

Pure Nash Equilibria in Resource Graph Games

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Abstract

This paper studies the existence of pure Nash equilibria in resource graph games, which are a general class of strategic games used to succinctly represent the players' private costs. There is a finite set of resources and the strategy set of each player corresponds to a set of subsets of resources. The cost of a resource is an arbitrary function that depends on the load vector of the resources in a specified neighborhood. As our main result, we give complete characterizations of the cost functions guaranteeing the existence of pure Nash equilibria for weighted and unweighted players, respectively.

1. For unweighted players, pure Nash equilibria are guaranteed to exist for any choice of the players' strategy space if and only if the cost of each resource is an arbitrary function of the load of the resource itself and linear in the load of all other resources where the linear coefficients of mutual influence of different resources are symmetric. This implies in particular that for any other cost structure there is a resource graph game that does not have a pure Nash equilibrium.
2. For weighted games where players have intrinsic weights and the cost of each resource depends on the aggregated weight of its users, pure Nash equilibria are guaranteed to exist if and only if the cost of a resource is linear in all resource loads, and the linear factors of mutual influence are symmetric, or there is no interaction among resources and the cost is an exponential function of the local resource load.
3. For the special case that the players' strategy sets are matroids, we show that pure Nash equilibria exist under a local monotonicity property, even when cost functions are player-specific. We point out an application of this result to bilevel load balancing games, which are motivated by the study of network infrastructures that are resilient against external attackers and internal congestion effects.
4. Finally, we discuss the computational complexity of deciding whether a given strategy profile is a pure Nash equilibrium and derive hardness results for network routing games and matroid games, respectively.

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1 Introduction

Multi-agent systems are characterized by the intricate interplay of the different and sometimes conflicting self-interests of a large number of independent individuals. In order to study the effects of selfish behavior on the overall state of these systems game-theoretic solution concepts are used, most notably the concept of a Nash equilibrium. Important questions for the analysis of multi-agents systems are, thus, under which conditions Nash equilibria exists and how they can be computed. For systems with a large number of players (as they frequently appear in multi-agent systems modelling economic, traffic, or telecommunication applications), the representation of the games becomes an important issue. For illustration, consider a system with n agents, each with m strategies. Encoding the payoffs of each agent in each of the m^n strategy profiles requires the encoding of nm^n rational number which is impractical even for modest sizes of n and m . Fortunately, for many multi-agents systems that arise from practical application, the agents' payoff have additional structure that allows for a succinct representation of the payoffs. Examples include extensive form games, congestion games (Rosenthal [32]), graphical games (Kearns et al. [23]), action graph games (Jiang et al. [21]), and local effect games (Leyton-Brown and Tennenholtz [25]).

A general class of games that includes several of the specific classes of games above is the class of *resource graph games* introduced by Jiang et al. [19]. In a resource graph game, we are given a finite set $N = \{1, \dots, n\}$ of players and a finite set $R = \{1, \dots, m\}$ of resources. The strategy set available to player i is a set $X_i \subseteq \{0, 1\}^m$ with a succinct representation.¹ Given a strategy profile $x = (\mathbf{x}_1, \dots, \mathbf{x}_n)$, let $\mathbf{x} = \sum_{i \in N} \mathbf{x}_i \in \mathbb{R}_{\geq 0}^m$ denote the configuration profile representing the total number of players using each resource in strategy profile x . Then, the private cost of player i is defined as

$$\pi_i(x) = \mathbf{x}_i^\top \mathbf{c}(\mathbf{x}) = \sum_{r \in R} x_{i,r} c_r(\mathbf{x}) \quad \text{for all } i \in N,$$

where $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ is an arbitrary function. For most applications, the function $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ itself has a succinct representation of the following form. For each resource r , let $B_r \subseteq R$ be an arbitrary subset of neighbors of $r \in R$ and assume that the function $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$, $\mathbf{x} \mapsto (c_1(\mathbf{x}), \dots, c_m(\mathbf{x}))$ has the property that for every resource $r \in R$ the cost c_r depends only on the configuration profile of the resources in B_r , i.e., $c_r(\mathbf{x}) = c_r(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}_{\geq 0}^m$ with $x_s = y_s$ for all $s \in B_r$. A graphical illustration of such a game is obtained by the graph that has the vertex set R and a directed edge from s to r if and only if $s \in B_r$. This is the intuition behind the name *resource graph games*, see Fig. 1 for an illustration.

Below, we illustrate classes of games that are special cases of resource graph games. We start with the class of unweighted congestion games introduced by Rosenthal [32] as a model for road traffic and for production with demand-dependent costs.

Example 1 (Unweighted congestion games). *When the neighborhood of each resource r contains only r , i.e., $B_r = \{r\}$ for all $r \in R$, the cost of each resource depends only on the number of players using it. We then obtain the class of (unweighted) congestion games as a special case of resource graph games.*

Another example of a subclass of resource graph games is the class of local effect games introduced by Leyton-Brown and Tennenholtz [25]. Compared to unweighted congestion games, they are less general in terms of the players' strategies since only singleton strategies are allowed;

¹Jiang et al. consider a polytopal representation, but the exact specifics how the set X_i is represented is less important in this work, as we focus on existence of equilibria and less on computational aspects.

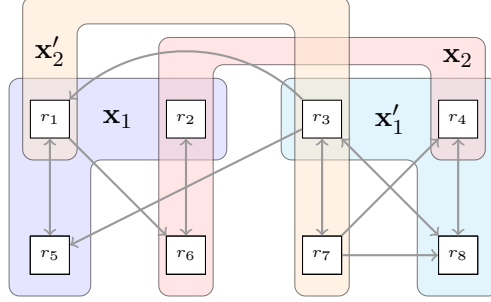


Figure 1: An example of a resource graph game with resource set $R = \{r_1, \dots, r_8\}$ visualizing the non-separable effects among resources w.r.t the cost function. The white rectangles represent the 8 resources, the (bi)directed edges represent the non-separability of costs, that is, a directed edge from node r_j to node r_i indicates that the function value $c_{r_i}(\mathbf{x})$ depends on the entry x_{r_j} , or equivalently, $r_j \in B_{r_i}$. The colored subsets of resources represent the strategies $\mathbf{x}_1, \mathbf{y}_1$ and $\mathbf{x}_2, \mathbf{y}_2$ of the two players $\{1, 2\}$.

in terms of the cost structure on the resources, they are more general since the cost of a resource may also depend on the load of other resources. We call a resource graph game a *singleton game*, if $\sum_{r \in R} x_{i,r} = 1$ for all $\mathbf{x}_i = (x_{i,r})_{r \in R} \in X_i$ and all $i \in N$.

Example 2 (Local effect games). *Local effect games are singleton resource graph games, where for every resource r , there is a function $f_r : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ and for every pair of resources $r, s \in R$ such that $s \in B_r$, there is a function $f_{r,s} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ such that $c_r(\mathbf{x}) = f_r(x_r) + \sum_{s \in B_r} f_{s,r}(x_s)$.*

The following class of action graph games is a generalization of local effect games. They are introduced by Jiang et al. [21] and generalize local effect games as they allow an arbitrary functional dependence of the cost of a resource on the load of all other resources, yet, they are a subclass of resource graph games. Jiang et al. discuss applications in modelling location games, congestion games, and anonymous games, but the class of games is universal as any strategic game can be represented as an action graph game. Thompson and Leyton-Brown [35] show how to use action graph games to compute equilibria in position auctions.

Example 3 (Action graph games). *The class of action graph games is equivalent to the class of singleton resource graph games.*

We now introduce a new class of games related to security games with congestion effects. Consider a load balancing setting where players choose one resource out of a set of resources. After observing the realized loads, a follower attacks the resources with maximum loads causing additional disutilities for the players choosing the attacked resources. Attacks may be thought of as either being actual attacks by a malicious player or as controls by a central authority to counter tax or fare evasion (see Correa et al. [7] for a related mathematical model of fare evasion without any congestion or load balancing effects). In both applications it is sensible to assume that the attacker has a budget B that is spent evenly among the resources with maximal load and that the leaders anticipate this strategy. This motivates the definition of the following class of *bilevel load balancing games* that, to the best of our knowledge, is new in the literature.

Example 4 (Bilevel load balancing games). *Bilevel load balancing games are singleton resource graph games, where for every resource r , the cost is of the form*

$$c_r(\mathbf{x}) := x_r + \kappa_r^*(\mathbf{x}), \text{ where } \kappa_r^*(\mathbf{x}) = \begin{cases} \frac{B}{|\arg \max_{r \in R} \{x_r\}|}, & \text{if } r \in \arg \max_{r \in R} \{x_r\}, \\ 0, & \text{else.} \end{cases} \quad (1)$$

Finally, we mention that the interdependence of costs of resources on the loads of other resources has a long history in non-atomic traffic models. Dafermos [8] proposes the use of such models to model the dependencies of the travel times on opposing directions of a two-lane road and on road segments leading to a common crossing. We obtain the following class of congestion games with non-separable costs as the natural atomic counterpart of the non-atomic traffic models with non-separable costs considered in the traffic literature (Dafermos [8, 9], Smith [34]).

Example 5 (Unweighted network congestion games with non-separable costs). *These games are resource graph games, where the set of resources R corresponds to the set of edges of a road network. For every player i , the strategy set X_i corresponds to the (indicator vectors of the edge set) of all paths between a source node o_i and a destination node d_i in the road network. One typically assumes that $\mathbf{c} : \mathbb{R}_{\geq 0}^R \rightarrow \mathbb{R}^m$ is monotonically non-decreasing, i.e., $c_r(\mathbf{x}) \leq c_r(\mathbf{y})$ for all $x, y \in X$ with $x_s \leq y_s$ for all $s \in R$.*

In the examples discussed above, a mixed Nash equilibrium is guaranteed to exist due to Nash's theorem [29]. However, as discussed, e.g., in Jiang and Leyton-Brown [20], pure Nash equilibria are more favorable as a solution concept as they are easier to implement in practice. We are interested in identifying maximal conditions on the cost functions that ensure the existence of pure Nash equilibria in resource graph games.

1.1 Our results

In this paper, we study the existence of pure Nash equilibria for resource graph games with respect to the non-separable cost structures. We call a non-empty set \mathcal{C} of cost functions *consistent*, if every resource graph game with cost functions from \mathcal{C} admits pure Nash equilibria. We only require a natural condition on \mathcal{C} , namely that \mathcal{C} is *closed under composition*. This means that for any two functions $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C}$ ($\mathbf{c}_1 = \mathbf{c}_2$ is allowed) acting on resource sets R_1 and R_2 with $|R_1| = m_1$ and $|R_2| = m_2$, respectively, the cost function $\mathbf{c}_1 \oplus \mathbf{c}_2 : \mathbb{R}_{\geq 0}^{m_1+m_2} \rightarrow \mathbb{R}^{m_1+m_2}$ defined as $\mathbf{c}_1 \oplus \mathbf{c}_2 = (\mathbf{c}_1, \mathbf{c}_2)$ also belongs to \mathcal{C} . This property naturally arises by composing two disjoint subsets R_1 and R_2 so that the cost structure within each set of the disjoint union is given by \mathbf{c}_1 and \mathbf{c}_2 , and there is no interaction between the loads and costs of resources contained in the two different sets. We obtain the following results.

