Signal analysis based on complex wavelet signs

Inzell 2012

Laurent Demaret, Peter Massopust, Martin Storath

Institute of Biomathematics - Helmholtz Zentrum München

September 18, 2012

Local signal analysis by wavelet transform

(Complex) wavelet at scale $a > 0$ and location $b \in \mathbb{R}$

$$\kappa_{a,b}(x) := \frac{1}{\sqrt{a}}\,\kappa\left(\frac{x-b}{a}\right)$$

Local signal analysis by wavelet transform

(Complex) wavelet at scale $a > 0$ and location $b \in \mathbb{R}$

$$\kappa_{a,b}(x) := \frac{1}{\sqrt{a}}\,\kappa\left(\frac{x-b}{a}\right)$$

Wavelet coefficient decomposition

$$\langle f, \kappa_{a,b}\rangle = |\langle f, \kappa_{a,b}\rangle| \cdot \frac{\langle f, \kappa_{a,b}\rangle}{|\langle f, \kappa_{a,b}\rangle|} = \underbrace{|\langle f, \kappa_{a,b}\rangle|}_{\text{amplitude}} \cdot \underbrace{\operatorname{sgn}\langle f, \kappa_{a,b}\rangle}_{\text{sign}}$$

Local signal analysis by wavelet transform

(Complex) wavelet at scale $a > 0$ and location $b \in \mathbb{R}$

$$\kappa_{a,b}(x) := \frac{1}{\sqrt{a}}\, \kappa\left(\frac{x-b}{a}\right)$$

Wavelet coefficient decomposition

$$\langle f, \kappa_{a,b} \rangle = |\langle f, \kappa_{a,b} \rangle| \cdot \frac{\langle f, \kappa_{a,b} \rangle}{|\langle f, \kappa_{a,b} \rangle|} = \underbrace{|\langle f, \kappa_{a,b} \rangle|}_{\text{amplitude}} \cdot \underbrace{\operatorname{sgn} \langle f, \kappa_{a,b} \rangle}_{\text{sign}}$$

Amplitudes of wavelet coefficients

$\updownarrow$

Classical local regularity

(e.g. Sobolev, Besov)

Local signal analysis by wavelet transform

(Complex) wavelet at scale $a > 0$ and location $b \in \mathbb{R}$

$$\kappa_{a,b}(x) := \frac{1}{\sqrt{a}}\,\kappa\left(\frac{x-b}{a}\right)$$

Wavelet coefficient decomposition

$$\langle f, \kappa_{a,b}\rangle = |\langle f, \kappa_{a,b}\rangle| \cdot \frac{\langle f, \kappa_{a,b}\rangle}{|\langle f, \kappa_{a,b}\rangle|} = \underbrace{|\langle f, \kappa_{a,b}\rangle|}_{\text{amplitude}} \cdot \underbrace{\operatorname{sgn}\langle f, \kappa_{a,b}\rangle}_{\text{sign}}$$

Amplitudes of wavelet coefficients
$\updownarrow$
Classical local regularity
(e.g. Sobolev, Besov)

Signs of wavelet coefficients
$\updownarrow$
?

Observations and approaches to wavelet signs

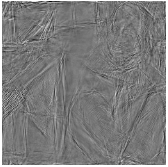
Importance of signs in signals

- Bandpass signals determined by sign information (Logan 1977)
- Signs important for image reconstruction (Oppenheim, Lim 1981)

Observations and approaches to wavelet signs

Importance of signs in signals

- Bandpass signals determined by sign information (Logan 1977)
- Signs important for image reconstruction (Oppenheim, Lim 1981)



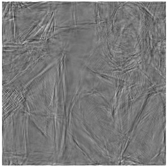
Shuffle of sign

Shuffle of amplitude

Observations and approaches to wavelet signs

Importance of signs in signals

- Bandpass signals determined by sign information (Logan 1977)
- Signs important for image reconstruction (Oppenheim, Lim 1981)



