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A UNIFIED FRAMEWORK FOR PRICING IN NONCONVEX RESOURCE ALLOCATION GAMES*

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Abstract. We consider a basic nonconvex resource allocation game, where the players' strategy spaces are subsets of $\mathbb{R}^m$ and cost functions are parameterized by some common vector $u \in \mathbb{R}^m$ and, otherwise, only depend on their own strategy choice. A strategy of a player can be interpreted as a vector of resource consumption and a joint strategy profile naturally leads to an aggregate consumption vector. Resources can be priced, that is, the game is augmented by a price vector $\lambda \in \mathbb{R}^m_{\geq 0}$ and players have quasi-linear overall costs, meaning that in addition to the original costs, a player needs to pay the corresponding price per consumed unit. We investigate the following question: for which aggregated consumption vectors u can we find prices λ that induce an equilibrium realizing the targeted consumption profile? For answering this question, we revisit a duality-based framework and derive a new characterization of the existence of such u and λ using convexification techniques. Our characterization implies the following result: If strategy spaces of players are bounded linear mixed-integer sets and the cost functions are linear or even concave, the equilibrium existence problem reduces to solving a well-structured LP. We then consider aggregate formulations assuming that cost functions are additive over resources and homogeneous among players. We derive a characterization of enforceable consumption vectors u, showing that u is enforceable if and only if u is a minimizer of a certain convex optimization problem with a linear functional. We demonstrate that this framework can unify parts of four largely independent streams in the literature: tolls in transportation systems, Walrasian equilibria, trading networks, and congestion control. Besides reproving existing results we establish new enforceability results for these domains as well.

Key words. pricing, nonconvex games, resource allocation, Nash equilibrium, Lagrangean duality, Wardrop equilibrium, Walras equilibrium

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1. Introduction.

Distributed resource allocation problems can be found in several application domains, including traffic and telecommunication networks. Here, a finite (or infinite) number of players interact strategically, each optimizing their individual objective function. The corresponding allocation of resources is then usually determined by an equilibrium solution of the underlying strategic game. A central question in all these areas concerns the problem of how to incentivize players in order to use the (scarce) resources optimally. One key approach in all named application areas is the concept of *pricing resources* according to their usage. Every resource comes with an anonymous price per unit of consumption and defining the "right" prices thus offers the chance of inducing equilibria with optimal or efficient resource usage. Prominent examples are toll pricing in transportation networks [2, 32, 50], congestion pricing in telecommunication networks [27, 43], and *market pricing* in economics [4, 5].

In this paper, we will introduce a generic nonconvex model of pricing in resource allocation games with quasi-linear costs that subsumes several of the above mentioned applications as a special case. The term *quasi-linear* refers to the standard assumption that the overall cost depends linearly on the prices. In the following, we first introduce the model formally and then give an overview of the main results and related work.

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1.1. The model. A resource allocation model is compactly described by a tuple $I = (N, E, X, g, \pi)$, where $N = \{1, \ldots, n\}$ describes a nonempty finite set of players and $E = \{1, \ldots, m\}$ denotes a nonempty finite set of resources. The set $X := \times_{i\in N} X_i$ describes the combined strategy space of the players where $X_i \subseteq \mathbb{R}^m$ is the nonempty strategy space of player $i \in N$. The vector $x_i = (x_{ij})_{j\in E} \in X_i$ is a strategy profile of player $i \in N$ and the entry $x_{ij} \in \mathbb{R}$ can be interpreted as the level of resource usage of player i for resource j. For every player $i \in N$, the function $g_i : \mathbb{R}^m \to \mathbb{R}^m, x_i \mapsto g_i(x_i)$ is mapping the strategy profile to a vector of the actual *resource consumption* of player i. The function $g = (g_i)_{i\in N}$ allows one to model player-specific characteristics such as weights. For both x_i and g_i negative values are allowed. We call the vector $x = (x_i)_{i\in N}$ a *strategy distribution*. Given $x \in X$, we can define the *load* on resource $j \in E$ as $\ell_j(x) := \sum_{i\in N} g_{ij}(x_i)$, where g_{ij} is the jth component of g_i. We denote the set of feasible loads, or the *load space*, by the Minkowski sum $\ell(X) := \sum_{i\in N} g_i(X_i)$ with $g_i(X_i) := \{g_i(x_i) \mid x_i \in X_i\}$.

In the following, we introduce properties of the players' cost functions needed for our main results. We assume that cost functions are parameterized by an *exogenously given vector* $u \in \mathbb{R}^m$ and depend on the own strategy vector only. We will consider cost minimization games, but all results carry over to utility maximization games by reversing the sign of the objective function. For $u \in \mathbb{R}^m$, the cost of a player $i \in N$ at the strategy profile $x_i \in X_i$ is given by $\pi_i(u, x_i)$ for some function $\pi_i : \mathbb{R}^m \times X_i \to \mathbb{R}$. For a model I, a vector $u \in \mathbb{R}^m$ leads to a special type of strategic game $G(u) = (N, X, (\pi_i(u, x_i))_{i\in N})$ which we call *decoupled* as the players' cost functions solely depend on their own strategy choice. For $G(u)$, the vector u can be interpreted as the induced load of an equilibrium x^*, that is, $u = \ell(x^*)$. In this regard, players are assumed to be *load taking* in the sense that they assume not being able to influence the global load vector u by their own strategy x_i. We will later also consider models in which a functional dependency of the strategy choice on the induced load is allowed.

1.2. Pricing in resource allocation games. Given a resource allocation model I, we are concerned with the problem of defining *prices* $\lambda_j \geq 0, j \in E$, on the resources in order to incentivize an efficient usage of the resources. If player i uses resource j at consumption level $g_{ij}(x_i)$, she needs to pay $\lambda_j g_{ij}(x_i)$. For any vector $u \in \mathbb{R}^m$, the quantities $\pi_i(u, x_i)$ and $\lambda^\intercal g_i(x_i)$ are assumed to be normalized to represent the same unit (say money in Euro) and we assume that the private cost functions are quasi-linear: $\pi_i(u, x_i) + \lambda^\intercal g_i(x_i)$. We write $G(u, \lambda) = (N, X, (\pi_i(u, x_i) + \lambda^\intercal g_i(x_i))_{i\in N})$ as the resulting *decoupled game augmented with prices* λ. If the parameter $u = (u_j)_{j\in E} \in \mathbb{R}^m$ represents a targeted load vector, then the task is to find prices $\lambda \in \mathbb{R}^m_{\geq 0}$ so that an equilibrium of the game $G(u, \lambda)$ realizes this load. Note that in analogue to the load taking property, players are assumed to be price taking, that is, they assume that their strategy has no influence on the prices λ (cf. [4, 27, 36]).

Definition 1.1 (enforceability). *Let I be a resource allocation model.*

1. *A vector $u \in \mathbb{R}^m$ is* enforceable *for I if there is a tuple $(x^*, \lambda^*) \in X \times \mathbb{R}^m_{\geq 0}$ such that the following two conditions are satisfied:*
   (a) $\ell_j(x^*) = u_j$ *for all* $j \in E$.
   (b) $x^*_i \in \arg\min_{x_i\in X_i}\{\pi_i(u, x_i) + (\lambda^*)^\intercal g_i(x_i)\}$ *for all* $i \in N$.
2. *A vector $u \in \mathbb{R}^m$ is* weakly enforceable *for I if there is a tuple (x^*, λ^*) that satisfies* (1b) *but* (1a) *is replaced with $\ell(x^*) \leq u$ and λ^* satisfies $\ell_j(x^*) < u_j \Rightarrow \lambda^*_j = 0$ for all $j \in E$.*

If the model I is clear from the context, we say that $u \in \mathbb{R}^m$ is (weakly) enforceable.

Condition (1a) requires that x^* realizes the targeted load $\ell(x^*) = u$ while condition (1b) implements x^* as a pure Nash equilibrium of the game $G(u, \lambda^*)$.

The concept of enforceability was first used by Fleischer, Jain, and Mahdian [15] in the context of toll pricing in nonatomic congestion games (cf. section 6). The definition of weakly enforceable load vectors is motivated by applications in market games (see the example described in section 7.1), for which outcomes are interesting that do not use the full targeted load at equilibrium and resources with slack have zero price.

1.3. Overview of results and organization of this paper. In section 2, we first revisit a duality-based framework based on a specific optimization problem which we call *master problem* $\mathrm{P}(u)$. We show in Theorem 2.3 that $u \in \mathbb{R}^m$ is enforceable by (x^*, λ^*) if and only if (x^*, λ^*) yields zero duality gap for the master problem and x^* satisfies $\ell(x^*) = u$. The if-direction of this characterization is well-known; see [3, 36, 43]. The only-if direction roughly corresponds to the first welfare theorem in economics saying that every pricing equilibrium maximizes social welfare; however, Theorem 2.3 asks for a stronger condition (strong duality) and the fact that an inequality must be tight leading to a complete characterization of enforceability.

Convexification. While the above result holds for general nonconvex problems, checking whether or not a nonconvex master problem exhibits zero duality gap may be difficult. In this regard, in section 3 we introduce a convex relaxation I^{conv} of I, where the strategy set and cost function of each player are convexified. In Theorem 3.2 we prove that u is enforceable for I via (x^*, λ^*) if and only if $(g(x^*), \lambda^*)$ enforces u for the convexified model I^{conv} and the respective cost functions coincide.

LP-based characterizations of enforceability. We then turn to a special class of models which allow for an LP-based characterization of enforceability. Theorem 4.2: If every Lagrangian-dual function value of the master problem can be obtained by minimizing the Lagrangian over a fixed and finite set of strategy distributions, then u is enforceable if and only if a corresponding LP has the same optimal objective value as $\mathrm{P}(u)$ and the latter admits optimal solutions x^* with $\ell(x^*) = u$. We then show in Corollary 4.5 that this theorem applies to the important special cases of *finite* and *concave* models, that are, models with either finite strategy spaces or strategy spaces with finitely generated convex hulls and concave costs. The dual of the above LP can be solved in polynomial time via the ellipsoid method if there is an efficient separation oracle. Thus, for several interesting game classes, one can efficiently decide whether or not u is enforceable. The complexity of the separation oracle versus the master problem $\mathrm{P}(u)$ can then be used to establish impossibility results for the enforceability using complexity-theoretic assumption (like $P \neq NP$). This connection has been discovered first by Roughgarden and Talgam-Cohen [41] in the context of pricing problems for Walrasian equilibria. Our more general characterization allows one to translate this approach to other domains (see, for instance, Proposition 7.9 below).

Aggregated games. We introduce in section 5 the class of *aggregated games* where cost functions are additive over resources and homogeneous among players—an assumption that is satisfied for most congestion games (cf. [40]). In Theorem 5.4, we derive a complete characterization of enforceability: u is enforceable if and only if u is a minimizer of a certain linear function over the convexified load space.

Application in congestion games. With these results and methods at hand, we first apply the framework to congestion games; see section 6. We consider the problem of defining tolls in order to enforce certain load vectors as equilibria. For nonatomic network games, we reprove and generalize in Corollary 6.2 an LP-based characterization of enforceable load vectors by Yang and Huang [50], Fleischer, Jain, and Mahdian [15], Karakostas and Kolliopoulos [26], and Marcotte and Zhu [32].

Then we turn to atomic congestion games where the strategy spaces of players are integral. For weighted congestion games with nondecreasing homogeneous cost functions, we apply Theorem 5.4 to show that any load vector that minimizes a strictly increasing and concave function over the feasible load space is enforceable. To the best of our knowledge, this is the first enforceability result for weighted congestion games with arbitrary nondecreasing cost functions. As a further consequence of Theorem 5.4, we show that every u minimizing a strictly increasing function over the feasible load space can be enforced if the convex hull of the feasible load space is sufficiently "well-behaved" (it must be box-integral and decomposable). It turns out that for a wide class of unweighted congestion games (matroid games, single-source network games, r-arborescences, matching games, and more) the convexified feasible load space is indeed well behaved leading to existence results of enforcing tolls (Corollary 6.7).

Then, we study the more challenging case of atomic congestion games with nondecreasing player-specific cost functions. We prove that for matroid congestion games, one can obtain an existence result using the integrality of a polymatroid intersection polytope (Corollary 6.8). To the best of our knowledge, this is the first existence result of enforcing tolls for congestion games with player-specific cost functions.

Further applications. We finally give in section 7 further applications related to the existence of prices supporting (Walrasian) equilibria in market games, trading networks, and congestion control in communication networks. Let us remark that for most of these application domains, except for trading networks, there already exist prior results based on formulating an appropriate master problem and using strong duality in order to generate corresponding prices. However, these approaches were developed separately (even in different decades) and are tailored to their specific setting. We give in this paper a unified view on these approaches which—apart from being more general—provides a more systematic treatment leading to new results for some of these application areas.

Besides the applications mentioned so far, our results also spurred some new interest in the domain of electricity markets exhibiting nonconvexities due to integrality conditions or nonconvex physical laws of the transmission network. Gröbel et al. [19] used our framework to devise an algorithm that computes equilibrium prices for such nonconvex models whenever such prices exist.

1.4. Related work. As outlined in the introduction, the topic of pricing resources concerns different streams of literature and it seems impossible to give a complete overview here. Lagrangian multipliers date back to the 18th century and their use in terms of *shadow prices* measuring the change of the optimal value function for marginal changes of the right-hand sides of constraints is well known—assuming some constraint qualification conditions; see, for instance, Boyd and Vandenberghe [6].