1. As our main result we show in Theorem 1 that a composition-closed set \mathcal{C} of cost functions is consistent if and only if for each $\mathbf{c} \in \mathcal{C}$ with $\mathbf{c} : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{R}^m$ for some $m \in \mathbb{N}$, there are arbitrary functions $f_1, \dots, f_m : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$ and a symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ such that

$$\mathbf{c}(\mathbf{x}) = (f_1(x_1), \dots, f_m(x_m))^\top + \mathbf{A}\mathbf{x}.$$

Our result implies in particular that every resource graph game with this cost structure has a pure Nash equilibrium. This generalizes a result of Leyton-Brown and Tennenholtz [25] who show that for the special case of local effect games with this cost structure, a pure Nash equilibrium exists. Our characterization also implies that for every other cost function $\tilde{\mathbf{c}}$ that does not adhere to this form, there is an unweighted resource graph game with costs defined by $\tilde{\mathbf{c}}$ that does not have a pure Nash equilibrium. For the proof of this result, we construct several highly symmetric resource graph games that allow to derive functional equations on the set of consistent cost functions that combined leave cost functions of the form above as the only possibility. Our results are also relevant for related work on the complexity of deciding the existence of pure Nash equilibria in action graph games (Jiang and Leyton-Brown [20]) as it implies that this computational problem is trivial for cost functions of the required form above.

2. We then study *weighted* resource graph games, a natural generalization of resource graph games, where every player i has an intrinsic weight w_i and their strategy set is $X_i = \{w_i \mathbf{x}_i : \mathbf{x}_i \in Y_i\}$, where $Y_i \subseteq \{0, 1\}^m$ is arbitrary. These games are relevant as a more fine-grained model for congestion games with non-separable costs, where the players have a different impact on the costs of the resources. We also provide a full characterization of the cost functions that are consistent for weighted resource graph games. Specifically, we show in Theorem 2 that a composition-closed set \mathcal{C} of continuous cost functions is consistent if and only if for each $\mathbf{c} \in \mathcal{C}$ with $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ for some $m \in \mathbb{N}$, either \mathbf{c} consists of separable exponential functions with a common exponent $\phi \in \mathbb{R}$, or there is a symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ and vector $\mathbf{b} \in \mathbb{R}^m$ such that $\mathbf{c}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$.
3. If the players' strategy spaces are restricted by some combinatorial property, then other non-separable cost functions are possibly consistent. In this regard, we consider matroid bases as such combinatorial domain. We show in Theorem 3 that pure Nash equilibria exist under a local monotonicity property, even when cost functions are player-specific. We demonstrate the applicability of this result by deriving an existence result of pure Nash equilibria for bilevel load balancing games as introduced in Example 4. This class of games is motivated by the study of network infrastructures facing external attackers and internal congestion effects.
4. Finally, in Section 6, we discuss the computational complexity of deciding whether a given strategy profile is a pure Nash equilibrium and derive hardness results for network routing games and matroid games, respectively.

1.2 Related work

Rosenthal [32] shows that every unweighted congestion game with separable costs has a pure Nash equilibrium. Milchtaich [28] proposes two generalizations of unweighted congestion games. In the first generalization, called weighted congestion games, each player has a weight and the cost of each resource depends on the aggregated weight of its users. In the second generalization, called congestion games with player-specific costs, every player has an individual cost function for each resource. Both generalizations alone still admit a pure Nash equilibrium for singleton congestion games, but the combination of both games fails to provide pure Nash equilibria, even for singleton games. The positive result for singletons is generalized by Ackermann et al. [2] to games, where the strategy set of each player corresponds to the set of basis of a matroid. Weighted congestion games with general strategy spaces may fail to have a pure Nash equilibrium (Goemans et al. [14], Libman and Orda [26]), but have a pure Nash equilibrium for affine costs or exponential costs (Harks and Klimm [15], Harks et al. [16], Fotakis et al. [12], Panagopoulou and Spirakis [30]). Local effect games are introduced by Leyton-Brown and Tennenholtz [25] who show that a pure Nash equilibrium exists when the mutual influence of different resources on the cost is linear and symmetric. Dunkel and Schulz [11] show that for these games the computation of a pure Nash equilibrium is PLS-complete. They also show that for both local effect games with non-linear mutual effects and weighted congestion games with arbitrary cost functions, it is NP-hard to decide whether a pure Nash equilibrium exists. The PLS-completeness of computing a pure Nash equilibrium in unweighted congestion games with affine costs due to Ackermann et al. [1] carries over to the weighted case.

Action graph games are introduced by Bhat and Leyton-Brown [3] and Jiang et al. [21] as a generalization of local-effect games. They show that every strategic game can be represented as an action graph game. Daskalakis et al. [10] give a fully polynomial-time approximation scheme (FPTAS) for computing an approximate mixed equilibrium in action graph games with constant

degree, constant treewidth, and a constant number of agent types. They also give several hardness results for the case that one of the conditions on the game is violated. Jiang and Leyton-Brown [20] show that for symmetric action graph games played on a graph of bounded treewidth, it can be decided efficiently whether a pure Nash equilibrium exists while the problem is NP-hard to decide in general.

Resource graph games are introduced as by Jiang et al. [19] as a further generalization of action graph games. Chan and Jiang [5] give an FPTAS for computing an approximate Nash equilibrium in resource graph games with a constant number of player types and further restrictions on the strategy sets.

Congestion games with non-atomic players where the load of one resource has an impact on the cost of another resource are usually called *congestion games with non-separable costs*. They were first proposed by Dafermos [8, 9]. She shows that the equilibrium condition can be formulated as an optimization problem, if the Jacobian of the cost function is symmetric. Smith [34] provides a variational inequality for the non-symmetric case. Perakis [31] studies the price of anarchy of non-atomic congestion games with linear non-separable costs of the form $c(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$.

Bilevel, Stackelberg, or Leader-Follower games, have been studied extensively over the last years. In these games, the players are partitioned into *leaders*, acting first, and *followers*, choosing their strategy only after the leaders' choices become apparent. Such hierarchical relationships appear in many real-world problems, e.g. in pricing or toll setting problems [24, 6, 17], security games [33, 22], fare evasion games [7], supply chain and marketing management [18], or in voting scenarios [36]. In the context of bilevel games with congestion effects, Castiglioni et al. [4] and Marchesi et al. [27] considered Stackelberg games with an underlying unweighted congestion game. However, they assume that there is only one leader and the leader participates in the same congestion game as the followers (but the leader's congestion cost functions may be different from the followers'). Depending on the structure of strategy spaces and congestion cost functions, they analyze the computational complexity of computing Stackelberg equilibria. In particular, they devise efficient algorithms for singleton strategy spaces, where either all followers have the same strategies [4], or the followers can be divided in "classes" having the same strategies [27]. The case of multiple leaders playing an unweighted or weighted congestion game subject to followers affecting the resource costs (as for instance by attacks as modeled in Example 4) is, to the best of our knowledge, completely open.

2 Preliminaries

For an integer $k \in \mathbb{N}$, let $[k] := \{1, \dots, k\}$. Let $N = [n]$ be a finite set of players and $R = [m]$ be a finite set of m resources. For each player i , the set of strategies available to player i is an arbitrary set $X_i \subseteq \{0, 1\}^m$. We call $x = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ with $x_i \in X_i$ for all $i \in N$ a strategy profile and $X = X_1 \times \dots \times X_n$ the strategy space. We use standard game theory notation; for a strategy profile $x \in X$, we write $x = (\mathbf{x}_i, x_{-i})$ meaning that \mathbf{x}_i is the strategy that player i plays in x and x_{-i} is the partial strategy profile of all players except i . Every strategy profile $x = (\mathbf{x}_1, \dots, \mathbf{x}_n) \in X$ induces a load vector $\mathbf{x} = \sum_{i \in [n]} \mathbf{x}_i \in \mathbb{R}_{\geq 0}^m$. For a set $S \subseteq R = [m]$, we denote by $\mathbf{1}_S$ the indicator vector of set S in \mathbb{R}^m . We are further given a cost function $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$. Usually, one assumes that \mathbf{c} has a succinct representation of the following form. For every resource $r \in R$, there is a neighborhood $B_r \subseteq R$ such that $c_r(\mathbf{x})$ is independent of x_s for all $s \notin B_r$, i.e., $c_r(\mathbf{x}) = c_r(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$ with $x_s = y_s$ for all $s \in B_r$. If this is the case, and $|B_r| \leq k$ for all $r \in R$ the function \mathbf{c} can be encoded by mn^k numbers since it suffices to specify for each $r \in R$ the value of $c_r(\mathbf{x})$ as a function of the n^k possible load vectors of the resources in B_r . Intuitively, the function \mathbf{c}

maps a load vector $\mathbf{x} \in \mathbb{R}_{\geq 0}^m$ to a cost vector $\mathbf{c}(\mathbf{x}) \in \mathbb{R}^m$, i.e., $\mathbf{c}(\mathbf{x}) = (c_1(\mathbf{x}), \dots, c_m(\mathbf{x}))^\top$ and for a resource $r \in R$ the cost experienced by players using e when the congestion vector is \mathbf{x} is $c_r(\mathbf{x})$. The strategic game $G = (N, X, (\pi_i)_{i \in N})$ where the private cost of player i in strategy profile $x \in X$ is defined as $\pi_i(x) = \mathbf{x}_i^\top \mathbf{c}(\mathbf{x}) = \sum_{r \in R} x_{i,r} c_r(\mathbf{x})$ is called a *resource graph game*.

We also consider a generalization of resource graph game to weighted players. In a *weighted resource graph game*, every player $i \in N$ has a weight $w_i \in \mathbb{R}_{> 0}$. The strategy set of player i is then defined as $X_i = \{w_i \mathbf{x}_i : \mathbf{x}_i \in Y_i\}$ where $Y_i \subseteq \{0, 1\}^m$ is arbitrary. Compared to unweighted resource graph games, in a weighted resource graph game, the set of vectors in the strategy of player i is multiplied with the scalar w_i . The weighted resource graph game is then the strategic game $G = (N, X, (\pi_i)_{i \in N})$ where π_i is defined as before.