Shuffle of sign

Shuffle of amplitude

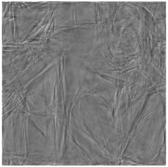
Phase congruency (Morrone, Owens 1987; Kovesi 1999)

- Heuristic approach to signal analysis based on Fourier-signs
- Successful application to edge detection

Observations and approaches to wavelet signs

Importance of signs in signals

- Bandpass signals determined by sign information (Logan 1977)
- Signs important for image reconstruction (Oppenheim, Lim 1981)



Shuffle of sign

Shuffle of amplitude

Phase congruency (Morrone, Owens 1987; Kovesi 1999)

- Heuristic approach to signal analysis based on Fourier-signs
- Successful application to edge detection

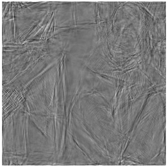
Wavelet sign in signal analysis (Kronland-Martinet, Grossmann, Morlet 1987; Mallat 1990)

- Lines of constant sign (phase) in scalogram $(a, b) \mapsto \langle f, \kappa_{a,b} \rangle$ converge towards singularities

Observations and approaches to wavelet signs

Importance of signs in signals

- Bandpass signals determined by sign information (Logan 1977)
- Signs important for image reconstruction (Oppenheim, Lim 1981)



Shuffle of sign

Shuffle of amplitude

Phase congruency (Morrone, Owens 1987; Kovesi 1999)

- Heuristic approach to signal analysis based on Fourier-signs
- Successful application to edge detection

Wavelet sign in signal analysis (Kronland-Martinet, Grossmann, Morlet 1987; Mallat 1990)

- Lines of constant sign (phase) in scalogram $(a, b) \mapsto \langle f, \kappa_{a,b} \rangle$ converge towards singularities

Wavelet sign as indicator of local symmetry (Holschneider 1995; Kovesi 1999)

Complex signature wavelets

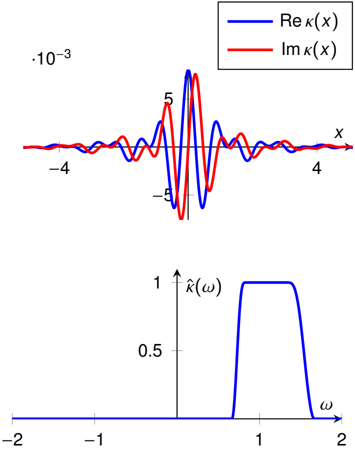
Complex valued function $\kappa \in \mathcal{S}(\mathbb{R};\mathbb{C})$ is called *signature wavelet* if

- frequency spectrum real and non-negative, i.e.,

$$\hat{\kappa} \geq 0,$$

- support of frequency spectrum compact and one-sided, i.e.,

$$\operatorname{supp}\,\hat{\kappa} \subset [c, d].$$



$0 < c < d < \infty$

The complex signature

Definition (Demaret, Massopust, Storath 2012)

The *signature* of f at location $b \in \mathbb{R}$ is defined by

$$\sigma f(b) = \lim_{a \to 0} \operatorname{sgn} \langle f, \kappa_{a,b} \rangle,$$

if the limit exists and has the same value for all signature wavelets κ; otherwise, we set

$$\sigma f(b) = 0.$$

$f \in \mathcal{S}'(\mathbb{R}; \mathbb{R})$ real-valued tempered distribution; $\operatorname{sgn} z = \frac{z}{|z|}$, $\operatorname{sgn} 0 = 0$

The complex signature

Definition (Demaret, Massopust, Storath 2012)

The *signature* of f at location $b \in \mathbb{R}$ is defined by

$$\sigma f(b) = \lim_{a \to 0} \operatorname{sgn} \langle f, \kappa_{a,b} \rangle,$$

if the limit exists and has the same value for all signature wavelets κ; otherwise, we set

$$\sigma f(b) = 0.$$

Idea

- b salient point $\quad \leftrightarrow \quad |\sigma f(b)| = 1$
- b regular point $\quad \leftrightarrow \quad \sigma f(b) = 0$