The first theorem, Theorem 2.3, relies on a decomposition property of the Lagrangian (for separable problems) and the use of Lagrange multipliers for pricing the resources. This approach is by no means new and has been developed in several facets before; see, for instance, Dantzig and Wolfe [12] and Bertsekas and Gallagher [3]. Dantzig and Wolfe [12] used this principle for their celebrated decomposition framework for solving certain linear (integer) programming problems. Bertsekas and Gallagher [3], Palomar and Chiang [36], and Scutari et al. [43] described how the Lagrangian of a general separable optimization problem can be decomposed into independent problems. These works already describe the close connection between strong duality and the existence of enforcing dual prices. In particular, Palomar and Chiang [36] point out that a priori no convexity conditions are needed for such a duality based characterization. In this paper, we contribute to this literature by deriving (computationally verifiable) conditions under which the master problem admits solutions with zero duality gap.

One further subtle difference of the above model to our formulation is the explicit parameterization of the cost functions $\pi_i(u, x_i)$ with respect to the vector u. This degree of freedom allows one to model dependencies of targeted load vectors with respect to the intrinsic cost—a prime example appears in nonatomic congestion games, where the cost function of an agent *only* depends on the aggregated load vector. Moreover, this dependency allows one to model *externalities* with respect to allocations which are not directly possible in the previous formulations.

Convexification of nonconvex models. The idea of convexifying a nonconvex economic model dates back to the late sixties starting with the work of Shapley and Shubik [46] and Starr [47]. Starr [47] considered a standard Arrow–Debreu exchange economy without convexity assumptions on production or consumption sets nor on the preference ordering. The analysis of the existence of competitive market equilibria is based on a *convexified economy* in which the convex hull of production or consumption sets and the convex hull of the epigraph with respect to the preference orderings are considered (see also later related works of Henry [23], Moore, Whinston, and Wu [33], and Svensson [48]). By separation arguments, this convexified economy permits a competitive equilibrium (called a synthetic convex equilibrium). A quasi-equilibrium lives in the original nonconvex model and is defined as a closest approximation within w.r.t. the synthetic equilibrium. With the Shapley–Folkman theorem the approximation guarantee can be parameterized in terms of the number of commodities or number of traders involved. The approach of convexifying a game in this paper is qualitatively similar to that of Starr with the difference that we use the separability of cost functions to define a master problem. This way, we obtain a complete characterization of enforceable vectors u by resorting to Lagrangian duality theory.

Motivated by applications in cognitive radio systems, Pang and Scutari [37, 38, 44] studied the existence, uniqueness, and computability of Nash equilibria for a class of nonconvex games. In these works, the idea that any equilibrium must satisfy first order optimality conditions is developed leading to a concept also termed *quasi-Nash equilibrium* as a relaxation of Nash equilibrium. It is shown that any Nash equilibrium must be a *quasi-Nash equilibrium*. Sufficient conditions for the existence of quasi-Nash equilibria are derived using regularization techniques. By exploiting the single-valuedness of the best response maps, the existence of Nash equilibria for the original game follows. Very recently and independently, Chao [7] and Hümbs, Martin, and Schewe [25] studied pricing problems for nonconvex models in the realm of electricity markets. Both models are less general than our model, because the utility functions are assumed to be linear (or concave in [7]) and only the presence of bounded linear mixed-integer sets leads to nonconvexity issues (see also the earlier work of Ruiz, Conejo, and Gabriel [42]).

2. Connection to Lagrangean duality in optimization. In the following, we prove our main results in the realm of cost minimization but it should be clear that all arguments carry directly over to the maximization case.

For a model I and $u \in \mathbb{R}^m$, we define the following minimization problem that we call the *master problem*:

$$
\begin{aligned}
&(\mathrm{P}(u)) \qquad && \inf_x \pi(u,x) \\
&(2.1) && \text{s.t.: } \ell(x) \le u, \\
& && x \in X,
\end{aligned}
$$

where the objective function is defined as $\pi(u,x) := \sum_{i\in N} \pi_i(u,x_i)$. The Lagrangian function for problem $\mathrm{P}(u)$ becomes $L(x,\lambda) := \pi(u,x) + \lambda^\intercal(\ell(x) - u)$, $\lambda \in \mathbb{R}^m_{\ge 0}$, and we can define the Lagrangian-dual as $\mu : \mathbb{R}^m_{\ge 0} \to \mathbb{R}$, $\mu(\lambda) = \inf_{x\in X} L(x,\lambda) = \inf_{x\in X}\{\pi(u,x) + \lambda^\intercal(\ell(x)-u)\}$. We assume that $\mu(\lambda) = -\infty$ if $L(x,\lambda)$ is not bounded from below on X. The *dual problem* is defined as

$$
\sup_{\lambda \ge 0} \mu(\lambda). \tag{2.2}
$$

Definition 2.1. *Problem* $\mathrm{P}(u)$ *has zero duality gap if there is* $\lambda^* \in \mathbb{R}^m_{\ge 0}$ *and* $x^* \in X$ *with* $\pi(u,x^*) = \mu(\lambda^*)$. *In this case, we say that the pair* (x^*,λ^*) *is primal-dual optimal.*

If problem $\mathrm{P}(u)$ has zero duality gap, the two solutions $\lambda^* \in \mathbb{R}^m_{\ge 0}$ and $x^* \in X$ are optimal for their respective problems (2.2) and $\mathrm{P}(u)$ and, thus, the appearing infima/suprema become a minimum/maximum, respectively. For this situation, we now recall a key structure, namely that $\min_{x\in X} L(x,\lambda)$ decomposes into independent problems, one for each player. This decomposition step is classical for separable optimization problems; see Bertsekas and Gallagher [3].

Lemma 2.2. *Let* $\lambda \in \mathbb{R}^m_{\ge 0}$. *For a problem of type* $\mathrm{P}(u)$, *the following holds true:*

$$
x^* \in \arg\min_{x\in X} L(x,\lambda) \Leftrightarrow x_i^* \in \arg\min_{x_i \in X_i} \{\pi_i(u,x_i) + \lambda^\intercal g_i(x_i)\} \quad \textit{for all } i \in N. \tag{2.3}
$$

We obtain the following result.

Theorem 2.3. *The following equivalences hold for I:*

1. *A vector* $u \in \mathbb{R}^m$ *is enforceable via* (x^*,λ^*) *if and only if* (x^*,λ^*) *has zero duality gap for* $\mathrm{P}(u)$ *and* x^* *satisfies* (2.1) *with equality.*
2. *A vector* $u \in \mathbb{R}^m$ *is weakly enforceable via* (x^*,λ^*) *if and only if* (x^*,λ^*) *has zero duality gap for* $\mathrm{P}(u)$.

Proof. For the proof we only show 2., since 1. follows from 2. as the additional condition $\ell(x^*) = u$ holds true for both statements of 1.

For 2.: $\Leftarrow$: Assume there are $\lambda^* \in \mathbb{R}^m_{\ge 0}, x^* \in X$ with $\ell(x^*) \le u$ so that $\mu(\lambda^*) = \pi(u,x^*)$. We obtain

$$
\begin{aligned}
\mu(\lambda^*) &= \min_{x\in X}\{\pi(u,x) + (\lambda^*)^\intercal(\ell(x)-u)\} \le \pi(u,x^*) + (\lambda^*)^\intercal(\ell(x^*)-u) \\
&\le \pi(u,x^*) = \mu(\lambda^*).
\end{aligned}
$$

Hence, all inequalities must be tight leading to $(\lambda^*)^\intercal(\ell(x^*)-u) = 0$ as claimed. It remains to prove condition (1b). With $x^* \in \arg\min_{x\in X} L(x,\lambda^*)$ we get

$$
x^* \in \arg\min_{x\in X} L(x,\lambda^*) \underset{Lem.\ 2.2}{\Leftrightarrow} x_i^* \in \arg\min_{x_i\in X_i}\{\pi_i(u,x_i) + (\lambda^*)^\intercal g_i(x_i)\} \quad \text{for all } i \in N.
$$

$\Rightarrow$: Let $u \in \mathbb{R}^m$ be weakly enforceable via $(x^*, \lambda^*) \in X \times \mathbb{R}^m_{\geq 0}$, that is, $\ell(x^*) \leq u, (\lambda^*)^\intercal(\ell(x^*) - u) = 0$ and $x_i^* \in \arg\min_{x_i \in X_i}\{\pi_i(u, x_i) + (\lambda^*)^\intercal g_i(x_i)\}$ for all $i \in N$. We calculate

$$\begin{aligned} \mu(\lambda^*) &= \inf_{x \in X}\{\pi(u,x) + (\lambda^*)^\intercal(\ell(x) - u)\} \\ &= \pi(u,x^*) + (\lambda^*)^\intercal(\ell(x^*) - u) \qquad (2.4) \\ &= \pi(u,x^*), \qquad (2.5) \end{aligned}$$

where (2.4) follows from Lemma 2.2 and (2.5) uses the condition $(\lambda^*)^\intercal(\ell(x^*) - u) = 0$. Hence, strong duality holds for the pair (x^*, λ^*). ∎

As mentioned before, the if-direction of the above characterizations are well known in the literature; see, e.g., [3, 27, 36, 43]. We remark here that the theorem does not rely on any assumption on the feasible sets X_i nor on the functions $\pi_i(u, x_i), i \in N$, but only on the duality gap of $\mathrm{P}(u)$. In the optimization literature, several classes of optimization problems are known to have zero duality gap even *without* convexity of feasible sets and objective functions; see, for instance, Zheng et al. [51].

In cost minimization games, the feasible sets X_i usually contain some sort of covering conditions on the resource consumption. For example, in network routing one needs to send some prescribed amount of flow. In network design problems, one is interested in installing just enough capacity on edges so as to support certain connectivity restrictions among given source-destination pairs; see [11, 18, 31]. In this regard, we introduce a natural candidate set of vectors u for which we know that any feasible solution satisfying (2.1) does so with equality.

Definition 2.4 (Fleischer, Jain, and Mahdian [15]). *A vector $u \in \mathbb{R}^m$ is called* minimal *w.r.t. $\ell(X)$ if $\{\tilde{u} \in \mathbb{R}^m : \tilde{u} \leq u\} \cap \ell(X) = \{u\}$ holds.*

The above concept has been introduced by Fleischer, Jain, and Mahdian [15] in the context of enforcing tolls for nonatomic congestion games. They used the term *minimally feasible*. An equivalent characterization of minimality is given in the following observation.

Remark 2.5. A vector $u \in \mathbb{R}^m$ is minimal w.r.t. $\ell(X)$ if and only if there exists a strictly increasing[1] function $h : \mathbb{R}^m \to \mathbb{R}$ such that $u \in \arg\min_{\tilde{u} \in \ell(X)} h(\tilde{u})$.

The if-direction of the statement clearly holds. For the only-if direction, one may simply define a function h as the sum of strictly increasing functions $h_j : \mathbb{R} \to \mathbb{R}, j \in E$, which are for each $j \in E$, respectively, sufficiently flat until the point u_j and then jump to a value large enough. The above characterization is, for instance, relevant for the aforementioned network design problems, where the edge installation cost is usually a strictly increasing cost function. Hence, an optimal vector minimizing the sum of installation costs (subject to configuration constraints) is minimal in the sense of Definition 2.4.

Corollary 2.6. *Let $u \in \mathbb{R}^m$ be minimal w.r.t. $\ell(X)$. Then, the following two statements are equivalent:*

1. *u is enforceable via price vector $\lambda^* \in \mathbb{R}^m_{\geq 0}$ and $x^* \in X$.*
2. *(x^*, λ^*) satisfies $\pi(u, x^*) = \mu(\lambda^*)$.*

The only difference in Theorem 2.3 is that by minimality of u, we get $\ell(x) = u$ for any feasible solution of $\mathrm{P}(u)$; therefore, tightness of inequality (2.1) is already satisfied.

[1] We call a function $h : \mathbb{R}^m \to \mathbb{R}$ strictly increasing if for all $u^*, \tilde{u} \in \mathbb{R}^m$ with $u^* \leq \tilde{u}$ and $u_j^* < \tilde{u}_j$ for at least one $j \in \{1, \ldots, m\}$ the inequality $h(u^*) < h(\tilde{u})$ holds.

3. Convexified games. So far, the strategy spaces $X_i, i \in N$ and the cost functions $\pi_i(u,\cdot), i \in N$, of a model I were not restricted and are allowed to be nonconvex. For example, integrality restrictions in $X_i \subseteq \mathbb{Z}^m, i \in N$, are allowed. In what follows, we connect I with a related convexified resource allocation model I^{conv}. With this convexification, it follows that the duals of the original master problem $\mathrm{P}(u)$ and that of the convexified model are equal. With this insight, the characterization of enforceable vectors u can (in some cases) be reduced to a more tractable convex problem. The overall idea of convexifying a (nonconvex) optimization problem is quite old and belongs to the broad field of *global optimization*. Let us refer here to standard textbooks of the late seventies such as that of Horst and Tuy [24, section 4.3] or Shapiro [45, section 5]. For an overview on duality theory of general nonconvex programs, we refer to the work of Lemaréchal and Renaud [30].

For $Z \subseteq \mathbb{R}^m$, denote $\operatorname{conv}(Z) := \cap\{K \subseteq \mathbb{R}^m | Z \subseteq K, K \text{ convex}\}$ the *convex hull* of Z.

