Basic relations valid for the Bernstein space B_σ^2 and their extensions to functions from larger spaces in terms of their distances from B_σ^2

Part 2: Foundations for a unified approach to extensions

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New Trends and Directions in Harmonic Analysis, Fractional Operator Theory, and Image Analysis

Summer School, Inzell, September 17–21, 2012

Preliminary reflections

The *Bernstein space* B_σ^p comprises all entire functions of exponential type σ whose restriction to $\mathbb{R}$ belongs to $L^p(\mathbb{R})$.

There exist numerous relations of the form

$$U(f) = V_\sigma(f) \quad \text{or} \quad U(f) \leq V_\sigma(f) \quad (f \in B_\sigma^p),$$

where U and V_σ are functionals.

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Examples

- The sampling formula:

$$U(f) = f(t), \qquad V_\sigma(f) = \sum_{k\in\mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \operatorname{sinc}\left(\frac{\sigma t}{\pi} - k\right)$$

- Bernstein's inequality in $L^2(\mathbb{R})$:

$$U(f) = \|f'\|_{L^2(\mathbb{R})}, \qquad V_\sigma(f) = \sigma\|f\|_{L^2(\mathbb{R})}$$

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Preliminary reflections, continued

Suppose that $f \notin B^p_\sigma$ but $U(f)$ and $V_\sigma(f)$ exists.
The previous relations may not hold any more. Write

$$U(f) = V_\sigma(f) + R_\sigma(f) \quad \text{or} \quad U(f) \le V_\sigma(f) + R_\sigma(f)$$

Interpretation

- $R_\sigma(f)$ is a *remainder* (engineers: *aliasing error*)
- $|R_\sigma(f)|$ is small when f is close to B^p_σ

Aims

1. Describe a hierarchy of spaces that extend B^p_σ
2. Introduce a metric for measuring the distance of f from B^p_σ
3. Estimate the distance in terms of properties of the spaces
4. Find representations for $R_\sigma(f)$ and give estimates in terms of the distance

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Fourier inversion classes and summability classes

Properties of members of Bernstein spaces:

• analytic • exp. growth • $L^p(\mathbb{R})$ membership

What are appropriate wider spaces?

Fourier inversion classes: $p \in [1,2]$

$$F^p := \left\{ f : \mathbb{R} \to \mathbb{C} \ : \ f \in L^p(\mathbb{R}) \cap C(\mathbb{R}), \ \widehat{f} \in L^1(\mathbb{R}) \right\}$$

$$f \in F^p \implies f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(v) e^{ivt}\, dv \qquad (t \in \mathbb{R})$$

ℓ^p summability class of step size h:

$$S_h^p := \left\{ f : \mathbb{R} \to \mathbb{C} \ : \ \big(f(hk)\big)_{k \in \mathbb{Z}} \in \ell^p(\mathbb{Z}) \right\}$$

$$B_\sigma^p|_{\mathbb{R}} \subset F^p \cap S_h^p$$

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Sobolev spaces

$\mathrm{AC}^{r-1}_{\mathrm{loc}}(\mathbb{R})$: the $(r-1)$-times locally abs. cont. functions on $\mathbb{R}$

$$W^{r,p}(\mathbb{R}) := \Big\{ f : \mathbb{R} \to \mathbb{C} \;:\; f = \phi \text{ a. e.}, \; \phi \in \mathrm{AC}^{r-1}_{\mathrm{loc}}(\mathbb{R}), \\ \phi^{(k)} \in L^p(\mathbb{R}), \; 0 \le k \le r \Big\}$$

This is a *Sobolev space* if endowed with the norm

$$\|f\|_{W^{r,p}(\mathbb{R})} := \left\{ \sum_{k=0}^{r} \|\phi^{(k)}\|^p_{L^p(\mathbb{R})} \right\}^{1/p},$$

Alternative description of $W^{r,p}(\mathbb{R})$

Proposition

For $p \in [1,2]$ we have

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Sobolev spaces, continued

The following statement is due to Nikol'skiĭ:

Let $f \in B^p_\sigma$ with $p \in [1, \infty)$. Then for any $h > 0$,

$$\left\{ h \sum_{k \in \mathbb{Z}} |f(hk)|^p \right\}^{1/p} \leq (1 + h\sigma) \|f\|_{L^p(\mathbb{R})} .$$