$f \in \mathcal{S}'(\mathbb{R}; \mathbb{R})$ real-valued tempered distribution; $\operatorname{sgn} z = \frac{z}{|z|}$, $\operatorname{sgn} 0 = 0$

Basic examples

- Bandlimited function f

$$\sigma f(b) = 0, \quad \text{for all } b \in \mathbb{R}.$$

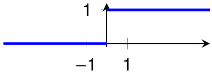
Basic examples

- Bandlimited function f

$$\sigma f(b) = 0, \quad \text{for all } b \in \mathbb{R}.$$

- Unit step U

$$\sigma U(b) = \begin{cases} i, & \text{for } b = 0, \\ 0, & \text{else.} \end{cases}$$



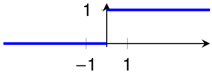
Basic examples

- Bandlimited function f

$$\sigma f(b) = 0, \quad \text{for all } b \in \mathbb{R}.$$

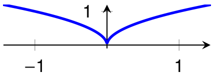
- Unit step U

$$\sigma U(b) = \begin{cases} i, & \text{for } b = 0, \\ 0, & \text{else.} \end{cases}$$



- Cusp $|\bullet|^\gamma$

$$\sigma(|\bullet|^\gamma)(b) = \begin{cases} -1, & \text{for } b = 0, \\ 0, & \text{else,} \end{cases}$$



where $0 < \gamma \leq 1$.

Signature at regular regions

Theorem (Demaret, Massopust, Storath 2012)

If f is smooth in a neighborhood of $b \in \mathbb{R}$ and if

$$f^{(k)}(b) = 0, \quad \text{for all } k \in \mathbb{N}_0,$$

then

$$\sigma f(b) = 0.$$

---

f real-valued, locally integrable function of polynomial growth

Signature at regular regions

Theorem (Demaret, Massopust, Storath 2012)

If f is smooth in a neighborhood of $b \in \mathbb{R}$ and if

$$f^{(k)}(b) = 0, \quad \text{for all } k \in \mathbb{N}_0,$$

then

$$\sigma f(b) = 0.$$

Sketch of proof

- Show that all moments of $G(u) = \langle f, \kappa_{\frac{1}{u},b} \rangle, u \in \mathbb{R}$, vanish.
- Conclude that either Re G and Im G have infinitely many sign changes or are equal to 0 for u large enough; thus

$$\lim_{u \to \infty} \operatorname{sgn} G(u) = \lim_{a \to 0} \operatorname{sgn} \langle f, \kappa_{a,b} \rangle$$

does not exist or is equal to 0. In either case, $\sigma f(b) = 0$.

---

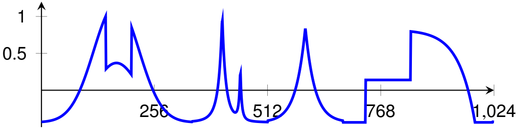
f real-valued, locally integrable function of polynomial growth

Locally polynomial signals

Corollary

If f coincides on an open set $U \subset \mathbb{R}$ with a polynomial then

$$\sigma f(b) = 0, \quad \text{for every } b \in U.$$



A piecewise polynomial signal from Wavelab (Donoho et. al.)