DEFINITION 3.1. *For a resource allocation model $I = (N, E, X, g, \pi)$ and load u, the associated convexified model I^{conv} is defined as*

$$I^{\text{conv}} = (N, E, X^{\text{conv}}, g^{\text{conv}}, (\phi_i)_{i\in N}),$$

where $X^{\text{conv}} := \times_{i\in N} \operatorname{conv}(g_i(X_i))$, $g^{\text{conv}}(\bar{x}) := \bar{x}$, and for all $\tilde{u} \in \mathbb{R}^m$

$$\phi_i(\tilde{u},\cdot) : \operatorname{conv}(g_i(X_i)) \to \mathbb{R} \cup \{-\infty\}$$

$$\bar{x}_i \mapsto \inf_{\alpha_{ik}, x_i^k} \left\{ \sum_{k=1}^{m+1} \alpha_{ik} \pi_i(u, x_i^k) \,\middle|\, \begin{array}{l} \sum_{k=1}^{m+1} \alpha_{ik} g_i(x_i^k) = \bar{x}_i, \alpha_i \in \Lambda, \\ x_i^k \in X_i, 1 \le k \le m+1 \end{array} \right\}, \tag{3.1}$$

in which $\Lambda := \{\alpha \in \mathbb{R}^{m+1}_{\ge 0} | \mathbf{1}^\intercal \alpha = 1\}$. Note that ϕ_i is constant in $\tilde{u} \in \mathbb{R}^m$.

THEOREM 3.2. *Let $x^* \in X$, $\lambda^* \in \mathbb{R}^m_{\ge 0}$. The following statements are equivalent.*
1. *u is enforceable for I via (x^*, λ^*).*
2. *$\phi_i, i \in N$ are real-valued functions, u is enforceable for I^{conv} via $(g(x^*), \lambda^*)$, and $\phi(u, g(x^*)) = \pi(u, x^*)$ holds.*

The equivalence remains true by replacing the term "enforceable" with "weakly enforceable".

Proof. We first derive another description of the dual for the master problem $\mathrm{P}(u)$ of I. We get for all $\lambda \in \mathbb{R}^m_{\ge 0}$

$$\mu(\lambda) = \inf_{x_i \in X_i, i \in N} \sum_{i\in N} \pi_i(u, x_i) + \lambda^\intercal(\ell(x) - u)$$

$$= \sum_{i\in N} \inf_{\alpha_{ik}, x_i^k} \left\{ \sum_{k=1}^{m+1} \alpha_{ik} \left[\pi_i(u, x_i^k) + \lambda^\intercal g_i(x_i^k)\right] \,\middle|\, \begin{array}{l} \alpha_i \in \Lambda, x_i^k \in X_i, \\ 1 \le k \le m+1 \end{array} \right\} - \lambda^\intercal u \tag{3.2}$$

$$= \inf_{\bar{x} \in X^{\text{conv}}} \sum_{i\in N} \phi_i(u, \bar{x}_i) + \lambda^\intercal(\ell(\bar{x}) - u), \tag{3.3}$$

where (3.2) follows by the linearity of $\alpha_i \mapsto \sum_{k=1}^{m+1} \alpha_{ik}[\pi_i(u, x_i^k) + \lambda^\intercal g_i(x_i^k)]$. Equation (3.3) follows as $\sum_{k=1}^{m+1} \alpha_{ik} \lambda^\intercal g_i(x_i^k) = \lambda^\intercal \sum_{k=1}^{m+1} \alpha_{ik} g_i(x_i^k)$ holds.

Now we are ready to prove 1.⇔2..

1.⇒2.: By Theorem 2.3, we have $\mu(\lambda^*) = \pi(u, x^*) > -\infty$, and thus, $\phi_i, i \in N$, need to be real-valued by the above description of μ. Subsequently, I^{conv} belongs to the resource allocation model defined in section 1.1. The dual $\mu^{\text{conv}}(\lambda)$ of the master problem $\mathrm{P}(u)$ of I^{conv} is then given by the expression in (3.3). We get $\pi(u,x^*) = \mu(\lambda^*) = \mu^{\text{conv}}(\lambda^*) \leq \phi(u,g(x^*))$, where the last inequality follows from weak duality. Since $\phi(u,g(x)) \leq \pi(u,x)$ for all $x \in X$, we get $\mu^{\text{conv}}(\lambda^*) = \phi(u,g(x^*))$ and $\phi(u,g(x^*)) = \pi(u,x^*)$. Thus the result follows by Theorem 2.3 and the fact that the load of x^* in I equals the load of $g(x^*)$ in I^{conv}.

2.⇒1.: As $\phi_i, i \in N$ are real-valued, I^{conv} belongs to the resource allocation model defined in section 1.1. Thus, the result follows by Theorem 2.3 together with $x^* \in X$, $\pi(u,x^*) = \phi(u,g(x^*))$, $\mu^{\text{conv}}(\lambda^*) = \mu(\lambda^*)$, and using that the respective induced loads of x^* in I and $g(x^*)$ in I^{conv} coincide. ∎

Remark 3.3. In the case of ϕ_i being real-valued, it follows that the function $g_i(X_i) \to \mathbb{R}$, $\bar{x}_i \to \inf_{x_i \in X_i, g_i(x_i) = \bar{x}_i} \pi_i(u,x_i)$ is also real-valued and has $\phi_i(u,\cdot)$ as its convex envelope which, in turn, implies that $\phi_i(u,\cdot)$ is convex on $\operatorname{conv}(g_i(X_i))$; see Horst and Tuy [24, section 4.3] for the concept of convex envelopes.

4. LP-based characterizations of enforceability. We now discuss a special class of models I, which allows for an LP-based characterization of enforceability. The main property needed is a special structure of the Lagrangian-dual function of the master problem $\mathrm{P}(u)$.

Assumption 4.1. For every $i \in N$, there exist $\{x_i^1, \dots, x_i^{k_i}\} \subseteq X_i$ for some $k_i \in \mathbb{N}$ such that the dual of $\mathrm{P}(u)$ may be represented as follows:

$$\mu(\lambda) = \min_{x_i \in \{x_i^1, \dots, x_i^{k_i}\}, i \in N} \sum_{i \in N} \pi_i(u,x_i) + \lambda^\intercal(\ell(x) - u).$$

Clearly, this assumption is fulfilled in the important case of *finite* models, where the strategy sets are finite point sets (see Figure 1 left). We will show in Corollary 4.5 that this assumption also holds for *concave* models, that is, models where the convex hull of each $X_i, i \in N$ is finitely generated and the functions $g_i, \pi_i(u,\cdot), i \in N$ are concave on $\operatorname{conv}(X_i)$ (see Figure 1 right).

For a model I fulfilling Assumption 4.1, we define the following LP in the variable $\alpha = (\alpha_i)_{i \in N}$:

$$(\mathrm{LP}(u)) \qquad \min_{\alpha} \sum_{i \in N} \pi_i^\intercal \alpha_i$$

$$(4.1) \qquad \text{s.t.: } \ell(\alpha) \leq u,$$

$$\alpha_i \in \Lambda_i, i \in N,$$

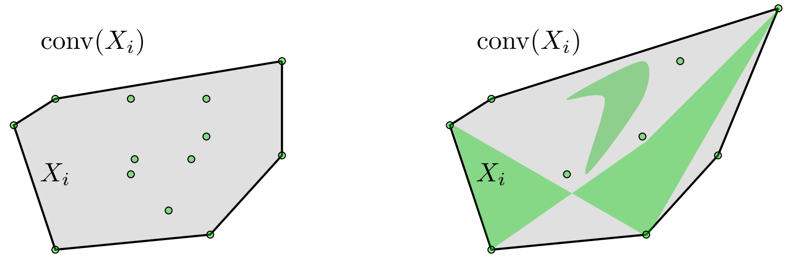

FIG. 1. *Left is the scenario of X_i consisting of a finite point set. Right, X_i may consist of connected components (in green) and isolated points but the convex hull is assumed to be finitely generated and additionally $g_i, \pi_i(u,\cdot)$ is assumed to be concave on* $\operatorname{conv}(X_i)$. *(Figure in color online.)*

where $\pi_i := (\pi_i(u, x_i^k))_{k\in\{1,\dots,k_i\}}$, $\ell(\alpha) := \sum_{i\in N}\sum_{k\in\{1,\dots,k_i\}} \alpha_{ik} g_i(x_i^k)$, and $\Lambda_i := \{\alpha_i \in \mathbb{R}^{k_i}_{\geq 0} | \mathbf{1}^\intercal \alpha_i = 1\}, i \in N$.

THEOREM 4.2. *Let I be a model for which Assumptions 4.1 holds. Then, the following statements are equivalent:*

1. *The vector u is enforceable for I.*
2. *There exists $x^* \in X$ with $\ell(x^*) = u$ and an optimal solution α^* of LP(u) such that $\pi(u, x^*) = \sum_{i\in N} \pi_i^\intercal \alpha_i^*$.*

The equivalence remains true by replacing the term "enforceable" with "weakly enforceable" and replacing the condition $\ell(x^*) = u$ in statement 2. by $\ell(x^*) \leq u$.

Proof. We first show that the duals of LP(u) and P(u) coincide. We get

$$\mu^{\mathrm{LP}}(\lambda) := \min_{\alpha_i\in\Lambda_i, i\in N} \sum_{i\in N} \pi_i^\intercal \alpha_i + \lambda^\intercal(\ell(\alpha) - u)$$

$$= \min_{x_i\in\{x_i^1,\dots,x_i^{k_i}\}, i\in N} \sum_{i\in N} \pi_i(u, x_i) + \lambda^\intercal(\ell(x) - u) \tag{4.2}$$

$$= \mu(\lambda), \tag{4.3}$$

where (4.2) follows by the linearity of the objective function w.r.t. α_i, $i \in N$, and (4.3) by Assumption 4.1.

Let us denote by $\pi^*, (\pi^{\mathrm{LP}})^*, \mu^*, (\mu^{\mathrm{LP}})^*$ the respective optimal values of P(u), LP(u), and their duals. By weak duality of P(u) and strong duality of LP(u), we get

$$\pi^* \geq \mu^* = (\mu^{\mathrm{LP}})^* = (\pi^{\mathrm{LP}})^*.$$

Now for 1.⇒ 2., we have $\pi^* = \pi(u, x^*) = \mu(\lambda^*) = \mu^*$ for a tuple $(x^*, \lambda^*) \in X \times \mathbb{R}^m_{\geq 0}$ with $\ell(x^*) = u$ by Theorem 2.3. Thus, $\pi^* = (\pi^{\mathrm{LP}})^*$ and there is an optimal solution α^* of LP(u) such that $\pi(u, x^*) = \sum_{i\in N} \pi_i^\intercal \alpha_i^*$. For the converse 2.⇒ 1., we observe that by $\pi(u, x^*) = \sum_{i\in N} \pi_i^\intercal \alpha_i^* = (\mu^{\mathrm{LP}})^* = \mu^*$, there exists a $\lambda^* \in \mathbb{R}^m_{\geq 0}$ with $\pi(u, x^*) = \mu^* = \mu(\lambda^*)$. Therefore, the statement follows by Theorem 2.3 and the assumption that $\ell(x^*) = u$ holds. ∎

In the following, we describe the consequences of Theorem 4.2 for the important cases of finite and concave models.

DEFINITION 4.3. *We call a model I*

1. finite *if $X_i = \{x_i^1, \dots, x_i^{k_i}\}$ for some $k_i \in \mathbb{N}$ and all $i \in N$;*
2. concave *if for all $i \in N$, there exist $\{x_i^1, \dots, x_i^{k_i}\} \subseteq X_i$ for some $k_i \in \mathbb{N}$ such that* $\mathrm{conv}(X_i) = \mathrm{conv}(\{x_i^1, \dots, x_i^{k_i}\})$. *Furthermore, the functions $g_i, i \in N$ are concave on* $\mathrm{conv}(X_i)$ *and $\pi_i(u, \cdot), i \in N$, can be extended to the domain* $\mathrm{conv}(X_i)$ *so that they are concave on* $\mathrm{conv}(X_i)$.

Remark 4.4. If for all $i \in N$, we have $X_i = \{(y_i, z_i) \in \mathbb{Z}^{m_i} \times \mathbb{R}^{l_i} | A_i y_i + B_i z_i \leq b_i\}$ with $m_i + l_i = m$, with X_i bounded, then $\mathrm{conv}(X_i)$ is generated by finitely many points. To see this, recall that the convex hull of the set X_i is a polytope (cf. Conforti, Cornuéjols, and Zambelli [8]) and thus can be represented as the convex hull of its vertices. Note that boundedness of X_i can also be relaxed in case that $\pi_i(u, x_i)$ is bounded and linear on X_i.