A strategy profile $x \in X$ is a *pure Nash equilibrium*, if $\pi_i(x) \leq \pi_i(\mathbf{y}_i, x_{-i})$ for all $i \in N$ and $\mathbf{y}_i \in X_i$. For a non-empty set \mathcal{C} set of cost functions, we are interested in establishing conditions on \mathcal{C} that ensure that every resource graph game with cost functions from \mathcal{C} admits pure Nash equilibria. We require a mild technical assumption on \mathcal{C} , namely that \mathcal{C} is *closed under composition* in the following sense. Let $\mathbf{c}, \mathbf{c}' \in \mathcal{C}$ with $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ and $\mathbf{c}' : \mathbb{R}_{\geq 0}^{m'} \rightarrow \mathbb{R}^{m'}$. Then, we require that the function $\mathbf{c} \oplus \mathbf{c}' : \mathbb{R}_{\geq 0}^{m+m'} \rightarrow \mathbb{R}^{m+m'}$ defined as $\mathbf{c} \oplus \mathbf{c}'(\mathbf{x}, \mathbf{y}) = (\mathbf{c}(\mathbf{x}), \mathbf{c}'(\mathbf{y}))$ is also contained in \mathcal{C} . This is an intuitive property of a set of functions \mathcal{C} for the following reasons. The cost functions \mathbf{c}, \mathbf{c}' each define a cost structure on sets of resources R, R' with $|R| = m$ and $|R'| = m'$. The cost function $\mathbf{c} \oplus \mathbf{c}' : \mathbb{R}_{\geq 0}^{m+m'} \rightarrow \mathbb{R}^{m+m'}$ then defines a cost structure on the disjoint union of R and R' , where the cost structure within each set of the disjoint union is given by \mathbf{c} or \mathbf{c}' , and there is no interaction between the loads and costs of resources contained in the two different sets. In particular, for any $k \in \mathbb{N}$ and any $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$, the k -fold disjoint union $\mathbf{c} \oplus \dots \oplus \mathbf{c} : \mathbb{R}_{\geq 0}^{km} \rightarrow \mathbb{R}^{km}$ is contained in \mathcal{C} . In the following, we denote the k -fold disjoint union of \mathbf{c} by \mathbf{c}^k . For a set \mathcal{C} of cost functions as above, we say that \mathcal{C} is *consistent for unweighted resource graph games*, if for every $\mathbf{c} \in \mathcal{C}$, we have that every unweighted resource graph game with costs given by \mathbf{c} has a pure Nash equilibrium. Recall that when $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ with $m \in \mathbb{N}$, then every unweighted resource graph game with costs given by \mathbf{c} has m resources. Consistency for weighted resource graph games is defined analogously.

3 Resource graph games with unweighted players

In this section, we consider unweighted resource graph games. Since in such a game, the load on each resource is a nonnegative integer, it is without loss of generality to assume that the domain of all cost functions is the non-negative integer lattice, that is, they are of the form $\mathbf{c} : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{R}^m$ for some $m \in \mathbb{N}$. Our main result gives a complete characterization of consistency for unweighted resource graph games.

Theorem 1. *Let \mathcal{C} be a set cost functions that is closed under composition. Then the following two statements are equivalent:*

1. \mathcal{C} is consistent for unweighted resource graph games.
2. For each $\mathbf{c} \in \mathcal{C}$ with $\mathbf{c} : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{R}^m$ for some $m \in \mathbb{N}$, there are functions $f_1, \dots, f_m : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$ and a symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ such that

$$\mathbf{c}(\mathbf{x}) = (f_1(x_1), \dots, f_m(x_m))^\top + \mathbf{A}\mathbf{x}. \quad (2)$$

In particular, the set \mathcal{C}^ of all cost functions of the form (2) is the unique maximal set of cost functions that is closed under composition and consistent for unweighted resource graph games.*

We subdivided the proof of both directions in the following subsections.

3.1 Proof of Theorem 1: 2. \Rightarrow 1.

We first prove that statement 2. of Theorem 1 implies consistency of \mathcal{C} . Observe that any composition of functions of form (2) is again of form (2). It is thus sufficient to show existence of a pure Nash equilibrium for any any unweighted resource graph game with a cost function of this form.

Lemma 1. *Let G be an unweighted resource graph game on m resources with cost function $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ given by $\mathbf{c}(\mathbf{x}) = (f_1(x_1), \dots, f_m(x_m))^\top + \mathbf{A}\mathbf{x}$, where $f_1, \dots, f_m : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$ are arbitrary functions and $\mathbf{A} \in \mathbb{R}^{m \times m}$ is a symmetric matrix. Then G has a pure Nash equilibrium.*

Proof. Fix an arbitrary unweighted resource graph game G whose cost is determined by $\mathbf{c} \in \mathcal{C}$ and an arbitrary strategy profile $x \in X$. Consider the process of adding the players to the game in order $1, \dots, n$ and let us sum their private costs. In the following, we write $\mathbf{x}_{\leq i} = \sum_{j \in N: j \leq i} \mathbf{x}_j$ for the load vector of players up to i . Let $P(x)$ be the sum of the private costs of the player added to the game when adding them in order $1, \dots, n$. We define the function $\mathbf{f} : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{R}^m$ as $\mathbf{f}(\mathbf{x}) = (f_1(x_1), \dots, f_m(x_m))^\top$ and obtain

$$P(x) = \sum_{i \in N} \mathbf{x}_i^\top \left[\mathbf{f}(\mathbf{x}_{\leq i}) + \mathbf{A}\mathbf{x}_{\leq i} \right].$$

We have

$$\sum_{i \in N} \mathbf{x}_i^\top \mathbf{f}(\mathbf{x}_{\leq i}) = \sum_{r \in R} \sum_{k=1}^{x_r} f_r(k)$$

as well as

$$\begin{aligned} \sum_{i \in N} \mathbf{x}_i^\top \mathbf{A}\mathbf{x}_{\leq i} &= \sum_{i \in N} \mathbf{x}_i^\top \mathbf{A} \left(\sum_{j \in N: j \leq i} \mathbf{x}_j \right) \\ &= \frac{1}{2} \sum_{i \in N} \sum_{j \in N} \mathbf{x}_i^\top \mathbf{A}\mathbf{x}_j + \frac{1}{2} \sum_{i \in N} \mathbf{x}_i^\top \mathbf{A}\mathbf{x}_i \\ &= \frac{1}{2} \mathbf{x}^\top \mathbf{A}\mathbf{x} + \frac{1}{2} \sum_{i \in N} \mathbf{x}_i^\top \mathbf{A}\mathbf{x}_i, \end{aligned}$$

where for the second equation we used the symmetry of \mathbf{A} . We obtain

$$P(x) = \sum_{r \in R} \sum_{k=1}^{x_r} f_r(k) + \frac{1}{2} \mathbf{x}^\top \mathbf{A}\mathbf{x} + \frac{1}{2} \sum_{i \in N} \mathbf{x}_i^\top \mathbf{A}\mathbf{x}_i.$$

This shows that $P(x)$ is invariant under a reordering of the players. Next, consider a deviation of an arbitrary player. Since $P(x)$ is invariant under a reordering of the players, it is without loss of generality to assume that player n deviates. We obtain

$$\begin{aligned} P(\mathbf{y}_i, x_{-i}) - P(x) &= \mathbf{y}_n^\top \left[\mathbf{f}(\mathbf{x}_{\leq n-1} + \mathbf{y}_n) + \mathbf{A}(\mathbf{x}_{\leq n-1} + \mathbf{y}_n) \right] \\ &\quad - \mathbf{x}_n^\top \left[\mathbf{f}(\mathbf{x}_{\leq n}) + \mathbf{A}(\mathbf{x}_{\leq n}) \right] \\ &= \pi_n(\mathbf{y}_i, x_{-i}) - \pi_n(x). \end{aligned}$$

We conclude that P is an exact potential function and, hence, G admits a pure Nash equilibrium. Since G was chosen arbitrarily, the result follows. \square

3.2 Proof of Theorem 1: 1. \Rightarrow 2.

In the following, we show that statement 2 of Theorem 1 is a necessary condition for the consistency of \mathcal{C} . We prove this by constructing for any given $\mathbf{c} \in \mathcal{C}$ a family of different resource graph games whose cost functions are 4-fold compositions of \mathbf{c} . All these games will have the following symmetry property, which we will use to establish that \mathbf{c} is indeed of the form (2).

Definition 1. Let $A, B \in \mathbb{R}$. We say a game $G = (N, X, (\pi_i)_{i \in N})$ is (A, B) -symmetric for players $i, j \in N$, if for any strategy profile $x \in X$, the following two statements are fulfilled:

- $\pi_i(x) = A$ and $\pi_j(x) = B$ or $\pi_i(x) = B$ and $\pi_j(x) = A$.
- There are $\mathbf{y}_i \in X_i$ and $\mathbf{y}_j \in X_j$ such that $\pi_i(\mathbf{y}_i, x_{-i}) = \pi_j(x)$ and $\pi_j(\mathbf{y}_j, x_{-j}) = \pi_i(x)$.

The following lemma shows a key property for (A, B) -symmetric games.

Lemma 2. If a game G is (A, B) -symmetric for players $i, j \in N$ and admits a pure Nash equilibrium, then $A = B$.

Proof. Let $x \in X$ be a pure Nash equilibrium for G . Because G is symmetric for i and j , there are $\mathbf{y}_i \in X_i$ and $\mathbf{y}_j \in X_j$ such that $\pi_i(\mathbf{y}_i, x_{-i}) = \pi_j(x)$ and $\pi_j(\mathbf{y}_j, x_{-j}) = \pi_i(x)$. Because x is a pure Nash equilibrium, we obtain

$$\pi_i(x) \leq \pi_i(\mathbf{y}_i, x_{-i}) = \pi_j(x) \leq \pi_j(\mathbf{y}_j, x_{-j}) = \pi_i(x)$$

and hence $\pi_i(x) = \pi_j(x)$. Note that symmetry of G implies $\{A, B\} = \{\pi_i(x), \pi_j(x)\}$ and therefore $A = B$. \square

We proceed to prove a first functional equation that needs to be satisfied for a set of consistent cost functions that is closed under composition. The equation states that a discrete version of the Jacobian of the cost function must be symmetric. For the proof, we construct a suitable (A, B) -symmetric game.

Lemma 3. Let \mathcal{C} be closed under composition and consistent for unweighted resource graph games. Then, for all $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{R}^m$ with $m \in \mathbb{N}$, we have

$$c_r(\mathbf{x} + \mathbf{1}_{\{r,s\}}) - c_r(\mathbf{x} + \mathbf{1}_r) = c_s(\mathbf{x} + \mathbf{1}_{\{r,s\}}) - c_s(\mathbf{x} + \mathbf{1}_s)$$

for all $r, s \in [m]$ and all $\mathbf{x} \in \mathbb{Z}_{\geq 0}^m$.