---

f real-valued, locally integrable function of polynomial growth

Jump discontinuites

Theorem (Demaret, Massopust, Storath 2012)

If f has a jump discontinuity at b, then

$$\sigma f(b) = \begin{cases} +i, & \text{if} \quad f(b-) < f(b+), \\ -i, & \text{if} \quad f(b-) > f(b+). \end{cases}$$

---

f real-valued, locally integrable function of polynomial growth

Jump discontinuites

Theorem (Demaret, Massopust, Storath 2012)

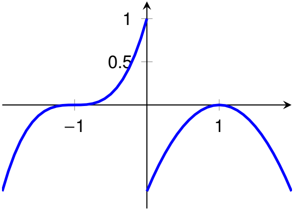
If f has a jump discontinuity at b, then

$$\sigma f(b) = \begin{cases} +i, & \text{if} \quad f(b-) < f(b+), \\ -i, & \text{if} \quad f(b-) > f(b+). \end{cases}$$

Example

$$f(x) = \begin{cases} (x+1)^3, & \text{for } x < 0, \\ -(x-1)^2, & \text{for } x \geq 0. \end{cases}$$

$$\sigma f(b) = \begin{cases} -i, & \text{if } b = 0, \\ 0, & \text{else.} \end{cases}$$



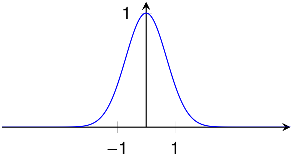
f real-valued, locally integrable function of polynomial growth

Differences to classical singular support

Gaussian function

$$f(x) = e^{-x^2}$$

$$\{0\} \subset \operatorname{supp} \sigma f \not\subset \operatorname{sing\,supp} f = \emptyset$$

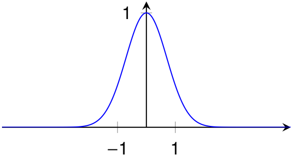


$\operatorname{sing\,supp} f = \{b \in \mathbb{R} : f \text{ not } C^\infty \text{ at } b\}$

Differences to classical singular support

Gaussian function

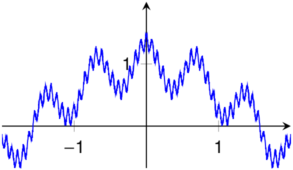
$$f(x) = e^{-x^2}$$



$$\{0\} \subset \operatorname{supp} \sigma f \not\subset \operatorname{sing\,supp} f = \emptyset$$

Weierstrass function

$$f(x) = \sum_{n=0}^{\infty} r^n \cos(t^n x)$$



$$\mathbb{R} = \operatorname{sing\,supp} f \not\subset \operatorname{supp} \sigma f = \emptyset.$$

---

sing supp $f = \{b \in \mathbb{R} : f \text{ not } C^\infty \text{ at } b\}$

Signature complementary to classical regularity

Fractional power of Laplacian

$$(-\Delta)^r f = \mathcal{F}^{-1}(|\bullet|^{2r} \cdot \hat{f})$$

Fractional Hilbert transform

$$\mathcal{H}^\alpha f = \mathcal{F}^{-1}(e^{-i\alpha\frac{\pi}{2}\cdot \mathrm{sgn}(\bullet)} \cdot \hat{f}).$$

$r, \alpha \in \mathbb{R}$; $s_f(b) := \sup\{s \in \mathbb{R} \,:\, \exists\, \varphi \in \mathcal{D}(\mathbb{R}),\, \varphi(b) \neq 0,\ \text{so that } \varphi f \in H^s(\mathbb{R})\}$

Signature complementary to classical regularity

Fractional power of Laplacian

$$(-\Delta)^r f = \mathcal{F}^{-1}(|\bullet|^{2r} \cdot \hat{f})$$

Fractional Hilbert transform

$$\mathcal{H}^\alpha f = \mathcal{F}^{-1}(e^{-i\alpha\frac{\pi}{2}\cdot\mathrm{sgn}(\bullet)} \cdot \hat{f}).$$

Local Sobolev regularity

Addition

$$s_{(-\Delta)^r f} = s_f - 2r$$

Invariant

$$s_{\mathcal{H}^\alpha f} = s_f$$

$r, \alpha \in \mathbb{R}$; $s_f(b) := \sup\{s \in \mathbb{R} \,:\, \exists\, \varphi \in \mathcal{D}(\mathbb{R}),\, \varphi(b) \neq 0,$ so that $\varphi f \in H^s(\mathbb{R})\}$