The proof includes the following more general result:

Proposition

Let $f \in W^{r,p}(\mathbb{R}) \cap C(\mathbb{R})$, $r \in \mathbb{N}$, $p \in [1, \infty]$. *Then for any* $h > 0$,

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For $p \in [1, 2]$, $(r, p) \neq (1, 1)$, the previous propositions imply

$$W^{r,p}(\mathbb{R}) \cap C(\mathbb{R}) \subset F^p \cap S^p_h .$$

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Lipschitz spaces

$$X_\infty := C(\mathbb{R}), \qquad X_p := L^p(\mathbb{R}), \quad p \in [1,\infty)$$

$$X_\infty^{(r)} := C^{(r)}(\mathbb{R}), \qquad X_p^{(r)} := W^{r,p}(\mathbb{R}) \quad (r \in \mathbb{N})$$

$$(\Delta_h^r f)(u) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(u+jh) \quad \text{(forward difference of order } r\text{)}$$

$$\omega_r(f;\delta;X_p) := \sup_{|h|\le\delta} \|\Delta_h^r f\|_{X_p} \quad \text{(modulus of smoothness of order } r\text{)}$$

The *Lipschitz spaces* are defined by

$$\mathrm{Lip}_r(\alpha; X_p) := \left\{ f \in X_p \,:\, \omega_r(f;\delta;X_p) = \mathcal{O}(\delta^\alpha),\ \delta \to 0+ \right\} \quad (0 < \alpha \le r)$$

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Lipschitz spaces, continued

Given $f \in X_p$, $1 \leq p \leq \infty$, there exists an $f_\sigma \in B_\sigma^p$ with

$$\|f - f_\sigma\|_{X_p} = \inf_{g \in B_\sigma^p} \|f - g\|_{X_p}.$$

Proposition (Junggeburth-Scherer-Trebels 1973)

For $f \in X_p$, $1 \leq p \leq \infty$ and $s < \alpha < r$, where $s, r \in \mathbb{N}_0$, the following assertions are equivalent:

(i) $f \in \mathrm{Lip}_r(\alpha; X_p)$,

(ii) $\left\|f - f_\sigma\right\|_{X_p} = \mathcal{O}(\sigma^{-\alpha}) \quad (\sigma \to \infty)$,

(iii) $f \in X_p^{(s)}$ *and* $\left\|f^{(s)} - f_\sigma^{(s)}\right\|_{X_p} = \mathcal{O}(\sigma^{-\alpha+s}) \quad (\sigma \to \infty)$,

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Wiener amalgam and modulation spaces

For $p, q \in [1, \infty]$ the *Wiener amalgam space* $W(L^p, \ell^q)$ comprises all measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{p,q} := \left\{ \sum_{n\in\mathbb{Z}} \left\{ \int_n^{n+1} |f(t)|^p \, dt \right\}^{q/p} \right\}^{1/q} = \left\| \|f\|_{L^p(n,n+1)} \right\|_{\ell^q} < \infty$$

(Wiener 1932, Cooper 1960, Holland 1975, Fournier-Stewart 1988)

$$W(L^p, \ell^q) \subset L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \qquad \text{if } q \le p$$

$$W(L^p, \ell^q) \supset L^p(\mathbb{R}) \cup L^q(\mathbb{R}) \qquad \text{if } q \ge p$$

The *modulation space* $M^{2,1}$ is given by (Feichtinger 1980):

$$M^{2,1} := \{ f \; : \; f := \widehat{g}, \; g \in W(L^2, \ell^1) \}$$

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The space $M_*^{2,1}$

Consider a dilation $f_h : t \mapsto f(ht)$. Then $\widehat{f_h}(v) = h^{-1}\widehat{f}(v/h)$.
The space $M^{2,1}$ is known to be *dilation invariant*, i. e.,

$$\sum_{n\in\mathbb{Z}} \frac{1}{h}\left\{ \int_n^{n+1} \left|\widehat{f}\left(\frac{v}{h}\right)\right|^2 dv \right\}^{1/2} < \infty \qquad (*)$$

for $f \in M^{2,1}$ and all $h > 0$.
The series $(*)$ need not converge uniformly with respect to h.