COROLLARY 4.5. *For a finite model I, the following statements are equivalent:*

1. *The vector u is enforceable for I.*
2. LP(u) *admits an integral optimal solution $\tilde{\alpha}$ for which* (4.1) *is tight.*

For a concave model I, the following statements are equivalent:

3. *The vector u is enforceable for I.*
4. LP(u) *admits an optimal solution $\tilde{\alpha}$ with $x_i^{\tilde{\alpha}} := \sum_{j=1}^{k_i} \tilde{\alpha}_{ij} x_i^j \in X_i, i \in N$, so that $\ell(x^{\tilde{\alpha}}) = u$ and $\pi(u, x^{\tilde{\alpha}}) = \sum_{i \in N} \pi_i^\intercal \tilde{\alpha}_i$.*

The equivalences remain true by replacing the term "enforceable" with "weakly enforceable" and removing the condition that (4.1) *needs to be tight in statement* 2. *and replacing $\ell(x^{\tilde{\alpha}}) = u$ in statement* 4. *by $\ell(x^{\tilde{\alpha}}) \leq u$.*

Proof. Finite models clearly fulfill Assumption 4.1. Thus, the statement for finite models follows by the equivalence of the assertions Theorem 4.2 (2.) $\Leftrightarrow$ Corollary 4.5 (2.). To observe that $\Rightarrow$ holds, note that $x_i^* = x_i^j$ for some $j \in \{1, \ldots, k_i\}$. By setting $\tilde{\alpha}_{ij} = 1$ and $\tilde{\alpha}_{il} = 0, l \neq j$, we may observe that $\pi(u, x^*) = \sum_{i \in N} \pi_i^\intercal \tilde{\alpha}_i$ and $\ell(\tilde{\alpha}) = \ell(x^*)$ hold. Thus, Corollary 4.5 (2.) follows as $\pi(u, x^*) = \sum_{i \in N} \pi_i^\intercal \alpha_i^*$ is assumed to be the optimal value of LP(u). The converse follows immediately.

For concave models, we verify first that Assumption 4.1 is satisfied. We calculate that

$$\mu(\lambda) = \inf_{x_i \in X_i, i \in N} \sum_{i \in N} \pi_i(u, x_i) + \lambda^\intercal (\ell(x) - u)$$

$$= \inf_{x_i \in \text{conv}(X_i), i \in N} \sum_{i \in N} \pi_i(u, x_i) + \lambda^\intercal (\ell(x) - u) \tag{4.4}$$

$$= \inf_{x_i \in \text{conv}(\{x_i^1, \ldots, x_i^{k_i}\}), i \in N} \sum_{i \in N} \pi_i(u, x_i) + \lambda^\intercal (\ell(x) - u) \tag{4.5}$$

$$= \inf_{x_i \in \{x_i^1, \ldots, x_i^{k_i}\}, i \in N} \sum_{i \in N} \pi_i(u, x_i) + \lambda^\intercal (\ell(x) - u) \tag{4.6}$$

Equations (4.4) and (4.6) follow as the function $x_i \mapsto \pi_i(u, x_i) + \lambda^\intercal g_i(x_i)$ is concave over the set $\text{conv}(X_i) = \text{conv}(\{x_i^1, \ldots, x_i^{k_i}\})$. Equation (4.5) follows as $\text{conv}(X_i) = \text{conv}(\{x_i^1, \ldots, x_i^{k_i}\})$. Furthermore, the argumentation shows that all infima are, in fact, minima as the last expression in (4.6) clearly attains the infimum.

It therefore suffices to show that the assertions Theorem 4.2 (2.) $\Leftrightarrow$ Corollary 4.5 (4.) are equivalent for concave models. Clearly, $\Leftarrow$ holds. For the other direction, we argue as follows: By $\text{conv}(X_i) = \text{conv}(\{x_i^1, \ldots, x_i^{k_i}\})$, there exists an $\tilde{\alpha}_i \in \Lambda_i$ with $x_i^* = \sum_{j=1}^{k_i} \tilde{\alpha}_{ij} x_i^j$ for all $i \in N$. As $\pi_i(u, \cdot), g_i, i \in N$ are concave, we get $\pi(u, x^*) \geq \sum_{i \in N} \pi_i^\intercal \tilde{\alpha}_i$ and $u = \ell(x^*) \geq \ell(\tilde{\alpha})$ which implies that $\tilde{\alpha}$ is optimal for LP(u) since $\pi(u, x^*) = \sum_{i \in N} \pi_i^\intercal \alpha_i^*$ is assumed to be the optimal value of LP(u). ∎

4.1. The LP complexity. LP(u) may, in general, involve (exponentially) many variables $\alpha_i, i \in N$, depending on the number $k := \sum_{i \in N} k_i$. A common approach is to dualize LP(u) to yield an LP with less variables at the cost of obtaining (exponentially) many constraints. In the following, we dualize the primal problem in the form $-\max\{-\sum_{i \in N} \pi_i^\intercal \alpha_i | \ell(\alpha) \leq u,\ \alpha_i \in \Lambda_i,\ i \in N\}$. The following steps are reminiscent to the standard dual LP of the Walrasian configuration LP (see, e.g., Bikchandani and Mamer [4]):

$$\text{(DP}(u)) \qquad \begin{aligned} \min_{\mu, \lambda} \ & \sum_{i \in N} \mu_i + \sum_{j \in E} \lambda_j u_j \\ \text{s.t.:} \ & \sum_{j \in E} g_{ij}(x_i^k) \lambda_j + \mu_i \geq -\pi_{ik} \quad \text{for all } i \in N, k = 1, \ldots, k_i, \\ & \mu_i \in \mathbb{R}, i \in N,\ \lambda_j \geq 0, j \in E. \end{aligned}$$

Note that $\mu_i, i \in N$ is not sign-constrained as it is the dual variable to $\sum_k \alpha_{ik} = 1, i \in N$. Moreover, recall that $g_{ij}(x_i^k) \in \mathbb{R}$ are just parameters in DP(u). The dual has $n+m$ many variables but exponentially many constraints, but if we have a polynomial time separation oracle, we can use the ellipsoid method to obtain a polynomial time algorithm (cf. Groetschel, Lovász, and Schrijver [20]). A standard way to obtain such an oracle is to assume an efficient *demand oracle*.

DEFINITION 4.6. *For a model I fulfilling Assumption* 4.1*, a demand oracle for player* $i \in N$ *gets as input prices* $\lambda \in \mathbb{R}^m_{\geq 0}$ *and outputs*

$$x_i(\lambda) \in \arg \min_{x_i \in \{x_i^1, \ldots, x_i^{k_i}\}} \{\pi_i(u, x_i) + \lambda^\intercal g_i(x_i)\}.$$

We obtain the following result for polynomial time computable demand oracles. Let us remark here that we assume that there is a succinct representation of the model I and hence of LP(u).

THEOREM 4.7. *Let I be a model which fulfills Assumption* 4.1*. If for all* $\lambda \in \mathbb{R}^m_{\geq 0}$ *and* $i \in N$ *the demand oracle* $x_i(\lambda)$ *can be computed in polynomial time, then, the optimal value of* LP(u) *can be computed in polynomial time.*

Proof. In order to use the ellipsoid method, we need to check whether we get a polynomial time separation oracle for the constraints:

$$\sum_{j \in E} g_{ij}(x_i^k)\lambda_j + \mu_i \geq -\pi_{ik}, \quad i \in N, \quad k = 1, \ldots, k_i.$$

With the demand oracle we can compute $\pi_i^*(\lambda) := \pi_i(u, x_i(\lambda)) + \lambda^\intercal g_i(x_i(\lambda))$. Now, if $\pi_i^*(\lambda) \geq -\mu_i$ for all $i \in N$, the current point (μ, λ) is feasible. Otherwise, suppose $\pi_i^*(\lambda) < -\mu_i$. As $x_i(\lambda) = x_i^k$ holds for some $k \in \{1, \ldots, k_i\}$, we get

$$\pi_i^*(\lambda) = \pi_i(u, x_i^k) + \sum_{j \in E} g_{ij}(x_i^k)\lambda_j = \pi_{ik} + \sum_{j \in E} g_{ij}(x_i^k)\lambda_j < -\mu_i,$$

which represents a violated inequality. ∎

4.2. Consequences and impossibility results. The characterization result in Theorem 4.2 together with the assumption of a polynomial time demand oracle can be used to establish nonexistence results based on complexity-theoretic assumptions like $P \neq NP$. If the optimal value of the master problem P(u) (which is also called the welfare maximization problem in some applications) is NP-hard to compute but there is a polynomial demand oracle, then, assuming $P \neq NP$, the guaranteed (weak) enforceability of u is ruled out since otherwise, we can just compute the optimal solution value of LP(u) in polynomial time (by solving the dual DP(u)) which corresponds to the optimal solution value of the master problem. The connection between the complexity of the demand problem and that of the master problem has been first observed by Talgam-Cohen and Roughgarden [41] for the case of pricing equilibria for Walrasian market settings. Our characterization results allow one to generalize this approach beyond market equilibria.

5. Aggregated formulations. Now we consider aggregated models I, where the private cost function of every player is assumed to be quasi-separable over the resources and of the following form: $\pi_i(u, x_i) = \sum_{j \in E} \pi_j(u) \cdot g_{ij}(x_i)$, where $\pi_j : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ denotes the per-unit cost on resource j mapping a vector u to the reals. For ease of presentation, we will assume that the per-unit cost of a resource is strictly positive. All results in this section can be extended to the case where the per-unit cost of resources may be zero; see the discussion at the end of this section.

For an aggregated model, we can assume w.l.o.g. that the resource consumption functions are the identity, i.e., $g_i(x_i) = x_i$, $i \in N$. Indeed, using that the private cost of a player solely depends on the resource consumption and not on the actual strategy, for an aggregated model $I = (N, E, X, g, \pi)$, we can define an isomorphic model $I' = (N, E, X', g', \pi')$, where the strategy set of a player i is given by $X'_i = g_i(X_i)$, the resource consumption function equals $g'_i(x_i) = x_i$ and the private cost is $\pi'_i(u, x_i) = \sum_{j\in E} \pi_j(u) \cdot x_{ij}$.

With these observations, it follows that the objective of the master problem $\mathrm{P}(u)$ of an aggregated model I does not depend on the specific decomposition of a strategy distribution $x \in X$ but only on its induced load $\ell(x)$. The same holds true for the corresponding convexified model I^{conv} as the latter also belongs to the class of aggregated formulations, because $\pi_i(u, x_i) = \phi_i(u, x_i)$, $i \in N$ holds, since $\pi_i(u, x_i)$ is linear in x_i. Note that $X^{\mathrm{conv}} = \times_{i\in N} \mathrm{conv}(X_i)$ as w.l.o.g. $g(x) = x$. These insights lead to the following optimization problems which will allow for a complete characterization of enforceability.

$$(\tilde{\mathrm{P}}(u)) \qquad \inf_{\tilde u} \pi(u)^\intercal \tilde u \quad \text{s.t.: } \tilde u \le u,\ \tilde u \in \ell(X),$$

$$(\tilde{\mathrm{P}}^{\mathrm{conv}}(u)) \qquad \inf_{\tilde u} \pi(u)^\intercal \tilde u \quad \text{s.t.: } \tilde u \le u,\ \tilde u \in \ell(X^{\mathrm{conv}}),$$

where $\pi(u) := (\pi_j(u))_{j\in E}$.

For the promised characterization (see Theorem 5.4), we need a lemma relating the master problem $\mathrm{P}(u)$ of I and I^{conv} with the new problems $(\tilde{\mathrm{P}}(u))$ and $(\tilde{\mathrm{P}}^{\mathrm{conv}}(u))$, respectively.

LEMMA 5.1. *For an aggregated model I, the following assertions are equivalent:*
1. *(x^*, λ^*) is primal-dual optimal for* $\mathrm{P}(u)$.
2. *$(\ell(x^*), \lambda^*)$ is primal-dual optimal for* $\tilde{\mathrm{P}}(u)$.

The analogous statements also hold for I^{conv} and $\tilde{\mathrm{P}}^{\mathrm{conv}}(u)$.

Proof. As I^{conv} belongs to the class of aggregated models, it is sufficient to show the statement for the original model I. To verify the stated equivalence, we first observe that the following description of $\pi(u, x)$ holds for any $x \in X$:

$$\pi(u, x) = \sum_{i\in N} \pi_i(u, x_i) = \sum_{i\in N}\sum_{j\in E} \pi_j(u) x_{ij} = \sum_{j\in E} \pi_j(u) \sum_{i\in N} x_{ij} = \pi(u)^\intercal \ell(x).$$

Thus, it suffices to show that the duals coincide. We denote by $\mu(\lambda)$ the dual of $\mathrm{P}(u)$, and by $\tilde\mu(\lambda)$ the dual of $\tilde{\mathrm{P}}(u)$, respectively, and observe

$$\begin{aligned}\mu(\lambda) &= \inf_{x\in X} \pi(u, x) + \lambda^\intercal(\ell(x) - u) = \inf_{x\in X} \pi(u)^\intercal \ell(x) + \lambda^\intercal(\ell(x) - u)\\ &= \inf_{\tilde u\in \ell(X)} \pi(u)^\intercal \tilde u + \lambda^\intercal(\tilde u - u) = \tilde\mu(\lambda).\end{aligned}$$ ☐

In the following, we want to better understand which vectors u are enforceable. As an immediate consequence of Lemma 5.1, we get by Theorem 2.3 that u is enforceable if and only if $\tilde{\mathrm{P}}(u)$ has zero duality gap and u is itself an optimal solution for $\tilde{\mathrm{P}}(u)$. Optimality of u for $\tilde{\mathrm{P}}(u)$ is fulfilled if and only if u is minimal w.r.t. $\ell(X)$ (cf. Definition 2.4). The only-if direction of the latter statement follows directly by the assumption that $\pi(u) \in \mathbb{R}^m_{>0}$, whereas the if direction follows because the feasible set of $\tilde{\mathrm{P}}(u)$ is given by $\ell(X) \cap \{\tilde u \mid \tilde u \le u\} = \{u\}$ provided u is minimal w.r.t. $\ell(X)$. Thus, so far we can deduce that minimality of u is already necessary for enforceability.