Proof. Let $\mathbf{c} \in \mathcal{C}$, $r, s \in [m]$ with $r \neq s$, let $\mathbf{x} \in \mathbb{Z}_{\geq 0}^m$ be arbitrary. Since \mathcal{C} is closed under composition, we have that $\mathbf{c}^4 : \mathbb{Z}_{\geq 0}^{4m} \rightarrow \mathbb{R}^{4m}$ is also contained in \mathcal{C} . Consider the following game with $4m$ resources and cost function \mathbf{c}^4 . For $k \in [4]$ and $t \in [m]$ we denote the k -th copy of resource t by t_k . For each original resource $t \in [m]$, there are x_t dummy players whose only strategy is $\sum_{k \in [4]} \mathbf{1}_{t_k}$. In addition there are two players 1 and 2 with strategy sets $X_1 = \{\mathbf{1}_{\{r_1, s_2\}}, \mathbf{1}_{\{s_3, r_4\}}\}$ and $X_2 = \{\mathbf{1}_{\{s_1, r_3\}}, \mathbf{1}_{\{r_2, s_4\}}\}$. See Figure 2(i) for a depiction of the strategy space.

It is straightforward to check that the above game is (A, B) -symmetric for players 1 and 2 with

$$A = c_r(\mathbf{x} + \mathbf{1}_{\{r,s\}}) + c_s(\mathbf{x} + \mathbf{1}_s) \quad \text{and} \quad B = c_r(\mathbf{x} + \mathbf{1}_r) + c_s(\mathbf{x} + \mathbf{1}_{\{r,s\}}).$$

Since \mathcal{C} is consistent, the thus defined game has a pure Nash equilibrium and we conclude $A = B$ by Lemma 2, which completes the proof of the lemma. \square

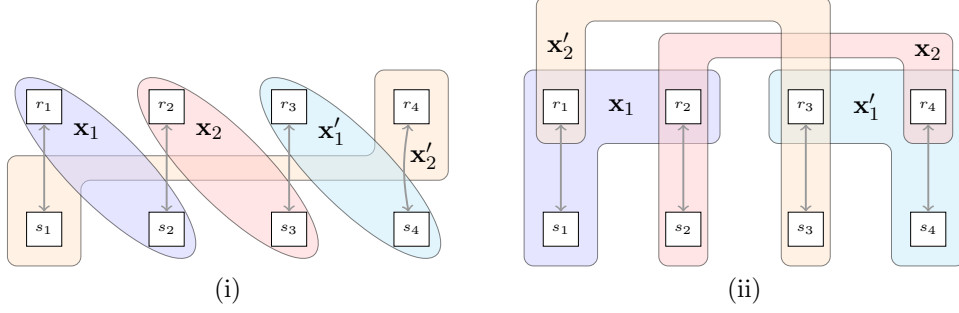


Figure 2: Games constructed for the proofs of Lemmas 3 and 4, respectively. Each clique represents a copy of the resource set (resources other than r, s and dummy players are omitted). Player 1 chooses among strategies \mathbf{x}_1 and \mathbf{x}'_1 , player 2 chooses among strategies \mathbf{x}_2 and \mathbf{x}'_2 .

The following two lemmas establish that the discrete Hessian of each c_r for $r \in [m]$ must be diagonal. For the proof of these two lemmas, we use the symmetry of the Jacobian shown in Lemma 3 together with suitably constructed (A, B) -symmetric games.

Lemma 4. *Let \mathcal{C} be closed under composition and consistent for unweighted resource graph games. Then, for all $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{R}^m$ with $m \in \mathbb{N}$, the following two functional equations are satisfied for all $r, s \in [m]$ with $r \neq s$ and all $\mathbf{x} \in \mathbb{Z}_{\geq 0}^R$ with $x_r > 0$:*

- (a) $c_r(\mathbf{x} + \mathbf{1}_s) - c_r(\mathbf{x}) = c_r(\mathbf{x} + \mathbf{1}_{\{r,s\}}) - c_r(\mathbf{x} + \mathbf{1}_r)$ and
- (b) $c_r(\mathbf{x} + 2 \cdot \mathbf{1}_s) - c_r(\mathbf{x} + \mathbf{1}_s) = c_r(\mathbf{x} + \mathbf{1}_s) - c_r(\mathbf{x})$.

Proof. Let $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{R}^m$, $r, s \in [m]$ with $r \neq s$, and $\mathbf{x} \in \mathbb{Z}_{\geq 0}^m$ with $x_r > 0$ be arbitrary. Since \mathcal{C} is closed under composition, the function $\mathbf{c}^4 : \mathbb{Z}_{\geq 0}^{4m} \rightarrow \mathbb{R}^{4m}$ is also contained in \mathcal{C} . Consider the following game with $4m$ resources and cost function \mathbf{c}^4 . For $k \in [4]$ and $t \in [m]$, we denote the k -th copy of resource t by t_k . For each original resource $t \in [m] \setminus \{r\}$, there are x_t dummy players whose only strategy is $\sum_{k \in [4]} \mathbf{1}_{t_k}$. There also are $x_r - 1$ dummy players for resource r whose only strategy is $\sum_{k \in [4]} \mathbf{1}_{r_k}$. In addition there are two players 1 and 2 with strategy sets

$$X_1 = \{\mathbf{1}_{\{r_1, s_1, r_2\}}, \mathbf{1}_{\{r_3, r_4, s_4\}}\} \quad \text{and} \quad X_2 = \{\mathbf{1}_{\{r_1, r_3, s_3\}}, \mathbf{1}_{\{r_2, s_2, r_4\}}\}.$$

See Figure 2(ii) for a depiction of the strategy space. It is easy to check that the above game is (A, B) -symmetric for players 1 and 2 with

$$\begin{aligned} A &= c_r(\mathbf{x} + \mathbf{1}_{\{r,s\}}) + c_s(\mathbf{x} + \mathbf{1}_{\{r,s\}}) + c_r(\mathbf{x}) \quad \text{and} \\ B &= c_r(\mathbf{x} + \mathbf{1}_s) + c_s(\mathbf{x} + \mathbf{1}_s) + c_r(\mathbf{x} + \mathbf{1}_{\{r,s\}}). \end{aligned}$$

By consistency of \mathbf{c} , the game must have a pure Nash Equilibrium and thus $A = B$ by Lemma 2. Subtracting the first and third term of A and the second term of B on both sides yields

$$c_s(\mathbf{x} + \mathbf{1}_{\{r,s\}}) - c_s(\mathbf{x} + \mathbf{1}_s) = c_r(\mathbf{x} + \mathbf{1}_s) - c_r(\mathbf{x}). \quad (3)$$

Applying Lemma 3 to the left-hand side of (3) yields

$$c_r(\mathbf{x} + \mathbf{1}_{\{r,s\}}) - c_r(\mathbf{x} + \mathbf{1}_r) = c_r(\mathbf{x} + \mathbf{1}_s) - c_r(\mathbf{x}),$$

which proves (a).

Applying Lemma 3 to the right-hand side of (3) instead, yields

$$c_s(\mathbf{x} + \mathbf{1}_{\{r,s\}}) - c_s(\mathbf{x} + \mathbf{1}_s) = c_s(\mathbf{x} + \mathbf{1}_s) - c_s(\mathbf{x} - \mathbf{1}_r + \mathbf{1}_s),$$

which is equivalent to (b) when substituting \mathbf{x} for $\mathbf{x} - \mathbf{1}_r + \mathbf{1}_s$ and then swapping the roles of r and s . \square

Lemma 5. *Let \mathcal{C} be closed under composition and consistent for unweighted resource graph games. Then, for all $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{R}^m$ with $m \in \mathbb{N}$ we have*

$$c_r(\mathbf{x} + \mathbf{1}_s) - c_r(\mathbf{x}) = c_r(\mathbf{x} + \mathbf{1}_{\{s,t\}}) - c_r(\mathbf{x} + \mathbf{1}_t)$$

for all $r, s, t \in [m]$ with r, s, t pairwise distinct and all $\mathbf{x} \in \mathbb{Z}_{\geq 0}^m$ with $x_r > 0$.

Proof. Let $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{R}^m$, $m \in \mathbb{N}$ be arbitrary, let $r, s, t \in [m]$ be pairwise distinct, and let $\mathbf{x} \in \mathbb{Z}_{\geq 0}^m$ with $x_r > 0$. Let $\mathbf{x}' = \mathbf{x}_r - \mathbf{1}_r$. Consider the following game with $4m$ resources and cost function \mathbf{c}^4 . For $k \in [4]$ and $u \in [m]$, we denote the k -th copy of resource u by u_k . For each resource $u \in [m] \setminus \{r\}$, there are x'_u dummy players whose only strategy is $\sum_{k \in [4]} \mathbf{1}_{u_k}$. In addition there are two players 1 and 2 with strategy sets

$$X_1 = \{\mathbf{1}_{\{r_1, s_2, t_2\}}, \mathbf{1}_{\{s_3, t_3, r_4\}}\} \quad \text{and} \quad X_2 = \{\mathbf{1}_{\{s_1, t_1, r_4\}}, \mathbf{1}_{\{r_2, s_4, t_4\}}\}.$$

See Figure 3 for a depiction of the strategy space. It is easy to check that the above game is (A, B) -symmetric for players 1 and 2 with

$$\begin{aligned} A &= c_r(\mathbf{x}' + \mathbf{1}_{\{r,s,t\}}) + c_s(\mathbf{x}' + \mathbf{1}_{\{s,t\}}) + c_t(\mathbf{x}' + \mathbf{1}_{\{s,t\}}), \quad \text{and} \\ B &= c_r(\mathbf{x}' + \mathbf{1}_r) + c_s(\mathbf{x}' + \mathbf{1}_{\{r,s,t\}}) + c_t(\mathbf{x}' + \mathbf{1}_{\{r,s,t\}}). \end{aligned}$$

By consistency of \mathcal{C} the game must have a pure Nash Equilibrium and thus $A = B$ by Lemma 2. By subtracting the second term of A and the first and third term of B from both sides we obtain

$$\begin{aligned} & c_r(\mathbf{x}' + \mathbf{1}_{\{r,s,t\}}) - \underbrace{(c_t(\mathbf{x}' + \mathbf{1}_{\{r,s,t\}}) - c_t(\mathbf{x}' + \mathbf{1}_{\{s,t\}}))}_{= c_r(\mathbf{x}' + \mathbf{1}_{\{r,s,t\}}) - c_r(\mathbf{x}' + \mathbf{1}_{\{r,s\}})} - c_r(\mathbf{x}' + \mathbf{1}_r) \\ &= \underbrace{c_s(\mathbf{x}' + \mathbf{1}_{\{r,s,t\}}) - c_s(\mathbf{x}' + \mathbf{1}_{\{s,t\}})}_{= c_r(\mathbf{x}' + \mathbf{1}_{\{r,s,t\}}) - c_r(\mathbf{x}' + \mathbf{1}_{\{r,t\}})}. \end{aligned}$$

Applying the identities indicated above, which follow from Lemma 3, and then using $\mathbf{x}' = \mathbf{x} - \mathbf{1}_r$ yields $c_r(\mathbf{x} + \mathbf{1}_s) - c_r(\mathbf{x}) = c_r(\mathbf{x} + \mathbf{1}_{\{s,t\}}) - c_r(\mathbf{x} + \mathbf{1}_t)$. \square

Given the form of the discrete Hessian established in Lemmas 4 and 5, we conclude now that the influence of the load on a resource s on the cost of some other resource r must be linear. This is formalized in the following lemma, which follows by inductively applying our previous results.