$M_*^{2,1}$ comprises all $f \in M^{2,1}$ such that $(*)$ converges uniformly on bounded subintervals of $(0,\infty)$. We have

$$M_*^{2,1} \subset M^{2,1} \quad \text{and} \quad M_*^{2,1} \subset \mathrm{Lip}_r(\tfrac{1}{2}, L^2(\mathbb{R}))$$

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Hardy spaces for strips

$$\mathcal{S}_d := \left\{ z \in \mathbb{C} \,:\, |\Im z| < d \right\}$$

$$H^p(\mathcal{S}_d) := \left\{ f \,:\, f \text{ analytic on } \mathcal{S}_d,\ \|f\|_{H^p(\mathcal{S}_d)} < \infty \right\}$$

$$\|f\|_{H^p(\mathcal{S}_d)} := \left[\sup_{0<y<d} \int_{\mathbb{R}} \frac{|f(t-iy)|^p + |f(t+iy)|^p}{2}\, dt \right]^{1/p}$$

Proposition

For $f \in H^p(\mathcal{S}_d)$:

$$\left\| f^{(j)} \right\|_{L^p(\mathbb{R})} \leq \frac{j!}{d^j} \|f\|_{H^p(\mathcal{S}_d)} \qquad (j \in \mathbb{N}_0)$$

and if $p \in [1,2]$, $0 < \delta < d$, $1/p + 1/p' = 1$, then

$$\widehat{f}(v) = e^{-\delta|v|} g(v) \qquad \text{a.e. on } \mathbb{R}, \quad \text{where } g \in L^{p'}(\mathbb{R}).$$

Hardy spaces for strips

$$\mathcal{S}_d := \left\{ z \in \mathbb{C} \,:\, |\Im z| < d \right\}$$

$$H^p(\mathcal{S}_d) := \left\{ f \,:\, f \text{ analytic on } \mathcal{S}_d,\ \|f\|_{H^p(\mathcal{S}_d)} < \infty \right\}$$

$$\|f\|_{H^p(\mathcal{S}_d)} := \left[\sup_{0<y<d} \int_{\mathbb{R}} \frac{|f(t-iy)|^p + |f(t+iy)|^p}{2}\, dt \right]^{1/p}$$

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The inclusions

So far, we have considered:

- The Fourier inversion class F^p and the summability class S_h^p
- The Sobolev space $W^{r,p}(\mathbb{R})$
- The Lipschitz space $\mathrm{Lip}_r(\alpha, X_p)$
- The modulation space $M^{2,1}$ and its subspace $M_*^{2,1}$
- The Hardy space $H^p(\mathcal{S}_d)$

For these spaces, the following inclusions hold, where $p \in [1,2]$:

$$B_\sigma^p|_{\mathbb{R}} \subsetneqq H^p(\mathcal{S}_d)|_{\mathbb{R}} \subsetneqq W^{r,p}(\mathbb{R}) \cap C(\mathbb{R}) \subsetneqq F^p \cap S_h^p \subsetneqq F^p \subsetneqq L^p(\mathbb{R}).$$

For $p = 2$ we have in addition:

$$W^{r,2}(\mathbb{R}) \cap C(\mathbb{R}) \subsetneqq M_*^{2,1} \subsetneqq M^{2,1} \subsetneqq F^2 \cap S_h^2$$

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Examples for verifying strict inclusions

$$f(t) = \operatorname{sinc}^2\left(\frac{t}{4\pi}\right) e^{it/2} \sum_{n=1}^{\infty} a_n e^{ik_n t} = \sum_{n=1}^{\infty} a_n \, \phi_n(t)\,,$$

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The graph of ψ is an equilateral triangle with base $[0, 1]$, height 1.

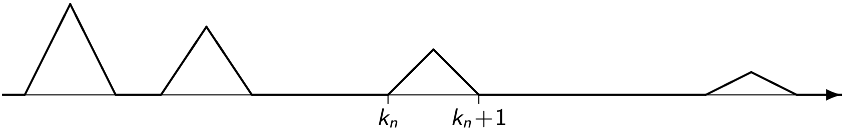

Figure: The graph of $\widehat{f}$.