Similarly, since I^{conv} also belongs to the class of aggregated models, the analogous statements are true for I^{conv}. By using Theorem 3.2 (connecting I with I^{conv}), we end up with u being enforceable for I if and only if $\tilde{\text{P}}^{\text{conv}}(u)$ has zero duality gap and u is minimal w.r.t. $\ell(X^{\text{conv}})$. This last characterization seems promising as problem $\tilde{\text{P}}^{\text{conv}}(u)$ has a linear objective, a convex domain $\ell(X^{\text{conv}}) = \text{conv}(\ell(X))$, and one linear constraint. Thus, for well-behaved sets $\text{conv}(\ell(X))$, e.g., polyhedral sets, it follows that problem $\tilde{\text{P}}^{\text{conv}}(u)$ always has zero duality gap. It turns out that enforceability of u is in this case equivalent to the property that the minimality of u w.r.t. $\ell(X)$ carries over to $\text{conv}(\ell(X))$. This latter property leads to the following refinement of minimal vectors u.

Definition 5.2 (concave and linear minimality).

1. *The vector u is called* concave-minimal *w.r.t. $\ell(X)$ if there exists a strictly increasing, differentiable, and concave function $h : \mathbb{R}^m \to \mathbb{R}$ such that $u \in \arg\min_{\tilde{u}\in\ell(X)} h(\tilde{u})$.*
2. *The vector u is called* linearly-minimal *w.r.t. $\ell(X)$ if u is concave-minimal for a linear function $h(\tilde{u}) = a^\intercal \tilde{u}$ with $a \in \mathbb{R}^m_{>0}$.*

Remark 5.3. Obviously we have the following relationship among the different notions of minimality of u w.r.t. $\ell(X)$:

$$\text{linearly-minimal} \subseteq \text{concave-minimal} \subseteq \text{minimal}.$$

Now we give a complete characterization of enforceability of u for I.

Theorem 5.4. *Let $u \in \ell(X)$ be a feasible load vector for the aggregated model I. Then the following statements are equivalent:*

1. *u is enforceable for I.*
2. *u is enforceable for I^{conv}.*
3. *$\tilde{\text{P}}(u)$ has zero duality gap and u is minimal w.r.t. $\ell(X)$.*
4. *$\tilde{\text{P}}^{\text{conv}}(u)$ has zero duality gap and u is minimal w.r.t. $\text{conv}(\ell(X))$.*
5. *u is linearly-minimal w.r.t. $\ell(X)$.*
6. *u is concave-minimal w.r.t. $\ell(X)$.*

Proof. The equivalence 1.$\Leftrightarrow$2. follows immediately by Theorem 3.2, the fact that $u \in \ell(X)$ and $\phi(u,x) = \pi(u,x) = \pi(u)^\intercal \ell(x)$ holds for any $x \in X$.

Next, we show 1.$\Leftrightarrow$3. which also shows 2.$\Leftrightarrow$4. since I^{conv} belongs to the class of aggregated models and $\ell(X^{\text{conv}}) = \text{conv}(\ell(X))$ holds.

1.$\Leftrightarrow$3.: By Theorem 2.3, a vector u is enforceable if and only if there exists $(x^*, \lambda^*) \in X \times \mathbb{R}_{\geq 0}$ primal-dual optimal for $\text{P}(u)$ of I with $\ell(x^*) = u$. By Lemma 5.1, this is again equivalent to the existence of $\lambda^* \in \mathbb{R}^m_{\geq 0}$ such that (u, λ^*) is primal-dual optimal for $(\tilde{\text{P}}(u))$. Since $\pi(u) \in \mathbb{R}^m_{>0}$, the optimality of u for $(\tilde{\text{P}}(u))$ is equivalent to u being minimal w.r.t. $\ell(X)$. Thus the existence of a primal-dual optimal $(u, \lambda^*) \in \ell(X) \times \mathbb{R}_{\geq 0}$ for $(\tilde{\text{P}}(u))$ is equivalent to $(\tilde{\text{P}}(u))$ having zero duality gap and u being minimal w.r.t. $\ell(X)$. It remains to show 1.$\Leftrightarrow$5. and 6.$\Leftrightarrow$5.. For 1.$\Leftrightarrow$5. we start with the following observation: By 1.$\Leftrightarrow$3., u is enforceable for I if and only if there exists $\lambda^* \in \mathbb{R}^m_{\geq 0}$ such that (u, λ^*) is primal-dual optimal for $(\tilde{\text{P}}(u))$, i.e.

$$\min_{\tilde{u}\in\ell(X)} \pi(u)^\intercal \tilde{u} + (\lambda^*)^\intercal(\tilde{u} - u) = \pi(u)^\intercal u.$$

By adding to both sides the term $(\lambda^*)^\intercal u$, this is again equivalent to

$$\min_{\tilde{u}\in\ell(X)} (\pi(u) + \lambda^*)^\intercal \tilde{u} = (\pi(u) + \lambda^*)^\intercal u. \tag{5.1}$$

1.⇒5.: With the above argumentation, if u is enforceable for I, (5.1) holds for a $\lambda^* \in \mathbb{R}^m_{\geq 0}$ and we can define $a := (\pi(u) + \lambda^*) \in \mathbb{R}^m_{>0}$, since $\pi(u) \in \mathbb{R}^m_{>0}$. Thus the so-defined vector a meets the requirements of Definition 5.2.

5.⇒1.: Let u be linearly-minimal w.r.t. $\ell(X)$. Then, there exists a vector $a \in \mathbb{R}^m_{>0}$ which meets the requirements of Definition 5.2. We can assume w.l.o.g. that $\pi(u) \leq a$ since we can just scale a sufficiently otherwise. By setting $\lambda^* := a - \pi(u) \in \mathbb{R}^m_{\geq 0}$ we obtain a nonnegative vector for which (5.1) holds.

6.⇒5.: Let $u \in \arg\min_{\tilde{u} \in \ell(X)} h(\tilde{u})$ be given. With the concavity of h, we get $u \in \arg\min_{\tilde{u} \in \mathrm{conv}(\ell(X))} h(\tilde{u})$. As $\mathrm{conv}(\ell(X))$ is obviously convex, for any $\tilde{u} \in \mathrm{conv}(\ell(X))$, the direction vector $(u - \tilde{u})$ belongs to the tangent cone of $\mathrm{conv}(\ell(X))$ at u. With the differentiability of h, we get as a first order necessary optimality condition

$$\nabla h(u)(u - \tilde{u}) \leq 0 \quad \text{for all } \tilde{u} \in \mathrm{conv}(\ell(X)).$$

Thus, as h is strictly increasing and concave in $\tilde{u}$, we have $a := \nabla h(u) \in \mathbb{R}^m_{>0}$ and the requirements of Definition 5.2 are fulfilled.

5.⇒6.: By Remark 5.3. ∎

From the latter proof follows directly another insight.

COROLLARY 5.5. *If λ^* enforces u for I, then every $\tilde{\lambda}$ on the half-ray*

$$\left\{ \tilde{\lambda} \in \mathbb{R}^m_{\geq 0} \mid \tilde{\lambda} = d \cdot (\pi(u) + \lambda^*) - \pi(u),\, d \in \mathbb{R}_{\geq 0} \right\}$$

also enforces u for I. In particular, if u is enforceable, then there exists a price-vector $\tilde{\lambda}$ which enforces u and $\tilde{\lambda}_j = 0$ holds for at least one $j \in E$.

We conclude the section with an example showing that not every minimal u is also linearly-minimal/enforceable even for a two player model with $X_i, i = 1, 2$ equal to strategy sets with three vectors each.

PROPOSITION 5.6. *There is an aggregated model I with minimal load u w.r.t. $\ell(X)$ that is not enforceable for I and thus not concave-/linearly-minimal w.r.t. $\ell(X)$.*

Proof. Consider the model constructed in Figure 2 involving two players $N = \{1, 2\}$ with eight resources $E = \{r_1, \ldots, r_8\}$. The strategy spaces are given as $X_1 = \{x_1, y_1, z_1\}$ with $x_1 = \mathbf{1}_{\{r_1,r_2,r_3,r_4\}}$, $y_1 = \mathbf{1}_{\{r_2,r_4,r_6\}}$, and $z_1 = \mathbf{1}_{\{r_1,r_3,r_7\}}$, where for a subset $S \subseteq E$, $\mathbf{1}_S$ denotes the indicator vector of set S in $\mathbb{R}^m$. For player 2, we have $X_2 = \{x_2, y_2, z_2\}$ with $x_2 = \mathbf{1}_{\{r_5,r_6,r_7,r_8\}}$, $y_2 = \mathbf{1}_{\{r_3,r_4,r_8\}}$, and $z_2 = \mathbf{1}_{\{r_1,r_2,r_5\}}$. Now, we argue that the vector $u = \mathbf{1}$ is minimal w.r.t. $\ell(X)$. To see this, observe that we have $x_1 + x_2 = u = \mathbf{1}$ but for every other strategy combination $s \in X$, there is at least one $r \in \{r_1, \ldots, r_8\}$ with $\ell_r(s) = 2$. For $w := 1/3(x_1 + y_1 + z_1) + 1/3(x_2 + y_2 + z_2) \in \mathrm{conv}(\ell(X))$ we get $w \leq \mathbf{1}$ and $w_r = 2/3$ for all $r \in \{r_5, \ldots, r_8\}$. Hence u is not minimal w.r.t. $\mathrm{conv}(\ell(X))$ and by Theorem 5.4 (4.-6.) neither enforceable nor concave-/linearly-minimal w.r.t. $\ell(X)$. ∎

Remark 5.7. In the case where the original costs $\pi_j(u)$ of some resources are zero, it follows directly that the properties described in Theorem 5.4 are still sufficient for enforceability of a load. But, in fact, the properties 3.–6. are not necessary anymore. That is due to the fact that the minimality of a load with respect to a zero-cost resource is not necessary anymore for enforceability. To conserve a full characterization of enforceability, one can relax the definition of minimality used in Theorem 5.4 by taking into account the cost-vector $\pi(u)$. More precisely one defines the following.

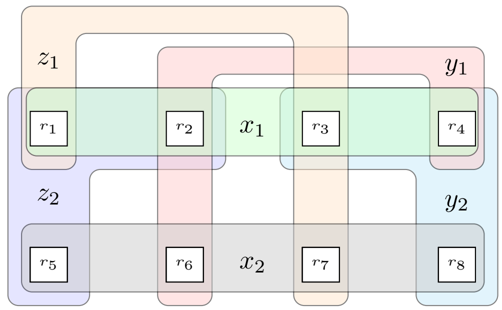

FIG. 2. *Representation of the strategy space of the game constructed in the proof of Proposition* 5.6.

DEFINITION 5.8 (relaxed minimality).

1. *The vector u is called* minimal *with respect to $\ell(X)$ and $\pi(u)$ if for all $\tilde{u} \in \{\bar{u} \in \mathbb{R}^m : \bar{u} \leq u\} \cap \ell(X)$, the equalities $\tilde{u}_j = u_j$ for all $j \in E(\pi(u)) := \{e \in E \mid \pi_e(u) > 0\}$ hold.*
2. *u is called* concave-minimal *w.r.t. $\ell(X)$ and $\pi(u)$ if there exists a nondecreasing,[2] differentiable, and concave function $h : \mathbb{R}^m \to \mathbb{R}$ such that $u \in \arg\min_{\tilde{u} \in \ell(X)} h(\tilde{u})$ and $\frac{\partial}{\partial u_j} h(u) = 0$ implies $\pi_j(u) = 0$ for all $j \in E$.*
3. *u is called* linearly-minimal *w.r.t. $\ell(X)$ and $\pi(u)$ if u is concave-minimal for a linear function $h(\tilde{u}) = a^\intercal \tilde{u}$ with $a \in \mathbb{R}^m_{\geq 0}$.*

Clearly, the respective relaxed definitions coincide with the original definition in the case of $\pi(u) \in \mathbb{R}^m_{>0}$, i.e., $E(\pi(u)) = E$. By using the above relaxed version of minimality, one gets a full characterization for the general case of arbitrary per unit-costs $\pi(u) \in \mathbb{R}^m_{\geq 0}$. The proof is conceptually exactly the same as the proof of Theorem 5.4.

6. Applications in congestion games.

We now demonstrate the applicability of our framework by deriving new existence results of tolls enforcing certain load vectors in congestion games. Moreover, we show how several known results in the literature follow directly.