Lemma 6. *Let \mathcal{C} be closed under composition and consistent for unweighted resource graph games. Then, for all $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{R}^m$ with $m \in \mathbb{N}$ we have*

$$c_r(\mathbf{x} + \mathbf{1}_s) - c_r(\mathbf{x}) = c_r(\mathbf{y} + \mathbf{1}_s) - c_r(\mathbf{y})$$

for all $r, s \in [m]$ with $r \neq s$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_{\geq 0}^m$ with $x_r, y_r > 0$.

Proof. Let $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{R}^m$, $r, s \in [m]$ with $r \neq s$, and $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_{\geq 0}^m$ with $x_r, y_r > 0$ be arbitrary. We show the lemma by induction on $k = \sum_{r' \in R} |x_{r'} - y_{r'}|$. If $k = 0$, then $x = y$ and the claim is trivially fulfilled. Thus assume $k > 0$. Without loss of generality, there is $t \in R$ with $x_t > y_t$. Let $\mathbf{x}' := \mathbf{x} - \mathbf{1}_t$. Note that $x'_r \geq y_r > 0$, so that Lemma 4 and Lemma 5 can be applied to \mathbf{x}' . We distinguish three cases for t .

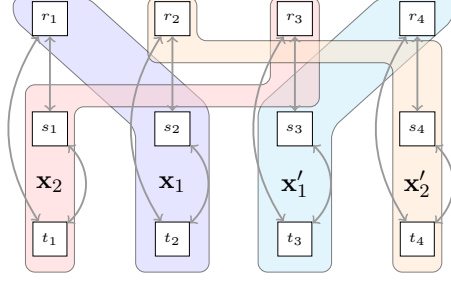


Figure 3: Game constructed for the proof of Lemma 5. Each clique represents a copy of the resource set (resources other than r, s, t and dummy players are omitted). Player 1 chooses among strategies \mathbf{x}_1 and \mathbf{x}'_1 , player 2 chooses among strategies \mathbf{x}_2 and \mathbf{x}'_2 .

Case $t = r$: In this case, $\mathbf{x} = \mathbf{x}' + \mathbf{1}_r$. We obtain

$$c_r(\mathbf{x} + \mathbf{1}_s) - c_r(\mathbf{x}) = c_r(\mathbf{x}' + \mathbf{1}_{\{r,s\}}) - c_r(\mathbf{x}' + \mathbf{1}_r) = c_r(\mathbf{x}' + \mathbf{1}_s) - c_r(\mathbf{x}').$$

where the second equality follows from Lemma 4.

Case $t = s$: In this case, $\mathbf{x} = \mathbf{x}' + \mathbf{1}_s$. We obtain

$$c_r(\mathbf{x} + \mathbf{1}_s) - c_r(\mathbf{x}) = c_r(\mathbf{x}' + 2 \cdot \mathbf{1}_s) - c_r(\mathbf{x}' + \mathbf{1}_s) = c_r(\mathbf{x}' + \mathbf{1}_s) - c_r(\mathbf{x}').$$

where the second equality follows from Lemma 4.

Case $t \in R \setminus \{r, s\}$: In this case, $\mathbf{x} = \mathbf{x}' + \mathbf{1}_t$. We obtain

$$c_r(\mathbf{x} + \mathbf{1}_s) - c_r(\mathbf{x}) = c_r(\mathbf{x}' + \mathbf{1}_{\{s,t\}}) - c_r(\mathbf{x}' + \mathbf{1}_t) = c_r(\mathbf{x}' + \mathbf{1}_s) - c_r(\mathbf{x}').$$

where the second equality follows from Lemma 5.

In either case, $c_r(\mathbf{x} + \mathbf{1}_s) - c_r(\mathbf{x}) = c_r(\mathbf{x}' + \mathbf{1}_s) - c_r(\mathbf{x}')$, which is equal to $c_r(\mathbf{y} + \mathbf{1}_s) - c_r(\mathbf{y})$ by the induction hypothesis because $\sum_{r' \in R} |x'_{r'} - y_{r'}| < k$. \square

Combining Lemmas 3 and 6, we observe that the interaction effects of distinct resources in the cost function \mathbf{c} must be linear and symmetric and thus condition (2) is indeed necessary for consistency. This is formalized in the following lemma, which completes the proof of Theorem 1.

Lemma 7. *Let \mathcal{C} be closed under composition and consistent for unweighted resource graph games. Then, for all $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{R}^m$ with $m \in \mathbb{N}$ there are m functions $f_1, \dots, f_m : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$ and a symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ such that $\mathbf{c}(\mathbf{x}) = (f_1(x_1), \dots, f_m(x_m))^\top + \mathbf{A}_r \cdot \mathbf{x}$ for all $\mathbf{x} \in \mathbb{Z}_{\geq 0}^m$.*

Proof. For $r \in R$ define $f_r : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ by $f_r(x) = c_r(x \cdot \mathbf{1}_r)$ for all $x \in \mathbb{R}_{\geq 0}$. The matrix $\mathbf{A} \in \mathbb{R}^{m \times m} = (a_{r,s})_{r,s \in [m]}$ is defined as

$$a_{r,s} = \begin{cases} c_r(\mathbf{1}_{\{r,s\}}) - c_r(\mathbf{1}_r) & \text{if } r \neq s, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $a_{r,s} = a_{s,r}$ and, hence \mathbf{A} is symmetric by Lemma 3.

Let $\mathbf{x} \in \mathbb{R}_{\geq 0}^m$ and $r \in [m]$ be arbitrary. For $r \in [m]$, we denote by \mathbf{A}_r its r -th row. We show that $c_r(\mathbf{x}) = f_r(x_r) + \mathbf{A}_r \cdot \mathbf{x}$ by induction on $k = \sum_{s \in R \setminus \{r\}} x_s$. For $k = 0$ the claim is true by definition of f_r . Thus assume $k > 0$ and let $s \in R \setminus \{r\}$ with $x_s > 0$. We obtain

$$\begin{aligned} c_r(\mathbf{x}) &= c_r(\mathbf{x} - \mathbf{1}_s) + c_r(\mathbf{1}_{\{r,s\}}) - c_r(\mathbf{1}_r) \\ &= f_r(x_r) + \mathbf{A}_r \cdot (\mathbf{x} - \mathbf{1}_s) + a_{r,s} \\ &= f_r(x_r) + \mathbf{A}_r \cdot \mathbf{x} \end{aligned}$$

where the first identity follows from Lemma 6 and the second identity follows from the induction hypothesis and the definition of $a_{r,s}$. \square

4 Resource graph games with weighted players

In this section, we establish necessary and sufficient conditions for consistency when each player $i \in N$ imposes a weight $w_i \in \mathbb{R}_{\geq 0}$ on the resources in their strategy. The characterization reveals two possible cases: A consistent set of cost functions either contains only affine functions with a symmetric Jacobian, or the cost functions of individual resources are exponential and separable (i.e., there is no interaction among distinct resources).

Theorem 2. *Let \mathcal{C} be a set of continuous cost functions that is closed under composition. Then \mathcal{C} is consistent for weighted resource graph games if and only if one of the following two statements is fulfilled:*

1. *For each $\mathbf{c} \in \mathcal{C}$ with $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ for some $m \in \mathbb{N}$ there is a symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ and a vector $\mathbf{b} \in \mathbb{R}^m$ such that $\mathbf{c}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$.*
2. *There is $\phi \in \mathbb{R}$ such that for all $\mathbf{c} \in \mathcal{C}$ with $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ there are $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$ such that $c_r(\mathbf{x}) = a_r \exp(\phi x_r) + b_r$ for all $r \in [m]$ and all $\mathbf{x} \in \mathbb{R}_{\geq 0}^m$.*

The two distinct cases arise due to the fact that *weighted congestion games* are a special case of weighted resource graph games, namely where the cost of each resource r depends on the load of r only, i.e., $B_r = \{r\}$ for all $r \in R$. For these games Harks and Klimm [15] provided a characterization that shows that consistent sets contain only affine or only exponential cost functions. In the following, we give prove the sufficiency and necessity of either of the two conditions.

4.1 Proof of Theorem 2: 1. or 2. \Rightarrow Consistency of \mathcal{C} :

We show sufficiency of conditions 1 or 2 in Theorem 2, respectively, for the consistency of \mathcal{C} . If condition 2 is fulfilled, then any weighted resource graph game with cost function $\mathbf{c} \in \mathcal{C}$ is a weighted congestion game with exponential costs. For these games, the existence of pure Nash equilibria has been established in [15], Theorem 5.1. It is therefore sufficient to show that condition 1 of Theorem 2 is also sufficient for consistency. The following lemma establishes the sufficiency of condition 1, following the same lines as the proof of Lemma 1.