Functions representable by a Fourier integral

Definition

For $q \in [1, \infty]$, the class G^q comprises all $f : \mathbb{R} \to \mathbb{C}$ such that

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(v) e^{itv} \, dv \qquad (*)$$

for some $\phi \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$.

The function ϕ associated with f is uniquely determined.
If $f \in F^p$, then $(*)$ holds with $\phi = \widehat{f}$. Since $\widehat{f} \in L^1(\mathbb{R}) \cap L^{p'}(\mathbb{R})$, we have

$$F^p \subset G^q \qquad \text{for } q \in [1, p'].$$

Furthermore,

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Definition of a norm

For $f \in G^q$ with associated ϕ,

$$\mathbf{|} f \mathbf{|}_q := \|\phi\|_{L^q(\mathbb{R})} = \left\{ \int_{\mathbb{R}} |\phi(v)|^q \, dv \right\}^{1/q}$$

defines a norm on G^q.

Indeed, for $f, g \in G^q$, $c \in \mathbb{C}$, we have:

(i) $\mathbf{|} f \mathbf{|}_q \geq 0$ and $\mathbf{|} f \mathbf{|}_q = 0 \iff f = 0$

(ii) $\mathbf{|} cf \mathbf{|}_q = |c| \, \mathbf{|} f \mathbf{|}_q$

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For $q = 2$, by the isometry of the L^2 Fourier transform,

$$\mathbf{|} f \mathbf{|}_2 = \|\phi\|_{L^2(\mathbb{R})} = \left\|\widehat{\phi}\right\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

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Distances from the Bernstein space B_σ^p

For $f, g \in G^q$,

$$\mathrm{dist}_q(f,g) := \| f - g \|_q$$

In particular, dist_q is a metric on F^p for all $q \in [p', \infty]$.

For $f \in G^q$, $p \in [1,2]$:

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Proposition

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Why a new norm?

The spaces of Hardy and Sobolev as well as the modulation space have their *individual* norm which makes them a Banach space.

- Want a norm that suits for all spaces of our hierarchy.
- The Banach space norms are too strong for measuring distances.

Example (Sobolev space)

There exist $f \in B_\sigma^2$ and $f_n \in W^{r,2}(\mathbb{R})$ such that

$$|f - f_n|_2 \to 0 \quad \text{but} \quad \|f - f_n\|_{W^{r,2}(\mathbb{R})} \to \infty \qquad (n \to \infty).$$

Recall that

$$|f|_2 = \|f\|_{L^2(\mathbb{R})} \quad \text{and} \quad \|f\|_{W^{r,2}(\mathbb{R})} = \left(\sum_{k=0}^{r} \left\| f^{(k)} \right\|_{L^2(\mathbb{R})}^2 \right)^{1/2}.$$

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Derivatives

Proposition

Let

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(v) e^{itv} \, dv$$

with $\phi \in L^1(\mathbb{R})$. *If* $v^r \phi(v)$ *belongs to* $L^1(\mathbb{R})$ *for some* $r \in \mathbb{N}$, *then* f *has derivatives up to order* r *in* $C_0(\mathbb{R})$ *and*

$$f^{(k)}(t) = \frac{i^k}{\sqrt{2\pi}} \int_{\mathbb{R}} v^k \phi(v) e^{itv} \, dv \qquad (k = 0, 1, \ldots, r).$$

An example

The previous proposition gave a *sufficient* condition for differentiability in G^q, namely $v\phi(v) \in L^1(\mathbb{R})$.
It is not necessary, but cannot be considerably relaxed.

Example

Consider

$$\phi(v) := \sqrt{\frac{2}{\pi}} \frac{1}{1+v^2} .$$

Then $\phi \in L^1(\mathbb{R})$ but $v\phi(v) \notin L^1(\mathbb{R})$ and

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(v) e^{itv}\, dv = e^{-|t|}.$$

Note that f is *not* differentiable at $t = 0$.

Distances of derivatives from the Bernstein space B_σ^p

The previous statements imply:

Proposition

Suppose that

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$\phi \in L^1(\mathbb{R})$. *If* $v^k \phi(v) \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$, *then for* $p \in [1, 2]$,

$$\mathrm{dist}_q(f^{(k)}, B_\sigma^p) = \begin{cases} \left\{ \int_{|v| \geq \sigma} |v^k \phi(v)|^q \, dv \right\}^{1/q}, & q < \infty, \\ \sup_{|v| \geq \sigma} |v^k \phi(v)|, & q = \infty, \end{cases}$$

where ϕ *is assumed to be continuous if* $q = \infty$.