6.1. Nonatomic congestion games.

We first present results for the case that the strategy spaces of players are convex subsets of $\mathbb{R}^m_{\geq 0}$. We are given a directed graph $G = (V, E)$ and a set of populations $N := \{1, \ldots, n\}$, where each population $i \in N$ has a demand $d_i > 0$ that has to be routed from a source $s_i \in V$ to a destination $t_i \in V$. In the *nonatomic* model, the demand interval $[0, d_i]$ represents a continuum of infinitesimally small agents each acting independently choosing a cost minimal s_i,t_i path. There are continuous cost functions $c_{ij} : \mathbb{R}^m \to \mathbb{R}_{\geq 0}, i \in N, j \in E$ which may depend on the population identity and also on the aggregate load vector—thus allowing for modeling *nonseparable* latency functions. A *flow* for population $i \in N$ is a nonnegative vector $x_i \in \mathbb{R}^{|E|}_{\geq 0}$ that lives in the flow polytope:

$$X_i = \left\{ x_i \in \mathbb{R}^m_{\geq 0} \mid \sum_{j \in \delta^+(v)} x_{ij} - \sum_{j \in \delta^-(v)} x_{ij} = \gamma_i(v) \text{ for all } v \in V \right\},$$

[2]We call a function $h : \mathbb{R}^m \to \mathbb{R}$ nondecreasing if for all $u^*, \tilde{u} \in \mathbb{R}^m$ with $u^* \leq \tilde{u}$ the inequality $h(u^*) \leq h(\tilde{u})$ holds.

where $\delta^+(v)$ and $\delta^-(v)$ are the arcs leaving and entering v, and $\gamma_i(v) = d_i$ if $v = s_i$, $\gamma_i(v) = -d_i$, if $v = t_i$, and $\gamma_i(v) = 0$, otherwise. We assume that every t_i is reachable in G from s_i for all $i \in N$, thus, $X_i \neq \emptyset$ for all $i \in N$. Given a combined flow $x \in X$, the cost of a path $P \in \mathcal{P}_i$, where $\mathcal{P}_i$ denotes the set of simple s_i, t_i paths in G, is defined as $c_{i,P}(\ell(x)) := \sum_{j \in P} c_{ij}(\ell(x))$. A *Wardrop equilibrium* x^* with path-decomposition $(x^*_{i,P})_{i \in N, P \in \mathcal{P}_i}$ is defined as follows:

$$c_{i,P}(\ell(x^*)) \leq c_{i,Q}(\ell(x^*)) \text{ for all } P, Q \in \mathcal{P}_i \text{ with } x^*_{i,P} > 0.$$

The interpretation here is that all agents are traveling along shortest paths given the overall load vector $\ell(x^*)$. One can reformulate the Wardrop equilibrium conditions using load vectors u stating that—given the load vector of a Wardrop equilibrium —every agent is traveling along a shortest path.

Lemma 6.1 (Dafermos [10]). *Let $x^* \in X$ and fix $u := \ell(x^*)$. Then, x^* is a Wardrop equilibrium if and only if*

$$x^*_i \in \arg\min_{x_i \in X_i} \left\{ \sum_{j \in E} c_{ij}(u) x_{ij} \right\} \quad \textit{for all } i \in N.$$

With this characterization, the model fits in our framework and we can apply our general existence result Theorem 2.3 and its direct consequences for minimal vectors as stated in Corollary 2.6.

Corollary 6.2 (Yang and Huang [50], Fleischer, Jain, and Mahdian [15], Karakostas and Kolliopoulos [26], Marcotte and Zhu [32]). *Every minimal vector u is enforceable.*

Proof. Define I with N, E, X as described above and $\pi_i(\tilde{u}, x_i) := \sum_{j \in E} c_{ij}(\tilde{u}) x_{ij}$, $\tilde{u} \in \mathbb{R}^m$, and $g_i(x_i) := x_i$ for every $i \in N$. With Lemma 6.1, the model fits into the framework and by the linearity of the objective in P(u), the master problem is an LP and thus satisfies strong duality. With the minimality of u, the result follows by Corollary 2.6. ∎

Note that the above result is more general than those of [15, 26, 50] as we allow for *arbitrary* player-specific cost functions $c_{ij}, i \in N, j \in E$. These previous works assumed less general *heterogeneous cost functions* of the form $c_{i,P}(x) = \sum_{j \in P} \alpha_i c_j(\ell_j(x)) + \lambda_j$, where $\alpha_i > 0$ represents a tradeoff parameter weighting the impact of money versus travel time. Fleischer, Jain, and Mahdian [15, sect. 6] also mention that their existence result holds for the more general case of nonseparable latency functions. Only Marcotte and Zhu [32] presented a formulation with general nonseparable latency functions and even extended the linear programming formulation (using a variational inequality formulation) to include the more general setting of infinitely many different user classes.

6.2. Atomic congestion games. Now we turn to atomic (resource-weighted) congestion games, a generalization of the model of Rosenthal [40]. An atomic congestion game is a strategic game $G^{\text{cg}} = (N, X, (\text{cost}_i(x))_{i \in N})$, where the set of strategies available to player $i \in N$ is given by $X_i \subseteq \times_{j \in E} \{0, d_{ij}\}$ for $d_{ij} > 0, i \in N, j \in E$. Note that by assuming $x_i \in \{0,1\}^m$ for all $i \in N$, that is, $d_{ij} \in \{0,1\}$, we obtain the standard congestion game model of Rosenthal.

The cost functions on resources are given by *player-specific functions* $c_{ij}(\ell_j(x)) \in \mathbb{R}, j \in E, i \in N$, where $\ell(x) = \sum_{i \in N} x_i$. If the cost functions only depend on the resource identity, that is, $c_j(\ell_j(x)), j \in E$, we speak of *homogeneous cost functions*.

The private cost of a player $i \in N$ for strategy distribution $x \in X$ is defined by $\text{cost}_i(x) := \pi_i(\ell(x), x_i)$ for a function $\pi_i : \mathbb{R}^m \times X_i \to \mathbb{R}$ defined as

$$\pi_i(\ell(x), x_i) := \sum_{j \in E} c_{ij}(\ell_j(x)) x_{ij}. \tag{6.1}$$

The definition in (6.1) shows that $\text{cost}_i(x)$ depends not only on the own strategy x_i but also on the aggregated load vector $\ell(x)$ which, in turn, depends on the strategies of all players. Thus, the class of atomic congestion games does not fit into the model considered so far. However, also in atomic congestion games, the question of whether or not the players can be incentivized by prices in order to realize a targeted load vector u is of particular interest; see, e.g., papers on toll pricing [15, 17, 50]. This leads to the following definition of enforceability for atomic congestion games.

Definition 6.3 (enforceability for atomic congestion games). *Consider an atomic congestion game $G^{cg} = (N, X, (\text{cost}_i(x))_{i \in N})$. The vector $u \in \mathbb{R}^m$ is enforceable for G^{cg} if there is a tuple $(x^*, \lambda^*) \in X \times \mathbb{R}^m_{\geq 0}$ such that the following two conditions are satisfied:*

1. *$\ell_j(x^*) = u_j$ for all $j \in E$.*
2. *$x_i^* \in \arg\min_{x_i \in X_i} \{\pi_i(\ell(x_i, x_{-i}^*), x_i) + (\lambda^*)^\intercal x_i\}$ for all $i \in N$.*

As stated previously, condition 1. requires that x^* realizes the targeted load $\ell(x^*) = u$ while condition 2. implements x^* as a pure Nash equilibrium of the atomic congestion game $G^{cg}(\lambda^*) := (N, X, (\text{cost}_i(x) + (\lambda^*)^\intercal x_i))$.

In order to apply the framework to the current setting, we define the model $I := (N, E, X, g, \pi)$ with N, X, E, π as described above and $g(x) = x$. In the case of nondecreasing cost functions $c_{ij}, j \in E, i \in N$, the following lemma shows that whenever u is enforceable for I via (x^*, λ^*), then u is also enforceable for G^{cg} via (x^*, λ^*).

Lemma 6.4. *Assume that cost functions $c_{ij}, i \in N, j \in E$ are nondecreasing. Let $x^* \in X$ and fix $u := \ell(x^*)$. Then x^* is a pure Nash equilibrium of the atomic congestion game $G^{cg}(\lambda^*)$ for $\lambda^* \in \mathbb{R}^m_{\geq 0}$ if*

$$x_i^* \in \arg\min_{x_i \in X_i} \left\{ \sum_{j \in E} c_{ij}(u_j) x_{ij} + (\lambda^*)^\intercal x_i \right\} \quad \textit{for all } i \in N. \tag{6.2}$$

Proof. Let $i \in N$ and $x_i \in X_i$. For x_i denote $E(x_i) := \{j \in E | x_{ij} = d_{ij}\}$ the support of x_i. We calculate

$$\begin{aligned}
\pi_i(\ell(x^*), x_i^*) + (\lambda^*)^\intercal x_i^* &= \sum_{j \in E(x_i^*)} (c_{ij}(u_j) + \lambda_j^*) x_{ij}^* \\
&\leq \sum_{j \in E(x_i)} (c_{ij}(u_j) + \lambda_j^*) x_{ij} \qquad (6.3) \\
&\leq \sum_{j \in E(x_i) \setminus E(x_i^*)} (c_{ij}(u_j + d_{ij}) + \lambda_j^*) x_{ij} + \sum_{j \in E(x_i) \cap E(x_i^*)} (c_{ij}(u_j) + \lambda_j^*) x_{ij} \qquad (6.4) \\
&= \pi_i(\ell(x_i, x_{-i}^*), x_i) + (\lambda^*)^\intercal x_i,
\end{aligned}$$

where (6.3) follows from the optimality of x_i^* for problem (6.2) and (6.4) follows from the monotonicity of the cost functions $c_{ij}, i \in N, j \in E$. ∎

Clearly, the condition (6.2) is only sufficient for enforcing u and not necessary. Fotakis and Spirakis [17] termed prices λ^* that induce equilibria x^* with the property stated in (6.2) *cost-balancing*.

Homogeneous cost functions. We now assume that cost functions are homogeneous, thus, the private cost of player $i \in N$ has the form

$$\pi_i(\ell(x), x_i) := \sum_{j \in E} c_j(\ell_j(x)) x_{ij}.$$

In order to state enforceability results for u, we can exploit the characterizations derived in section 5. Again we assume for the sake of simplicity that the per-unit cost of a resource is positive, i.e., $c_j(u) > 0$, and refer to Remark 5.7 for the more general case of nonnegative costs. Since the set of feasible load vectors $\ell(X)$ is finite, $\operatorname{conv}(\ell(X))$ is a polytope and, hence, $(\tilde{\mathrm{P}}^{\mathrm{conv}}(u))$ has zero duality gap. Thus, Theorem 5.4 together with Lemma 6.4 yields the following result.

Corollary 6.5. *For $u \in \mathbb{R}^m$ the following are equivalent:*
1. *u is concave-minimal w.r.t. $\ell(X)$.*
2. *u is minimal w.r.t. $\operatorname{conv}(\ell(X))$.*

Moreover, u is enforceable for the atomic congestion game G^{cg} with homogeneous nondecreasing cost functions by cost-balancing prices if and only if u is concave-minimal w.r.t. $\ell(X)$.

This enforceability result holds, in particular, for weighted congestion games and *arbitrary* positive nondecreasing homogeneous cost functions $c_j, j \in E$ and arbitrary strategy spaces X. This is interesting as weighted congestion games without tolls may fail to admit pure Nash equilibria if cost functions are neither affine nor exponential [21]. Another interesting consequence of Corollary 6.5 is the fact that it is sufficient for enforceability of u to check whether or not the minimality w.r.t. $\ell(X)$ carries over to $\operatorname{conv}(\ell(X))$. In this regard, Proposition 5.6 implies the first example of a minimal u w.r.t. $\ell(X)$ that is not enforceable for an asymmetric congestion game with two players.

Corollary 6.6. *Not every minimal u w.r.t. $\ell(X)$ is enforceable for congestion games, even for games with only two players.*

Proof. We use the construction of the proof of Proposition 5.6 to define a congestion game with two players. We assign cost functions c_j that are nondecreasing and satisfy $c_j(z) = c_j(u_j) > 0$ for all $z \geq u_j$. Then any $\lambda \in \mathbb{R}^m_{\geq 0}$ enforcing u is also cost-balancing and thus Corollary 6.5 together with Proposition 5.6 imply the wanted counterexample. ∎

For several interesting unweighted congestion games (e.g., single-source routing games), the property of minimality w.r.t. $\ell(X)$ carrying over to $\operatorname{conv}(\ell(X))$ follows by polyhedral theory arguments. More precisely, the main idea is to check whether or not the polytope $\operatorname{conv}(\ell(X)) \cap \{\tilde{u} \in \mathbb{R}^m | \tilde{u} \leq u\}$ is integral and has the integer decomposition property (IDP), that is, every integral load vector contained in the polytope can be decomposed into a feasible strategy distribution. A powerful tool to recognize integrality of polyhedra is the notion of *total-dual-integrality* (TDI) of linear systems (see Edmonds and Giles [14]). A rational system of the form $Az \geq b$ with $A \in \mathbb{Q}^{r \times m}$ and $b \in \mathbb{Q}^r$ is TDI if for every integral $c \in \mathbb{Z}^m$, the dual of $\min_z\{c^\intercal z | Az \geq b\}$ given by $\max_z\{z^\intercal b | A^\intercal z = c, z \geq 0\}$ has an integral optimal solution (if the problem admits a finite optimal solution). It is known that for TDI systems with integral

$b\in\mathbb{Z}^r$, the corresponding polyhedron is integral. A system $Az\geq b$ is *box-TDI* if the system $Az\geq b, w\leq z\leq u$ is TDI for all integral w,u. We obtain the following result.

