markets with continuously differentiable and strictly concave market reaction function and continuously differentiable and convex production cost functions are demand allocation games satisfying assumptions EXT, DMR, and HOM and thus possess a PNE. For games with two players (originally studied by Cournot), a PNE exists even if HOM is violated.

Singleton Congestion Games. The class of congestion games is a well-studied class of games introduced by Rosenthal [25]. As congestion games with weighted players and/or player-specific costs may fail to have a PNE (see the counterexamples given in [6,7,14] for weighted congestion games and [16,17] for games with player-specific costs) many authors focused on singleton strategies. Here, a PNE is guaranteed to exists, even when players are weighted (see [1,4,5,9,26]) or costs are player-specific (see [13,16]). However, games with weighted players *and* player-specific costs need not possess a PNE [16].

In many situations, however, the assumption that the demand of each player is *fixed* is unrealistic. In a previous work [8], we studied congestion games with elastic demands. In that work, we show that affine or certain exponential cost functions yield the existence of a PNE. We did not study, however, the case of player-specific costs. Demand allocation games include *singleton congestion games with variable demands and player-specific costs* as a special case. In such games, the incentive of each player i to use higher demands is stimulated by a reward function $U_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$ that defines the reward received from the chosen demand. Given a strategy profile (a, d), the utility of player i is defined as $u_i(a, d) = U_i(d_i) - c_i^{a_i}(\ell^{a_i}(a, d))$, where $\ell^{a_i}(a, d) = \sum_{j \in N : a_j = a_i} d_j$ is the load of resource a_i under strategy profile (a, d). Singleton congestion games with variable demands and player-specific costs are demand allocation games. If reward functions are continuously differentiable and strictly concave functions and for each player all costs functions are equal, continuously differentiable, non-decreasing and convex they satisfy assumptions EXT, DMR, and HOM and thus possess a PNE. For two-player games, we can drop assumption HOM and still get the existence of a PNE.

Games in Biology. Consider population behavior in biology as described in the introduction. The food patches correspond to the actions and the population-specific costs ${c_i}^{a_i}(\ell^{a_i}(a,d))$ capture the rivalry for food supply. The size of population i is given by an inverse demand function, say $f_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is decreasing in the population specific costs. Thus, defining $v_i(d_i, \ell^{a_i}(a,d)) = \int_0^{d_i} f_i(z) - {c_i}^{a_i}(\ell^{a_i}(a,d) - d_i + z)\, dz$ models the tradeoff between food supply and population size, see also Milchtaich (cf. [16]) for a detailed discussion of congestion games used in biology. His actual model, however, involves fixed demands only.

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