Lemma 8. *Let G be a weighted resource graph game on m resources with cost function $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ given by $\mathbf{c}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{m \times m}$ is a symmetric matrix and $\mathbf{b} \in \mathbb{R}^m$ is a vector. Then G has a pure Nash equilibrium.*

Proof. Fix an arbitrary weighted resource graph game G whose cost is determined by $\mathbf{c} \in \mathcal{C}$ and an arbitrary strategy profile $x \in X$. As in the proof of Theorem 1, let $P(x)$ be the sum of the

private costs of the players when adding them to the game in order $1, \dots, n$. We again write $\mathbf{x}_{\leq i} = \sum_{j \in N: j \leq i} \mathbf{x}_j$ for the load vector of the players up to i . We then obtain

$$P(x) = \sum_{i \in N} \mathbf{x}_i^\top [\mathbf{A}\mathbf{x}_{\leq i} + \mathbf{b}].$$

Similarly to the proof of Theorem 1, we calculate

$$\begin{aligned} \sum_{i \in N} \mathbf{x}_i^\top \mathbf{A}\mathbf{x}_{\leq i} &= \sum_{i \in N} \mathbf{x}_i^\top \mathbf{A} \left(\sum_{j \in N: j \leq i} \mathbf{x}_j \right) \\ &= \frac{1}{2} \sum_{i \in N} \sum_{j \in N} \mathbf{x}_i^\top \mathbf{A}\mathbf{x}_j + \frac{1}{2} \sum_{i \in N} \mathbf{x}_i^\top \mathbf{A}\mathbf{x}_i \\ &= \frac{1}{2} \mathbf{x}^\top \mathbf{A}\mathbf{x} + \frac{1}{2} \sum_{i \in N} \mathbf{x}_i^\top \mathbf{A}\mathbf{x}_i, \end{aligned}$$

as in the unweighted case where we again used the symmetry of \mathbf{A} . We obtain

$$P(x) = \frac{1}{2} \mathbf{x}^\top \mathbf{A}\mathbf{x} + \frac{1}{2} \sum_{i \in N} \mathbf{x}_i^\top [\mathbf{A}\mathbf{x}_i + \mathbf{b}].$$

This shows that P is independent of the ordering of the player and the remainder of the proof is equivalent to Theorem 1. \square

4.2 Proof of Theorem 2: Consistency of $\mathcal{C} \Rightarrow 1.$ or $2.$:

We now prove the necessity of the conditions given in Theorem 2. By a slight adaptation of the constructions in Section 3, we obtain the following stronger version of Lemmas 3 and 6 for cost functions that are consistent for weighted players.

Lemma 9. *Let \mathcal{C} be closed under composition and consistent for weighted resource graph games. Then, for all $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ with $m \in \mathbb{N}$ the following functional equations are satisfied:*

- (a) $c_r(\mathbf{x} + \varepsilon \cdot \mathbf{1}_{\{r,s\}}) - c_r(\mathbf{x} + \varepsilon \cdot \mathbf{1}_r) = c_s(\mathbf{x} + \varepsilon \cdot \mathbf{1}_{\{r,s\}}) - c_f(\mathbf{x} + \varepsilon \cdot \mathbf{1}_s)$ for all $r, s \in R$ and all $\mathbf{x} \in \mathbb{R}_{\geq 0}^m$ and all $\varepsilon > 0$ and
- (b) $c_r(\mathbf{x} + \varepsilon \cdot \mathbf{1}_s) - c_r(\mathbf{x}) = c_r(\mathbf{y} + \varepsilon \cdot \mathbf{1}_s) - c_r(\mathbf{y})$ for all $r, s \in R$ with $r \neq s$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}_{\geq 0}^m$ with $x_r, y_r > 0$ and all $\varepsilon > 0$.

Proof (Sketch). We follow the same constructions used to establish Lemmas 3, 4, 5, and 6. However, we set $w_1 = w_2 = \varepsilon$ and adjust the weights of the dummy players for each resource such that the load on the resource equals the corresponding coordinate of \mathbf{x} . \square

The following lemma follows from the characterization of consistent functions for weighted congestion games with separable cost functions due to Harks and Klimm [15].

Lemma 10. *Let \mathcal{C} be a set of continuous functions that is closed under composition and consistent for weighted resource graph games. Then, for all $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ with $m \in \mathbb{N}$ one of the following statements is true:*

- (a) For all $S \subseteq R$ and all $\mathbf{z} \in \mathbb{R}_{\geq 0}^m$ there are $a_{S,\mathbf{z}}, b_{S,\mathbf{z}} \in \mathbb{R}$ such that $\sum_{r \in S} c_r(\mathbf{z} + \lambda \mathbf{1}_S) = a_{S,\mathbf{z}} \lambda + b_{S,\mathbf{z}}$ for all $\lambda \geq 0$.

(b) There is $\phi \in \mathbb{R}$ such that for all $S \subseteq R$ and all $\mathbf{z} \in \mathbb{R}_{\geq 0}^m$ there are $a_{S,\mathbf{z}}, b_{S,\mathbf{z}} \in \mathbb{R}$ such that $\sum_{r \in S} c_r(\mathbf{z} + \lambda \mathbf{1}_S) = a_{S,\mathbf{z}} \exp(\phi \lambda) + b_{S,\mathbf{z}}$ for all $\lambda \geq 0$.

Proof. Let $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ with $m \in \mathbb{N}$ be arbitrary. For $S \subseteq R$ and $\mathbf{z} \in \mathbb{R}_{\geq 0}^m$ define $c_{S,\mathbf{z}} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ by $c_{S,\mathbf{z}}(\lambda) = \sum_{r \in S} c_r(\mathbf{z} + \lambda \mathbf{1}_S)$. Let $\mathcal{C}' = \{c_{S,\mathbf{z}} : S \subseteq R, \mathbf{z} \in \mathbb{R}_{\geq 0}^m\}$ be the set of all functions arising in this way. We show that any weighted congestion game with separable cost functions on k resources where each resource has a cost function $c' \in \mathcal{C}'$ is isomorphic to a weighted resource graph game with cost function \mathbf{c}^k on km resources. Since \mathcal{C} is closed under composition, the function \mathbf{c}^k is contained in \mathcal{C} , and, hence, consistency of \mathcal{C} for weighted resource graph games implies the consistency of \mathcal{C}' for weighted congestion games. It is known ([15], Theorem 5.1) that a set of continuous functions is consistent for weighted congestion games if and only if it contains only affine functions (as described in case (a) of the lemma) or it only contains only exponential functions (as described in case (b) of the lemma). Hence the lemma follows from the following construction.

Consider any weighted congestion game G' with arbitrary player set N' , weights w'_i for each $i \in N'$, strategies $X'_i = \{w'_i \cdot \mathbf{x}_i : \mathbf{x}_i \in Y_i\}$ with $Y_i \subseteq \{0, 1\}^k$, and resource set $R' = \{r'_1, \dots, r'_k\}$ such that for all $r \in R'$ the cost function $c'_r : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ of resource r is of the form $c'_r = c_{S,\mathbf{z}}$ for some $S \subseteq R$ and $\mathbf{z} \in \mathbb{R}_{\geq 0}^m$. In what follows, we construct an isomorphic weighted resource graph game G with player set N , mk resources, and cost function \mathbf{c}^k . For $j \in [k]$ let $S_j \subseteq R$ and $\mathbf{z}_j = (z_{j,1}, \dots, z_{j,m}) \in \mathbb{R}_{\geq 0}^m$ be such that $c'_{r_j} = c_{S_j, \mathbf{z}_j}$. We define $N = N' \cup \{(r, j) : r \in R, j \in [k]\}$, i.e., the set of players N of G contains the player set N' of the original congestion game plus mk additional dummy players. Each dummy player (r, j) can only play strategy $\mathbf{1}_{r_j}$ where r_j is the j -th copy of resource $r \in [m]$. That dummy player has a weight $w_{(r,j)} = z_{j,r}$. Each normal player $i \in N'$ has the same weight $w_i = w'_i$ as in the original congestion game. For each strategy $\mathbf{x}'_i \in X'_i$ of player i in the original congestion game, there is a strategy $\mathbf{x}_i \in X_i$ that arises from \mathbf{x}'_i by replacing each resource $r_j \in R'$ by the set of resources $S_j \subseteq R$, i.e., $\mathbf{X}_i = \{\sum_{j \in [k]} x'_{i,j} \mathbf{1}_{S_j} : \mathbf{x}'_i = (x'_{i,1}, \dots, x'_{i,k}) \in X'_i\}$. Thus, there is a one-to-one correspondence between strategy profiles \mathbf{x}' for G' and strategy profiles \mathbf{x} of G and it is easy to see that by construction, the private cost of player $i \in N'$ is the same for \mathbf{x}' in G' and the corresponding profile \mathbf{x} in G . \square

Equipped with Lemmas 9 and 10, we can show that the impact of the load of resource s on the cost of resource r needs to be linear and symmetric. In addition, the impact is non-existent if case (a) of Lemma 10 does not hold. This is formalized in the following lemma.

Lemma 11. *Let \mathcal{C} be a set of continuous functions that is closed under composition and consistent for weighted resource graph games. Then, for all $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ with $m \in \mathbb{N}$ and all $r, s \in [m]$ with $r \neq s$ there is $a_{r,s} = a_{s,r}$ such that*

$$c_r(\mathbf{z} + \lambda \mathbf{1}_s) - c_r(\mathbf{z}) = a_{r,s} \lambda$$

for all $\mathbf{z} \in \mathbb{R}_{\geq 0}^m$ with $z_r > 0$ and all $\lambda \geq 0$. Moreover, if case (a) of Lemma 10 does not hold, then $a_{r,s} = 0$ for all $r, s \in R$.

Proof. Let $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ with $m \in \mathbb{N}$ be arbitrary. Let $r, s \in [m]$ with $r \neq s$ and let $\mathbf{z} \in \mathbb{R}_{\geq 0}^m$ with $z_r > 0$. Using Lemma 9 we obtain

$$\begin{aligned} \underbrace{c_r(\mathbf{z} + \lambda \mathbf{1}_{\{r,s\}}) + c_s(\mathbf{z} + \lambda \mathbf{1}_{\{r,s\}})}_{h_0(\lambda)} &= c_r(\mathbf{z} + \lambda \mathbf{1}_r) + c_r(\mathbf{z} + \lambda \mathbf{1}_{\{r,s\}}) - c_r(\mathbf{z} + \lambda \mathbf{1}_r) \\ &\quad + c_s(\mathbf{z} + \lambda \mathbf{1}_f) + c_f(\mathbf{z} + \lambda \mathbf{1}_{\{r,s\}}) - c_s(\mathbf{z} + \lambda \mathbf{1}_s) \\ &= \underbrace{c_r(\mathbf{z} + \lambda \mathbf{1}_r)}_{h_1(\lambda)} + \underbrace{c_s(\mathbf{z} + \lambda \mathbf{1}_s)}_{h_2(\lambda)} + 2 \underbrace{(c_r(\mathbf{z} + \lambda \mathbf{1}_{\{r,s\}}) - c_r(\mathbf{z} + \lambda \mathbf{1}_r))}_{h_4(\lambda)} \end{aligned}$$

for all $\lambda \geq 0$. We apply Lemma 10 to the expressions $h_0(\lambda), h_1(\lambda), h_2(\lambda)$ and distinguish two cases.

If we are in case (a) of Lemma 10, then all three expressions are affine functions of λ and we conclude that also h_4 must be affine in λ , i.e., there is $a, b \in \mathbb{R}$ such that $c_r(\mathbf{z} + \lambda \mathbf{1}_{\{r,s\}}) - c_r(\mathbf{z} + \lambda \mathbf{1}_r) = a\lambda + b$. By part (a) of Lemma 9, we observe that this equality also holds (for the same values of a and b) when swapping the roles of s and r . Applying part (b) of Lemma 9, we observe that $b = 0$ and a is independent of z , thus proving the statement of the lemma for this case.