COROLLARY 6.7. *Let G^{cg} be an atomic congestion game with homogeneous, positive, and nondecreasing cost functions. Let u be minimal w.r.t. $\ell(X)$. If the set $\operatorname{conv}(\ell(X))$ satisfies IDP and can be described by a box-TDI system $Az\geq b$ with $b\in\mathbb{Z}^r$, then u is enforceable.*

Proof. As argued above it is sufficient to show that the minimality of u w.r.t. $\ell(X)$ carries over to $\operatorname{conv}(\ell(X))$. By the assumed box-TDI property of the latter set, the polytope $\operatorname{conv}(\ell(X))\cap\{\tilde{u}\in\mathbb{R}^m|\tilde{u}\leq u\}$ has only integral vertices. Thanks to the IDP property these vertices can each be decomposed into a feasible strategy distribution. Therefore, the vertices lie in $\ell(X)$. Thus by the minimality of u w.r.t. $\ell(X)$, it follows that the vertices need to be equal to u and thus $\operatorname{conv}(\ell(X))\cap\{\tilde{u}\in\mathbb{R}^m|\tilde{u}\leq u\}=\{u\}$ which shows that u is minimal w.r.t. $\operatorname{conv}(\ell(X))$; cf. Definition 2.4. ∎

In particular, IDP and the above box-TDI property hold for the following *unweighted* congestion games:

1. Network games with a common source and multiple sinks.
2. r-arborescence congestion games.
3. Intersection of strongly base-orderable matroids.
4. Symmetric totally-unimodular games including matching games.
5. Asymmetric matroid games.

We would like to emphasize that Del Pia, Ferris, and Michini [39] and Kleer and Schäfer [28, 29] were the first to use the concept of polyhedral theory, especially the notions of TDI and unimodularity in the context of congestion games, e.g., for efficiently computing pure Nash equilibria [28, 29, 39] and bounding the price of stability [28, 29]. To the best of our knowledge, the only previous results for the existence of enforcing tolls are due to Fotakis and Spirakis [17] and Fotakis, Karakostas, and Kolliopoulos [16]. Fotakis and Spirakis [17] proved that any acyclic integral flow in an s,t digraph can be enforced. It is not hard to see that the notion of minimality of u w.r.t. $\ell(X)$ exactly corresponds to the set of acyclic integral s,t flows. Paccagnan et al. [35] investigated the price of anarchy of tolls for unweighted congestion games.

Matroid congestion games with player-specific cost functions. Now we turn to matroid congestion games with player-specific, nondecreasing, and separable cost functions $c_{ij}(\ell_j(x)), i\in N, j\in E$. It will be convenient to represent the set of incidence vectors of the bases of a player-specific matroid via an integral base polyhedron $\mathcal{B}_{f_i}\subseteq\{0,1\}^m$ of an *integral polymatroid*, that is,

$$X_i=\mathcal{B}_{f_i}=\Big\{x_i\in\{0,1\}^m\mid x_i(U)\leq f_i(U)\text{ for all }U\subseteq E,\ x_i(E)=f_i(E)\Big\},$$

where, for a set $U\subseteq E$, we write $x_i(U)=\sum_{j\in U}x_{ij}$. The integral set function $f_i:2^E\to\mathbb{Z}$ is assumed to be submodular, monotone, and normalized. An integral set function $f:2^E\to\mathbb{Z}$ is *submodular* if $f(U)+f(V)\geq f(U\cup V)+f(U\cap V)$ for all $U,V\in 2^E$; f is *monotone* if $f(U)\leq f(V)$ for all $U\subseteq V\subseteq E$; and f is *normalized* if $f(\emptyset)=0$.

COROLLARY 6.8. *Consider a matroid congestion game with nondecreasing player-specific and separable cost functions. Any $u\in\ell(X)$ is enforceable.*

Proof. With Lemma 6.4 and Theorem 2.3, it suffices to show that the master problem $\mathrm{P}(u)$ of $I=(N,E,X,g,\pi)$ with $\pi_i(\tilde{u},x_i):=\sum_{j\in E}c_{ij}(\tilde{u})x_{ij},\tilde{u}\in\mathbb{R}^m$ satisfies strong duality and admits an optimal solution x^* with $\ell(x^*)=u$. The latter follows directly as $X_i, i \in N$, are matroid bases and with $u \in \ell(X)$ any $x \in X$ with $\ell(x) \leq u$ already fulfills $\ell(x)=u$. Concerning the strong duality of $\mathrm{P}(u)$, the fractional relaxation of $\mathrm{P}(u)$ is an LP, and therefore, it suffices to show that the relaxation admits integral optimal solutions. To see this, we first lift all integral base polyhedra $\mathcal{B}_{f_i} \subset \mathbb{Z}^m$ to the higher dimensional space $\bar{\mathcal{B}}_{f_i} \subset \mathbb{Z}^{n\cdot m}$ by introducing n copies $E_i, i \in N$, of the elements E leading to $\bar{E} := \dot{\cup}_{i\in N} E_i$ with $E_i = \{e_1^i, \ldots e_m^i\}, i \in N$. The domain of the integral polymatroid function f_i is extended to $\bar{E}$ as follows: $\bar{f}_i(S) := f_i(E_i \cap S)$ for all $S \subseteq \bar{E}$. This way $\bar{f}_i(S)$ remains an integral polymatroid rank function on the lifted space $\mathbb{Z}^{n\cdot m}$. Note that for $\bar{x}_i \in \bar{\mathcal{B}}_{f_i}$, we have $\bar{x}_i \in \mathbb{Z}^{n\cdot m}$ and with $f_i(\{\emptyset\})=0$, we get $\bar{x}_{ij}=0$ for all $j \in \bar{E} \setminus E_i$. By this construction, we get $x_i \in \mathcal{B}_{f_i} \Leftrightarrow \bar{x}_i \in \bar{\mathcal{B}}_{f_i}$. Now we define the Minkowski sum $\bar{\mathcal{B}}_1 := \sum_{i\in N} \bar{\mathcal{B}}_{f_i} \subset \mathbb{Z}^{n\cdot m}$, which is again an integral polymatroid base polyhedron. By this construction we can represent all collections of integral base vectors by a single integral polymatroid base polyhedron.

It remains to also handle the capacity constraint $\ell(x) \leq u$ (note that this is not a box constraint for polymatroid $\bar{\mathcal{B}}_1$). For $S \subseteq \bar{E}$, we define $S_E := \{j \in E | \ \exists\ i \in N \text{ with } e_j^i \in S\}$ as the union of those original element indices (in E) for which S contains at least one copy. With this definition, we define a second polymatroid $\bar{\mathcal{B}}_2 := \{\bar{x} \in \mathbb{Z}^{n\cdot m} | \ \bar{x}(S) \leq h(S) \text{ for all } S \subseteq \bar{E},\ \bar{x}(\bar{E}) = h(\bar{E})\}$, where for $S \subseteq \bar{E}$ $h(S) := \sum_{j\in S_E} u_j$. One can easily verify that h is an integral polymatroid function. Now observe that for the sets $\{e_1^j, \ldots, e_n^j\}, j \in E$, we exactly get the capacity constraint $\bar{x}(\{e_1^j, \ldots, e_n^j\}) \leq u_j, j \in E$. Altogether, with the feasibility of u, the fractional relaxation of problem $\mathrm{P}(u)$ can be reduced to the problem of minimizing the linear objective $\sum_{i\in N}\sum_{j\in E} c_{ij}(u)\bar{x}_{ij}$ over the fractional relaxation of the intersection of $\bar{\mathcal{B}}_1$ and $\bar{\mathcal{B}}_2$. By the fundamental result of Edmonds [13, Thm. (35)], this fractional relaxation is an integral polytope. Note that there are strongly polynomial time algorithms for the polymatroid intersection problem (see Cunningham and Frank [9]). ☐

To the best of our knowledge, this is the first existence result of enforceable tolls in congestion games with player-specific cost functions.

7. Further applications. We now discuss further applications mentioned in the introduction and show that they all fit into the framework.

7.1. Market equilibria. Suppose there are items $E = \{1, \ldots, m\}$ for sale and there is a set $N = \{1, \ldots, n\}$ of buyers interested in buying some of the items. For every subset $S \subseteq E$ of items, player i experiences value $w_i(S) \in \mathbb{R}$ giving rise to a *valuation function* $w_i : 2^m \to \mathbb{R}, i \in N$, where 2^m represents the set of all subsets of E. The market manager wants to determine a price vector $\lambda^* \in \mathbb{R}^m_{\geq 0}$ for selling the items such that every player receives a subset $S_i^* \subseteq E$ maximizing her quasi-linear utility $S_i^* \in \arg\max_{S_i \subseteq E}\{w_i(S_i) - \sum_{j\in S_i} \lambda_j^*\}$ and unsold items have prices equal zero. The tuple $((S_i^*)_{i\in N}, \lambda^*)$ is known as a *Walrasian or competitive equilibrium*.

To show that this example fits into our framework, we construct the following resource allocation model $I = (N, E, X, g, \pi)$. Let $X_i = \{0,1\}^m, i \in N$, be the set of incidence vectors of the set E. The cost function $\pi_i := -v_i$ is given by the negative valuation function $v_i : \mathbb{R}^m \times X_i \to \mathbb{R}, (\tilde{u}, x_i) \mapsto w_i(E(x_i))$ for all $\tilde{u} \in \mathbb{R}^m$, where $E(x_i) := \{j \in E : x_{ij} = 1\}$. The resource consumption function are linear $g_i(x_i) = x_i, i \in N$. We get the following characterization.

LEMMA 7.1. *The tuple $((S_i^*)_{i\in N}, \lambda^*)$ is a Walrasian equilibrium if and only if the tuple (x^*, λ^*) with $E(x_i^*) = S_i^*, i \in N$, weakly enforces $\mathbf{1} \in \mathbb{R}^m$ for I.*

Proof. Note that the condition $\ell(x^*) \le \mathbf{1}$ ensures that every item goes to at most one player. Thus, we get

$$
\begin{aligned}
& & x_i^* &\in \arg\max_{x_i \in X_i} \{v_i(\mathbf{1}, x_i) - (\lambda^*)^\intercal x_i\} \\
\text{(by definition of } v_i) & \Leftrightarrow & x_i^* &\in \arg\max_{x_i \in X_i} \{w_i(E(x_i)) - (\lambda^*)^\intercal x_i\} \\
& \Leftrightarrow & S_i^* &\in \arg\max_{S_i \subseteq E} \left\{ w_i(S_i) - \sum_{j \in S_i} \lambda_j^* \right\}. \qquad \square
\end{aligned}
$$

With this analogy to I we can analyze problem $\mathrm{P}(u)$ for $u = \mathbf{1}$ (for the maximization variant) in more detail:

$$
(\mathrm{P}(\mathbf{1})) \quad \max_x \sum_{i \in N} v_i(\mathbf{1}, x_i) \quad \text{s.t.: } \ell(x) \le \mathbf{1}, \quad x_i \in \{0,1\}^m, i \in N,
$$

$$
(\mathrm{LP}(\mathbf{1})) \quad \max_\alpha \sum_{i \in N} v_i^\intercal \alpha_i \quad \text{s.t.: } \ell(\alpha) \le \mathbf{1}, \quad \alpha_i \in \Lambda_i, i \in N,
$$

where $v_i := (v_i(\mathbf{1}, x_i))_{x_i \in X_i}$ and $\ell(\alpha) := \sum_{i \in N} \sum_{k \in \{1,\dots,k_i\}} \alpha_{ik} x_i^k$.

Clearly, $X_i, i \in N$ consists of finitely many ($k_i = 2^m$) points and thus we can apply Corollary 4.5 to obtain a full characterization of the existence of Walrasian equilibria (which leads precisely to the characterization of Bikchandani and Mamer [4]).

COROLLARY 7.2 (Bikchandani and Mamer [4]). *Walrasian equilibria exist if and only if* $\mathrm{LP}(\mathbf{1})$ *admits integral optimal solutions.*

Multiple items and valuations with externalities. Now we introduce a multi-item model that allows for several items of the same type and some degree of externalities of allocations. There is a finite set $E = \{1, \dots, m\}$ of *item types* where each item is available multiple times. The strategy $x_i \in X_i \subseteq \mathbb{R}^m$ describes the amount of each item that player $i \in N$ buys. Suppose that valuations of players are additive over items, that is, $v_i(\ell(x), x_i) := \sum_{j \in E} v_{ij}(\ell_j(x)) x_{ij}$, where $v_{ij} : \mathbb{R} \to \mathbb{R}_{\ge 0}$ is the nonnegative value player i gets from receiving an item of type j assuming that item j is sold $\ell_j(x)$ many times. A pair (x^*, λ^*) is a Walrasian equilibrium if for all $i \in N$, we have $x_i^* \in \arg\max_{x_i \in X_i} \{v_i(\ell(x_i, x_{-i}^*), x_i) - (\lambda^*)^\intercal x_i\}$. This formulation is not directly comparable to the one before. On the one hand, additivity of valuations over items is less general. On the other hand, several items of the same type can be sold and we allow for a functional dependency of the valuations with respect to the load $\ell_j(x)$. Such dependency may be interesting for situations, where the value $v_{ij}(\cdot)$ of receiving item type j drops as other players also receive the same type—this is referred to as a *setting with negative externalities*.

Assumption 7.3. The function $v_{ij} : \mathbb{R} \to \mathbb{R}_{\ge 0}$ is nonincreasing for all $i \in N, j \in E$.

Under this assumption, we want to determine in the following whether or not a load u may be realized by a Walrasian equilibrium (x^*, λ^*), that is, $\ell(x^*) = u$. It follows by the same argumentation as for atomic congestion games (Lemma 6.4) that it is sufficient to show that $u \in \mathbb{R}^m$ is enforceable for $I = (N, E, X, g, \pi)$ with $g_i(x_i) = x_i$ and $\pi_i(\tilde{u}, x_i) := -v_i(\tilde{u}, x_i)$ for all $\tilde{u} \in \mathbb{R}^m$. This insight allows us to deduce the following statements.