For $k = 0$, we recover the result on $\mathrm{dist}_q(f, B_\sigma^p)$ for $f \in G^q$.

Application to functions from F^p and $W^{r,p}(\mathbb{R})$

In the following: $p \in [1,2], \quad q \in [1, p']$

Corollary

a) If $f \in F^p$, then

$$\operatorname{dist}_q(f, B_\sigma^p) = \begin{cases} \left\{ \int_{|v|\geq\sigma} |\widehat{f}(v)|^q \, dv \right\}^{1/q}, & q < \infty, \\ \sup_{|v|\geq\sigma} |\widehat{f}(v)|, & q = \infty. \end{cases}$$

b) If $f \in W^{r,p}(\mathbb{R}) \cap C(\mathbb{R})$ and $v^r \widehat{f}(v) \in L^1(\mathbb{R})$, then

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Derivative-free estimates for the distance from B_σ^p

Proposition

a) Let $f \in F^1$ and $q \in [1, \infty]$. Then

$$\mathrm{dist}_q(f, B_\sigma^1) \le d_{r,1,q} \begin{cases} \left\{ \int_\sigma^\infty \left[\omega_r\left(f; v^{-1}; L^1(\mathbb{R})\right)\right]^q dv \right\}^{1/q}, & q < \infty, \\ \omega_r\left(f; \sigma^{-1}; L^1(\mathbb{R})\right), & q = \infty. \end{cases}$$

b) Let $f \in F^p$, $p \in (1, 2]$ and $q \in [1, p']$. Then

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The constants $d_{r,p,q}$ depend on r, p and q only.

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Proof of statement a)

$$\Delta_h^r f(u) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(u+kh)$$

$$\Longrightarrow (\Delta_h^r f)\widehat{\ }(v) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} e^{ikhv} \widehat{f}(v) = \left(e^{ihv} - 1\right)^r \widehat{f}(v)$$

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$$\Longrightarrow\; (\Delta_h^r f)\widehat{\ }(v) \;=\; \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} e^{ikhv} \widehat{f}(v) = \left(e^{ihv} - 1\right)^r \widehat{f}(v)$$

For $h = \pi/v$, we obtain

$$\widehat{f}(v) = \frac{1}{(-2)^r \sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\Delta_{\pi/v}^r f\right)(u) e^{-ivu} du.$$

$$\Longrightarrow \left|\widehat{f}(v)\right| \leq \frac{1}{2^r \sqrt{2\pi}} \omega_r\left(f; \frac{\pi}{|v|}; L^1(\mathbb{R})\right) \leq \frac{(1+\pi)^r}{2^r \sqrt{2\pi}} \omega_r\left(f; |v|^{-1}; L^1(\mathbb{R})\right)$$

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Recall $\quad \mathrm{dist}_q(f, B_\sigma^1) = \left\{ \int_{|v| \geq \sigma} \left|\widehat{f}(v)\right|^q dv \right\}^{1/q}$

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$$\Longrightarrow \operatorname{dist}_q(f, B_\sigma^1) \le \frac{(1+\pi)^r}{2^r \sqrt{2\pi}} \left\{ \int_{|v| \ge \sigma} \left[\omega_r\left(f; v^{-1}; L^1(\mathbb{R})\right)\right]^q dv \right\}^{1/q} \qquad \square$$

Distances of Lipschitz functions from B^2_σ

Recall

$$\mathrm{Lip}_r(\alpha; X_p) := \left\{ f \in X_p \ : \ \omega_r(f;\delta;X_p) = \mathcal{O}(\delta^\alpha),\ \delta \to 0+ \right\} \quad (0 < \alpha \le r)$$

The estimates in terms of the modulus of smoothness imply:

Corollary

Let $f \in F^2 \cap \mathrm{Lip}_r(\alpha; L^2(\mathbb{R}))$ *and* $q \in [1,2]$ *such that* $1/q - 1/2 < \alpha \le r$. *Then*

$$\mathrm{dist}_q(f, B^2_\sigma) = \mathcal{O}\left(\sigma^{-\alpha-1/2+1/q}\right) \qquad (\sigma \to \infty).$$