If we are in case (b) of Lemma 10, then all three expressions are exponential functions of the form $a \exp(\phi\lambda) + b$ for some $\phi \in \mathbb{R}$ and we conclude that also h_4 must be of this form, i.e., there is $a', b' \in \mathbb{R}$ such that $c_r(\mathbf{z} + \lambda \mathbf{1}_{\{r,s\}}) - c_r(\mathbf{z} + \lambda \mathbf{1}_r) = a' \exp(\phi\lambda) + b'$. Applying part (b) of Lemma 9, we conclude that $a' = b' = 0$, thus proving the statement of the lemma for this case. \square

We are now ready to establish the necessity of condition 1 or 2 of Theorem 2, concluding the proof of the theorem.

Lemma 12. *Let \mathcal{C} be a set of continuous functions that is closed under composition and consistent for weighted resource graph games. Then, one of the following statements is true:*

1. *For each $\mathbf{c} \in \mathcal{C}$ with $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ for some $m \in \mathbb{N}$ there is a symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ and a vector $\mathbf{b} \in \mathbb{R}^m$ such that $\mathbf{c}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$.*
2. *There is $\phi \in \mathbb{R}$ such that for all $\mathbf{c} \in \mathcal{C}$ with $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ there are $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$ such that $c_r(\mathbf{x}) = a_r \exp(\phi x_r) + b_r$ for all $r \in [m]$ and all $\mathbf{x} \in \mathbb{R}_{\geq 0}^m$.*

Proof. Let $\mathbf{c} \in \mathcal{C}$, $\mathbf{c} : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}^m$ with $m \in \mathbb{N}$ be arbitrary. By Lemma 11, for all $r, s \in [m]$, there is $a_{r,s}$ such that $c_r(\mathbf{z} + \lambda \mathbf{1}_s) - c_r(\mathbf{z}) = a_{r,s}\lambda$ for all $\mathbf{z} \in \mathbb{R}_{\geq 0}^m$ with $z_r > 0$ and all $\lambda \geq 0$. Let r_1, \dots, r_m be an arbitrary ordering of the resources in R with $r_m = r$. Defining $\mathbf{x}^{(0)} = \mathbf{x}$ and $\mathbf{x}^{(i)} = \mathbf{x}^{(i-1)} - x_{r_i} \cdot \mathbf{1}_{r_i}$ for $i \in [m]$ we obtain

$$\begin{aligned} c_r(\mathbf{x}) &= c_r(x_r \cdot \mathbf{1}_r) + \sum_{i=1}^{m-1} c_r(\mathbf{x}^{(i-1)}) - c_r(\mathbf{x}^{(i)}) \\ &= c_r(x_r \cdot \mathbf{1}_r) + \sum_{s \in R \setminus \{r\}} a_{r,s}(x_s - 1). \end{aligned}$$

In case (a) of Lemma 10, we conclude that $c_r(x_r \cdot \mathbf{1}_r)$ is an affine function of x_r . This implies that there is a symmetric matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ and a vector $\mathbf{b} \in \mathbb{R}^m$ such that $c_r(\mathbf{x}) = \mathbf{A}_{r,\cdot} \mathbf{x} + b_r$ for all $r \in R$ and all $\mathbf{x} \in \mathbb{R}_{\geq 0}^m$ with $x_r > 0$. Since \mathcal{C} is closed under composition, all functions $\mathbf{c} \in \mathcal{C}$ have this property thus we retrieve case 1 of Lemma 12.

In case (b) of Lemma 10, we conclude that $c_r(x_r \cdot \mathbf{1}_r)$ is an exponential function of x_r . By Lemma 11, we then have that $a_{r,s} = 0$ and we thus obtain that $c_r(\mathbf{x}) = a_r \exp(\phi x_r) + b_r$ for all $r \in [m]$ for some constant $a_r, b_r, \phi \in \mathbb{R}$. As \mathcal{C} is closed under composition, this implies that all functions $\mathbf{c} \in \mathcal{C}$ have this property, and we retrieve case 2 of Lemma 12. \square

5 Resource graph games on matroids

While our previous characterizations hold for *arbitrary strategy spaces* of the players, we now turn to *restricted strategy spaces*. Specifically, we consider *matroidal* strategy spaces, where the strategy set X_i of player $i \in N$ corresponds to the set of incidence vectors of bases of a player-specific matroid $M_i = (R, \mathcal{B}_i)$, $i \in N$ defined on the resource set R .

A *resource graph game on matroids* is then represented by the tuple $G = (N, X, C)$, where $C = (\mathbf{c}_i)_{i \in N}$ denotes the vector of *player-specific non-separable cost functions*. The private cost of player i under strategy profile $x \in X$ is defined as $\pi_i(x) = \mathbf{x}_i^\top \mathbf{c}_i(\mathbf{x})$. In order to specify assumptions on the functions $\mathbf{c}_i(\mathbf{x}), i \in N$, we introduce the concept of *player types*. Let \mathcal{B} denote the set of feasible matroid base systems represented in binary vectors over $\{0, 1\}^m$. For example $U_k \in \mathcal{B}$, where U_k is the base set of the uniform matroid of rank $1 \leq k \leq m$. For a matroid game $G = (N, X, C)$, we say that player $i \in N$ is of type $T \in \mathcal{B}$, if the matroid base system for i is given by $X_i = T$. Now we define a general notion of local monotonicity of non-separable functions.

Definition 2 (Local Monotonicity). *A function $\mathbf{c} : \mathbb{Z}_{\geq 0}^m \rightarrow \mathbb{R}^m$ is locally monotone, if for all $T \in \mathcal{B}, r \in R$, there are non-decreasing functions $\nu_{T,r} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$, such that for all $\mathbf{t} \in T$, all $\mathbf{z} \in \mathbb{Z}_+^m$, and all $r, s \in R$ with $\mathbf{u} = \mathbf{t} + \mathbf{1}_s - \mathbf{1}_r \in T$ and $\nu_{T,r}(t_r) \leq \nu_{T,s}(u_s)$ it holds that*

$$\mathbf{t}^\top \mathbf{c}(\mathbf{t} + \mathbf{z}) \leq \mathbf{u}^\top \mathbf{c}(\mathbf{u} + \mathbf{z}). \quad (4)$$

The locality aspect of Definition 2 arises, because the condition relates the cost of playing strategies \mathbf{t} and \mathbf{u} , which differ only by exchanging the entries of two elements. While this definition is quite abstract, we will provide an application and illustrating example in the realm of bilevel load balancing games, formalized in Theorem 4 and its proof given below. The main idea for our existence proof is to use an associated matroid congestion game with *separable* player-specific non-decreasing cost functions $\nu(\mathbf{x}) := (\nu_i(\mathbf{x}))_{i \in N}$ in order to construct a Nash equilibrium for the original game with non-separable functions. The proof of the theorem is given in Section 5.1.

Theorem 3. *Let $G = (N, X, C)$ be a resource graph game on matroids with $C = (\mathbf{c}_i)_{i \in N}$ and locally monotone cost functions $\mathbf{c}_i, i \in N$. Then, the following statements hold.*

1. *Any profile $x \in X$ that is a Nash equilibrium for the matroid congestion game $G = (N, X, \nu)$ is also a Nash equilibrium for the resource graph game $G = (N, X, C)$.*
2. *Nash equilibria for the resource graph game $G = (N, X, C)$ do exist.*

As a consequence of Theorem 3, we obtain the following application to the class of bilevel load balancing games on matroids. The proof of the theorem is given in Section 5.2.

Theorem 4. *Bilevel load-balancing games on matroids possess pure Nash equilibria.*

5.1 Proof of Theorem 3

We recap a trivial property of an equilibrium for the matroid game with separable cost functions.

Lemma 13. *Let $x \in X$ be an equilibrium for the matroid congestion game $G = (N, X, \nu)$, with player-specific separable and non-decreasing cost functions $\nu(\mathbf{x}) := (\nu_i(\mathbf{x}), i \in N)$. Then, for any $y \in X$ with $\mathbf{y}_i = \mathbf{x}_i + \mathbf{1}_s - \mathbf{1}_r$ for some $i \in N$ and $\mathbf{y}_j = \mathbf{x}_j$ for all $j \neq N \setminus \{i\}$, we have $\nu_{i,r}(x_r) \leq \nu_{i,s}(y_s)$.*

Proof. Assume by contradiction $\nu_{i,r}(x_r) > \nu_{i,s}(y_s)$. Then, by the monotonicity and separability of ν_i , we have $\pi_i(y) < \pi_i(x)$, contradiction. \square

Now we prove Theorem 3.

Proof. For 1.:

Let $x \in X$ be an equilibrium for the matroid congestion game $G = (N, X, \nu)$. Now we evaluate the cost of a player $i \in N$ when switching from $\mathbf{x}_i \in X_i$ to some $\mathbf{y}_i \in X_i$ for the game $G = (N, X, C)$:

$$\pi_i(x) = \mathbf{x}_i^\top \mathbf{c}_i(\mathbf{x}) = \sum_{r \in \text{supp}(\mathbf{x}_i)} c_{i,r}(\mathbf{x}) = \sum_{r \in \text{supp}(\mathbf{x}_i) \setminus \text{supp}(\mathbf{y}_i)} c_{i,r}(\mathbf{x}) + \sum_{r \in \text{supp}(\mathbf{x}_i) \cap \text{supp}(\mathbf{y}_i)} c_{i,r}(\mathbf{x}).$$

Because X_i consists of the bases of a matroid, the profile \mathbf{y}_i can be decomposed into a sequence of single-element exchanges of the form

$$\mathbf{y}_i = \mathbf{x}_i + \sum_{j=1}^k (\mathbf{1}_{s_j} - \mathbf{1}_{r_j})$$

with $r_1, \dots, r_k, s_1, \dots, s_k \in R$, $r_j \neq r_{j'}$ and $s_j \neq s_{j'}$ for all $j \neq j'$, and such that $\mathbf{y}_i^\ell = \mathbf{x}_i + \sum_{j=1}^\ell (\mathbf{1}_{s_j} - \mathbf{1}_{r_j}) \in X_i$ for all $1 \leq \ell \leq k$. We denote by $\mathbf{y}^\ell := (\mathbf{y}_i^\ell, x_{-i})$ the corresponding profile, where only player i changed the strategy according to \mathbf{y}_i^ℓ . We prove by induction over $1 \leq \ell \leq k-1$ that $\pi_i(\mathbf{y}^\ell) \leq \pi_i(\mathbf{y}^{\ell+1})$. For $\ell = 1$, we get

$$\pi_i(x) = \mathbf{x}_i^\top \mathbf{c}_i(\mathbf{x}) \leq \mathbf{y}_i^1 \mathbf{c}_i(\mathbf{y}^1) = \pi_i(\mathbf{y}^1),$$

where the inequality follows by $\nu_{i,s_1}(y_{s_1}^1) \geq \nu_{i,r_1}(x_{r_1})$ and (4). Here, we used the local monotonicity property of Definition 2 by identifying $\mathbf{x}_{-i} = \mathbf{u}$, $\mathbf{x}_i = \mathbf{t}$ and $\mathbf{y}_i^1 = \mathbf{y}$. Now we consider the inductive step $\ell \rightarrow \ell + 1$:

$$\pi_i(\mathbf{y}^\ell) = \mathbf{y}_i^\ell \mathbf{c}_i(\mathbf{y}^\ell) \leq \mathbf{y}_i^{\ell+1} \mathbf{c}_i(\mathbf{y}^{\ell+1}) = \pi_i(\mathbf{y}^{\ell+1}),$$

which again follows by $\nu_{i,s_{\ell+1}}(y_{s_{\ell+1}}^\ell) \geq \nu_{i,r_{\ell+1}}(y_{r_{\ell+1}}^\ell) = \nu_{i,r_{\ell+1}}(x_{r_{\ell+1}})$ and (4).