COROLLARY 7.4. *Any $u \in \ell(X)$ can be realized by a Walrasian equilibrium if*

1. *$X_i \subseteq \{0,1\}^m$ corresponds to the independence set of a matroid for any $i \in N$;*
2. *$X_i \subseteq \{0,1\}^m$ corresponds to the set of bases of a matroid for any $i \in N$;*
3. *the valuations are homogeneous $v_{ij} = v_j, i \in N, j \in E$.*

For the first case, the same construction as used in the proof of Corollary 6.8 adjusted to independence sets instead of bases shows that $\mathrm{P}(u)$ admits strong duality. Thus, the statement follows by Theorem 2.3 since the nonnegativity of the valuations $v_{ij}(u) \geq 0$ and the fact that $X_i, i \in N$ are independence sets of matroids imply that $\mathrm{P}(u)$ also admits optimal solutions $x^* \in X$ with $\ell(x^*) = u$. The second case is a direct consequence of Corollary 6.8. Regarding the third case, one may immediately verify that any (x^*, λ^*) with $\lambda^* = (v_j(u))_{j \in E}$ and $\ell(x^*) = u$ is primal-dual optimal for $\mathrm{P}(u)$, and thus, the statement follows again by Theorem 2.3.

7.2. Trading networks. A bilateral trading network is represented by a directed multigraph $G = (N, E)$, where N is the set of vertices and $E = \{e_1, \ldots, e_m\}$ the set of edges. Each vertex corresponds to a player and each edge $e = (s, b)$ represents a bilateral trade that can take place between s and b. For each $e = (s, b) \in E$, the source vertex s corresponds to the seller and the sink vertex b corresponds to the buyer in the trade. For $i \in N$, let $\delta^+(i)$ and $\delta^-(i)$ be the set of outgoing and incoming edges of vertex $i \in N$ and as usual we denote the set of all edges incident to i by $\delta(i) = \delta^+(i) \cup \delta^-(i)$. For a set of edge prices $\lambda_e \geq 0, e \in E$, we can associate with each possible trade $e = (s, b) \in E$ a price $\lambda_e \geq 0$ with the understanding that the buyer b pays λ_e to the seller s. Given prices $\lambda \in \mathbb{R}^m_{\geq 0}$, an outcome of the game is a set of *realized trades* $S \subseteq E$. The quasi-linear utility of a player $i \in N$ is defined as the value gained from trades plus the monetary income from sells minus the cost of bought trades. The value of realized trades is given by a function $\bar{w}_i : 2^{\delta(i)} \to \mathbb{R}$. We can extend $\bar{w}_i$ to 2^m by taking $w_i : 2^m \to \mathbb{R}, S \mapsto \bar{w}_i(S \cap \delta(i))$. The overall utility for $S \subseteq E$ and $\lambda \in \mathbb{R}^m_{\geq 0}$ is defined as $w_i(S) + \sum_{e \in \delta^+(i) \cap S} \lambda_e - \sum_{e \in \delta^-(i) \cap S} \lambda_e$. A price vector $\lambda^* \in \mathbb{R}^m_{\geq 0}$ and a set of realized trades $S^* \subseteq E$ with

$$S^* \in \arg\max_{S \subseteq E} \left\{ w_i(S) + \sum_{e \in \delta^+(i) \cap S} \lambda^*_e - \sum_{e \in \delta^-(i) \cap S} \lambda^*_e \right\} \text{ for all } i \in N$$

constitutes a competitive equilibrium. The main difference to the Walrasian equilibrium model is that players can simultaneously act as buyers and sellers in different trades. We will cast this problem in the framework by constructing an equivalent model $I = (N, E, X, g, \pi)$. For each player i, we have a vector $x_i \in \{-1, 0, 1\}^m$ with the understanding that $x_{ie} = -1$ if $e \in \delta^+(i)$ and trade e is realized as seller, $x_{ie} = 1$ if $e \in \delta^-(i)$ and trade e is realized as buyer and $x_{ie} = 0$ if $e \notin \delta(i)$ or e is not realized. We thus define $X_i = \{x_i \in \{-1, 0, 1\}^m | x_{ie} = 0, e \notin \delta(i)\}, i \in N$. To complete the description of I, we assume that the resource consumption functions are given as $g_i(x_i) = x_i$ for all $i \in N$ and we define the cost function $\pi_i := -v_i$ by the negative utility function of player $i \in N$ on X_i for all $\tilde{u} \in \mathbb{R}^m$:

$$v_i(\tilde{u}, x_i) := w_i(\{e \in \delta(i) : |x_{ie}| = 1\}). \tag{7.1}$$

With this construction, we have a one-to-one correspondence between $x_i \in X_i$ and sets $S_i \subseteq \delta(i)$ via $E(x_i) := \{e \in \delta(i) : |x_{ie}| = 1\}$. We obtain the following characterizations on the existence of competitive equilibria using the notation $E(x) := \cup_{i \in N} E(x_i)$.

LEMMA 7.5. *Consider a bilateral trading network. Let I be the associated resource allocation model and $x^* \in X$. Then, the following statements are equivalent:*

1. $(E(x^*), \lambda^*) \in E \times \mathbb{R}^m_{\geq 0}$ *is a competitive equilibrium for the bilateral trading network.*
2. *The vector* $u = \mathbf{0}$ *is enforceable via* $(x^*, \lambda^*) \in X \times \mathbb{R}^m_{\geq 0}$ *for* I.

The proof is similar to that of Lemma 7.1. With this characterization, we can again analyze problem $\mathrm{P}(u)$ for $u = \mathbf{0}$ (for the maximization variant) in more detail:

$$(\mathrm{P}(\mathbf{0})) \qquad \max_x \sum_{i \in N} v_i(\mathbf{0}, x_i) \quad \text{s.t.: } \ell(x) \leq \mathbf{0}, \quad x_i \in X_i, i \in N,$$

$$(\mathrm{LP}(\mathbf{0})) \qquad \max_\alpha \sum_{i \in N} v_i^\intercal \alpha_i \quad \text{s.t.: } \ell(\alpha) \leq \mathbf{0}, \quad \alpha_i \in \Lambda_i, i \in N,$$

where $v_i := (v_i(\mathbf{0}, x_i))_{x_i \in X_i}$ and $\ell(\alpha) := \sum_{i \in N} \sum_{k \in \{1,\dots,k_i\}} \alpha_{ik} x_i^k$.

Note that every $X_i, i \in N$ consists of finitely many points ($k_i := 3^{|\delta(i)|}$). Thus, Corollary 4.5 gives a complete characterization of competitive equilibria.

COROLLARY 7.6. *Competitive equilibria for bilateral trading networks exist if and only if* $LP(\mathbf{0})$ *admits integral optimal solutions* α *with* $\ell(\alpha) = \mathbf{0}$.

7.3. Congestion control in communication networks. We consider a model of Kelly, Maulloo, and Tan [27] in the domain of TCP-based congestion control. We are given a directed *capacitated* graph $G = (V, E, u)$, where V are the nodes, E with $|E| = m$ is the edge set, and $u \in \mathbb{R}^m_{\geq 0}$ denote the edge capacities. There is a set of players $N = \{1, \dots, n\}$ and every $i \in N$ is associated with an end-to-end pair $(s_i, t_i) \in V \times V$ and a bandwidth utility function $U_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ measuring the received benefit from sending net flow from s_i to t_i. As in congestion games, a *flow* for $i \in N$ is a nonnegative vector $x_i \in \mathbb{R}^{|E|}_{\geq 0}$ that lives in the flow polyhedron:

$$X_i = \left\{ x_i \in \mathbb{R}^m_{\geq 0} | \sum_{j \in \delta^+(v)} x_{ij} - \sum_{j \in \delta^-(v)} x_{ij} = 0 \text{ for all } v \in V \setminus \{s_i, t_i\} \right\},$$

where $\delta^+(v)$ and $\delta^-(v)$ are the arcs leaving and entering v. We assume $X_i \neq \emptyset$ for all $i \in N$ and we denote the net flow reaching t_i by $\mathrm{val}(x_i) := \sum_{j \in \delta^+(s_i)} x_{ij} - \sum_{j \in \delta^-(s_i)} x_{ij}, i \in N$. The goal in price-based congestion control is to determine edge prices $\lambda_j^*, j \in E$ so that a strategy distribution x^* is induced as an equilibrium respecting the network capacities u and, hence, avoiding congestion. Assuming that resource consumption is linear, that is, $g_i(x_i) = x_i, i \in N$, the equilibrium condition amounts to $x_i^* \in \arg\max_{x_i \in X_i} \{U_i(\mathrm{val}(x_i)) - (\lambda^*)^\intercal x_i\}$ for all $i \in N$. By considering the corresponding model $I = (N, E, X, g, \pi)$ with $\pi_i(\tilde{u}, x_i) := -U_i(\mathrm{val}(x_i))$ for all $\tilde{u} \in \mathbb{R}^m$, we obtain the following result for concave bandwidth utility functions.

THEOREM 7.7 (Kelly, Maulloo, and Tan [27]). *For concave bandwidth utility functions* $U_i, i \in N$, *every capacity vector* $u \in \mathbb{R}^m_{\geq 0}$ *is weakly enforceable.*

Proof. With the concavity of $U_i, i \in N$, problem $\mathrm{P}(u)$ is a convex optimization problem over a polytope and hence satisfies Slater's constraint qualification conditions for strong duality. Thus, Theorem 2.3 implies the result. ∎

Let us turn to models, where the flow polyhedron X_i is intersected with $\mathbb{Z}^m_{\geq 0}$. Most of the previous works in the area of congestion control assume either that there is only a single path per (s_i, t_i) pair or as in Kelly, Maulloo, and Tan [27], the flow is allowed to be fractional. Allowing a fully fractional distribution of the flow, however, is not possible in some applications—the notion of data packets as indivisible units seems more realistic. The issue of completely fractional routing versus integrality requirements has been explicitly addressed by Orda, Rom, and Shimkin [34], Harks and Klimm [22] and Wang et al. [49]. We obtain the following result for single source flow models.

COROLLARY 7.8. *Let the bandwidth utility functions $U_i, i \in N$ be nondecreasing, identical, and linear and assume that all players share the same source $s_i = s, i \in N$. Then, for integral routing models with strategy spaces $X_i' = X_i \cap \mathbb{Z}^m_{\geq 0}$, every capacity vector $u \in \mathbb{Z}^m_{\geq 0}$ is weakly enforceable.*

Proof. For problem $\mathrm{P}(u)$, we can w.l.o.g. change the instance by introducing a supersink and connect all t_i's to the sink with large enough integral capacity. This way, we obtain an ordinary s-t max-flow problem for which the LP-formulation $\mathrm{LP}(u)$ is known to be integral. Thus, the result follows from Corollary 4.5. ∎

While the above result seems to require restrictive assumptions (linear identical bandwidth utilities and a common source), we show in the following that already for two source-sink pairs with identical linear capped bandwidth utilities, enforceability is not guaranteed, unless $P = NP$. A capped linear function $f : \mathbb{R} \to \mathbb{R}$ has the form $f(x) = ax$ for $x \leq x^{\max}$ and $f(x) = ax^{\max}$ for $x \geq x^{\max}$.

PROPOSITION 7.9. *Unless $P = NP$, there is an instance with only two players with different source sink pairs $(s_i, t_i), i \in \{1, 2\}$, and nondecreasing, identical, and linear capped bandwidth utilities $U_i, i \in \{1, 2\}$, for which there is a vector $u \in \mathbb{Z}^m_{\geq 0}$ that is not weakly enforceable.*

Proof. Having capped bandwidth utilities implies that there are only finitely many strategies per player. Thus, we can use the LP complexity result of Theorem 4.7: It remains to prove that the master problem $\mathrm{P}(u)$ is NP-hard and that the demand problem is polynomial time solvable. The demand problem becomes $\max_{x_i \in X_i}\{\mathrm{val}(x_i) - \lambda^\intercal x_i\}$, which is just a max flow problem. For the master problem, it is not hard to see that we can reduce from the two-directed disjoint path problem. For an instance of two-directed disjoint path, we associate the given two source-sink pairs naturally with those of two players $\{1, 2\}$ and assume $u = \mathbf{1}$ and $U(\mathrm{val}(x_i)) = \mathrm{val}(x_i), i \in \{1, 2\}$ with a cap at any value larger equal 1. This way, due to the integrality of the flows, there is a solution to the disjoint path problem if and only if the objective value of the master problem is 2. ∎

8. Conclusions.

We introduced a generic resource allocation problem and studied the question of enforceability of certain load vectors u via (anonymous) pricing of resources. We derived a characterization of enforceable load vectors via studying the duality gap of an associated optimization problem. We further derived a characterization of enforceability connecting a general nonconvex setting with a convexified model. Understanding duality gaps of optimization problems is an active research area; see, for instance, the progress on duality for nonlinear mixed-integer programming (cf. Baes, Oertel, and Weismantel [1]). Thus, our general characterization yields the opportunity to translate progress in this field to economic situations mentioned in the applications.

The characterization of enforceable load vectors for models, where the players' private cost function depends on the induced load of all players, deserves further research. While the characterizations based on the u-parameterized form of cost functions did partially carry over to specialized congestion games (see section 6.2), for a more general formulation new approaches are required.

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