Since dist_2 is the Euclidean distance, the charcterization of Lip-functions gives:

Proposition

Let $f \in F^2$ *and* $0 < \alpha \le r$. *Then*

$$f \in \mathrm{Lip}_r(\alpha, L^2(\mathbb{R})) \iff \mathrm{dist}_2(f, B^2_\sigma) = \mathcal{O}\left(\sigma^{-\alpha}\right) \qquad (\sigma \to \infty).$$

Distances of Lipschitz functions from B_σ^2

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$$\mathrm{dist}_q(f, B_\sigma^2) = \mathcal{O}\left(\sigma^{-\alpha-1/2+1/q}\right) \qquad (\sigma \to \infty).$$

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Distance of $M_*^{2,1}$-functions from B_σ^2

Proposition

a) If $f \in M_*^{2,1}$, then

$$\operatorname{dist}_q(f, B_\sigma^2) = \begin{cases} o(1), & q = 1, \\ \mathcal{O}\left(\sigma^{-1+1/q}\right), & q \in (1,2] \end{cases} \qquad (\sigma \to \infty).$$

b) If $f \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$ and $f^{(r)} \in M_*^{2,1}$, then for $q \in [1,2]$,

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Distance of functions in Sobolev spaces from B^2_σ

Proposition

Let $f \in F^2 \cap W^{r,2}(\mathbb{R})$ *and* $q \in [1,2]$. *Then*

$$\mathrm{dist}_q(f, B^2_\sigma) \le c_{r,q} \left\| f^{(r)} \right\|_{L^2(\mathbb{R})} \cdot \sigma^{-r-1/2+1/q} .$$

If, in addition, $v\widehat{f}(v) \in L^1(\mathbb{R})$, *then for* $r \ge 1/2 + 1/q$,

$$\mathrm{dist}_q(f', B^2_\sigma) \le c_{r-1,q} \left\| f^{(r)} \right\|_{L^2(\mathbb{R})} \cdot \sigma^{-r+1/2+1/q} .$$

Values for the constants $c_{r,q}$ are known. In particular,

$$c_{r,1} = \sqrt{\frac{2}{2r-1}} \quad \text{and} \quad c_{r,2} = 1.$$

Idea of proof

$$f \in W^{r,2}(\mathbb{R}) \implies v^r \widehat{f}(v) = g(v) \quad (g \in L^2(\mathbb{R}))$$

and

$$\|g\|_{L^2(\mathbb{R})} = \left\|f^{(r)}\right\|_{L^2(\mathbb{R})}$$

$$\implies \operatorname{dist}_q(f, B^2_\sigma) = \left\{ \int_{|v|\geq\sigma} \left|\frac{g(v)}{v^r}\right|^q dv \right\}^{1/q}$$

Now use Hölder's inequality for estimating the integral in terms of σ and $\|f^{(r)}\|_{L^2(\mathbb{R})}$.

Analogously for $\operatorname{dist}_q(f', B^2_\sigma)$.

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Distance of functions in Hardy spaces from B^2_σ

Proposition

Let $f \in H^2(\mathcal{S}_d)$ and $q \in [1,2]$. Then

$$\mathrm{dist}_q(f, B^2_\sigma) \le \gamma_q\, d^{1/2-1/q}\, e^{-d\sigma}\, \|f\|_{H^2(\mathcal{S}_d)}\,,$$

$$\mathrm{dist}_q(f', B^2_\sigma) \le \gamma'_q\, d^{1/2-1/q}\, \sigma\, e^{-d\sigma}\, \|f\|_{H^2(\mathcal{S}_d)} \quad (\sigma \ge 1/d).$$

Values for γ_q and γ'_q are known. In particular, $\gamma_2 = \gamma'_2 = \sqrt{2}$.

Idea of proof

$$f \in H^2(\mathcal{S}_d) \implies \widehat{f}(v) = e^{-\delta|v|}g(v) \quad (0 < \delta < d,\, g \in L^2(\mathbb{R}))$$

and

$$\|g\|_{L^2(\mathbb{R})} \le \sqrt{2}\,\|f\|_{H^2(\mathcal{S}_d)}$$

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Thank you for your attention