For 2.: We use 1. together with the fact that pure Nash equilibria do exist for the matroid congestion game $G = (N, X, \nu)$ with non-decreasing separable player-specific cost functions, see Ackermann, Röglin and Vöcking [2]. \square

5.2 Proof of Theorem 4

In the following lemma, we verify that bilevel load balancing games as introduced in Example 4 satisfy the conditions of Definition 2 and thus possesses Nash equilibria. Instead of only considering singleton strategies as in Example 4, we allow bases of matroids that may be player-specific.

Lemma 14. *The cost function defined in (1) is locally monotone.*

Proof. We need to show that there are functions $\nu(\mathbf{x}) := (\nu_T(\mathbf{x}))_{T \in \mathcal{B}}$ such that (4) holds under the assumptions stated in Definition 2. Let us rewrite (4) in the current setting:

$$\begin{aligned} \mathbf{t}^\top \mathbf{c}(\mathbf{t} + \mathbf{z}) &= \sum_{g \in \text{supp}(\mathbf{t})} t_g + z_g + \kappa_g^*(\mathbf{t} + \mathbf{z}) \\ &\leq \sum_{g \in \text{supp}(\mathbf{u})} u_g + z_g + \kappa_g^*(\mathbf{u} + \mathbf{z}) = \mathbf{u}^\top \mathbf{c}(\mathbf{u} + \mathbf{z}). \end{aligned} \tag{5}$$

We claim that by setting $\nu_{T,g}(x) = \nu_{T,g}(x) = x$ for all $x \in \mathbb{Z}_+, T \in \mathcal{B}, g \in R$ inequality (5) holds. Let $\mathbf{t} \in T$, $\mathbf{z} \in \mathbb{Z}_+^m$, and consider $r, s \in R$ with $\mathbf{u} = \mathbf{t} + \mathbf{1}_s - \mathbf{1}_r \in T$ and $\nu_{T,r}(t_r + z_r) = t_r + z_r \leq \nu_{T,s}(u_s + z_s) = u_s + z_s$. By $t_r + z_r \leq u_s + z_s$ and $t_g + z_g = u_g + z_g$ for all $g \notin \{r, s\}$, we get

$$\sum_{g \in \text{supp}(\mathbf{t})} t_g + z_g \leq \sum_{g \in \text{supp}(\mathbf{u})} u_g + z_g.$$

Thus, it suffices to show that

$$\sum_{g \in \text{supp}(\mathbf{t})} \kappa_g^*(\mathbf{t} + \mathbf{z}) \leq \sum_{g \in \text{supp}(\mathbf{u})} \kappa_g^*(\mathbf{u} + \mathbf{z}). \quad (6)$$

For $\mathbf{w} \in \mathbb{Z}_+^R$ define $S(\mathbf{w}) := \arg \max\{w_g | g \in R\}$. We distinguish two cases.

- Case 1: $s \notin S(\mathbf{u} + \mathbf{z})$.

With $u_s + z_s \geq t_r + z_r$, we get $r \notin S(\mathbf{t} + \mathbf{z})$ and thus $\sum_{g \in \text{supp}(\mathbf{t})} \kappa_g^*(\mathbf{t} + \mathbf{z}) = \sum_{g \in \text{supp}(\mathbf{u})} \kappa_g^*(\mathbf{u} + \mathbf{z})$, hence (6) follows.

- Case 2: $s \in S(\mathbf{u} + \mathbf{z})$.

We consider the two sub-cases of whether or not s was already an argmax element under $\mathbf{t} + \mathbf{z}$ or not.

- Case 2(a): $s \in S(\mathbf{t} + \mathbf{z})$.

This case implies $S(\mathbf{u} + \mathbf{z}) = \{s\}$ and thus we get

$$\sum_{g \in \text{supp}(\mathbf{u})} \kappa_g^*(\mathbf{u} + \mathbf{z}) = \kappa_s^*(\mathbf{u} + \mathbf{z}) = B \geq \sum_{g \in \text{supp}(\mathbf{t})} \kappa_g^*(\mathbf{t} + \mathbf{z}).$$

- Case 2(b): $s \notin S(\mathbf{t} + \mathbf{z})$.

This case implies $S(\mathbf{u} + \mathbf{z}) = S(\mathbf{t} + \mathbf{z}) \cup \{s\}$. Let us consider two further sub-cases: $|S(\mathbf{t} + \mathbf{z}) \cap \text{supp}(\mathbf{t})| = 0$ or $|S(\mathbf{t} + \mathbf{z}) \cap \text{supp}(\mathbf{t})| \geq 1$.

For $|S(\mathbf{t} + \mathbf{z}) \cap \text{supp}(\mathbf{t})| = 0$, we trivially get

$$\sum_{g \in \text{supp}(\mathbf{u})} \kappa_g^*(\mathbf{u} + \mathbf{z}) \geq 0 = \sum_{g \in \text{supp}(\mathbf{t})} \kappa_g^*(\mathbf{t} + \mathbf{z}).$$

For $|S(\mathbf{t} + \mathbf{z}) \cap \text{supp}(\mathbf{t})| \geq 1$, we get in the case $r \notin S(\mathbf{t} + \mathbf{z})$

$$\begin{aligned} \sum_{g \in \text{supp}(\mathbf{u})} \kappa_g^*(\mathbf{u} + \mathbf{z}) &= \frac{B \cdot |S(\mathbf{u} + \mathbf{z}) \cap \text{supp}(\mathbf{u})|}{|S(\mathbf{u} + \mathbf{z})|} = \frac{B \cdot (|S(\mathbf{t} + \mathbf{z}) \cap \text{supp}(\mathbf{t})| + 1)}{|S(\mathbf{t} + \mathbf{z})| + 1} \\ &\geq \frac{B \cdot (|S(\mathbf{t} + \mathbf{z}) \cap \text{supp}(\mathbf{t})|)}{|S(\mathbf{t} + \mathbf{z})|} = \sum_{g \in \text{supp}(\mathbf{t})} \kappa_g^*(\mathbf{t} + \mathbf{z}). \end{aligned}$$

In case $r \in S(\mathbf{t} + \mathbf{z})$, we have $|S(\mathbf{t} + \mathbf{z})| = |S(\mathbf{u} + \mathbf{z})|$ and $|S(\mathbf{t} + \mathbf{z}) \cap \text{supp}(\mathbf{t})| = |S(\mathbf{u} + \mathbf{z}) \cap \text{supp}(\mathbf{u})|$ and thus

$$\begin{aligned} \sum_{g \in \text{supp}(\mathbf{u})} \kappa_g^*(\mathbf{u} + \mathbf{z}) &= \frac{B \cdot |S(\mathbf{u} + \mathbf{z}) \cap \text{supp}(\mathbf{u})|}{|S(\mathbf{u} + \mathbf{z})|} = \frac{B \cdot (|S(\mathbf{t} + \mathbf{z}) \cap \text{supp}(\mathbf{t})|)}{|S(\mathbf{t} + \mathbf{z})|} \\ &= \sum_{g \in \text{supp}(\mathbf{t})} \kappa_g^*(\mathbf{t} + \mathbf{z}). \quad \square \end{aligned}$$

This lemma together with the existence result in part 2 of Theorem 3 implies Theorem 4.

6 Complexity of verifying equilibria

Theorem 5. *It is NP-hard to determine whether a given strategy profile of a resource graph game is a pure Nash equilibrium. This holds even when restricted to games where there is only a single player, the cost function c fulfills the requirements of Theorem 1 and one of the following conditions is fulfilled:*

1. \mathcal{S}_1 is a partition matroid on R .
2. \mathcal{S}_1 is the set of s - t -paths in a directed graph and $|B_r| = 1$ for all $r \in R$.

Proof. In case 1: We reduce from 3-SAT. Given a 3-SAT instance on m clauses, let z_{ij} denote the j th literal of clause i . We let $R = \{z_{ij} : i \in [m], j \in [3]\}$ be the set of resources of the game. The strategy space of the single player 1 is given by the bases of a partition matroid such that $S \subseteq R$ is a basis of the matroid if and only if $|S \cap \{z_{i1}, z_{i2}, z_{i3}\}| = 1$ for all $i \in [m]$. We further define the cost function $c_r(\mathbf{x}) = \sum_{s \in \Delta(r)} x_s$ where $\Delta(r)$ is the set of literals that contradict the literal r (i.e., $s \in \Delta(r)$ if and only if s and r are literals of the same variable but with different signs). It is easy to see that player 1 has a strategy of cost 0 if and only if there is a truth assignment fulfilling all clauses of the 3-SAT instance.

In case 2: We reduce from FORBIDDEN PAIRS s - t -PATH, which is known to be NP-hard [13]: Given a digraph $G = (V, E)$, two nodes $s, t \in V$, and a collection of edge pairs $\{e_1, e'_1\}, \dots, \{e_k, e'_k\}$, does there exist an s - t -path P such that $|P \cap \{e_i, e'_i\}| \leq 1$ for all $i \in [k]$?

We construct the resource graph game as follows: Let $R = E$ be the resource set. For $e \in E$ and $\mathbf{x} \in \mathbb{R}^E$ define $c_e(\mathbf{x}) = x_{e'_i}$ if $e = e_i$ for some $i \in [k]$, $c_e(x) = x_{e_i}$ if $e = e'_i$ for some $i \in [k]$ and $c_e(x) = 0$ otherwise. The game has a single player whose strategy space corresponds to the set of s - t -paths in G . It is easy to see that player 1 has a strategy of cost 0 if and only if path avoiding all forbidden pairs. \square

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