Signature complementary to classical regularity

Fractional power of Laplacian

$$(-\Delta)^r f = \mathcal{F}^{-1}(|\bullet|^{2r} \cdot \hat{f})$$

Fractional Hilbert transform

$$\mathcal{H}^\alpha f = \mathcal{F}^{-1}(e^{-i\alpha\frac{\pi}{2}\cdot\mathrm{sgn}(\bullet)} \cdot \hat{f}).$$

Local Sobolev regularity

Addition

$$s_{(-\Delta)^r f} = s_f - 2r$$

Invariant

$$s_{\mathcal{H}^\alpha f} = s_f$$

Signature

Invariant

$$\sigma\left((-\Delta)^r f\right) = \sigma f$$

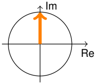
Rotation

$$\sigma\left(\mathcal{H}^\alpha f\right) = e^{i\alpha\frac{\pi}{2}} \cdot \sigma f$$

$r, \alpha \in \mathbb{R}$; $s_f(b) := \sup\{s \in \mathbb{R} \,:\, \exists\, \varphi \in \mathcal{D}(\mathbb{R}),\, \varphi(b) \neq 0,$ so that $\varphi f \in H^s(\mathbb{R})\}$

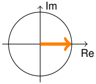
Geometric interpretation

Step right 



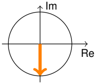
Signature $+i$

Cusp upwards 



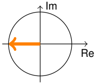
Signature $+1$

Step left 



Signature $-i$

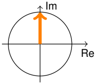
Cusp downwards 



Signature -1

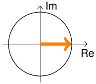
Geometric interpretation

Step right



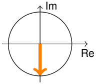
Signature $+i$

Cusp upwards



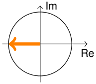
Signature $+1$

Step left



Signature $-i$

Cusp downwards



Signature -1

Imaginary signature
$\updownarrow$
locally antisymmetric

Real signature
$\updownarrow$
locally symmetric

Discretization (I)

Reminder: If $\sigma f(b) \neq 0$, then

$$\sigma f(b) = \lim_{j \to \infty} \operatorname{sgn} \langle f, \kappa_{a_j,b} \rangle, \qquad \text{for } a_j \to 0.$$

Discretization (I)

Reminder: If $\sigma f(b) \neq 0$, then

$$\sigma f(b) = \lim_{j\to\infty} \operatorname{sgn} \langle f, \kappa_{a_j,b} \rangle, \qquad \text{for } a_j \to 0.$$

Observations

- Cesàro-sequence converges to signature,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \operatorname{sgn} \langle f, \kappa_{a_j,b} \rangle = \sigma f(b).$$

Discretization (I)

Reminder: If $\sigma f(b) \neq 0$, then

$$\sigma f(b) = \lim_{j\to\infty} \operatorname{sgn} \langle f, \kappa_{a_j,b}\rangle, \qquad \text{for } a_j \to 0.$$

Observations

- Cesàro-sequence converges to signature,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \operatorname{sgn} \langle f, \kappa_{a_j,b}\rangle = \sigma f(b).$$

- Modulus of Cesàro-sequence tends to 1,

$$\lim_{N\to\infty} \Big|\frac{1}{N} \sum_{j=1}^{N} \operatorname{sgn} \langle f, \kappa_{a_j,b}\rangle\Big| = |\sigma f(b)| = 1.$$

Discretization (II)

In practice: finite number of scale samples $\{a_j\}_{j=1}^N$

Idea

Consider the finite sum

$$\overline{w}_b := \frac{1}{N} \sum_{j=1}^{N} \operatorname{sgn} \langle f, \kappa_{a_j,b} \rangle,$$

as N-th element of Cesàro-sequence

- $|\overline{w}_b| \ll 1 \rightarrow$ estimate signature as 0
- $|\overline{w}_b| \approx 1 \rightarrow$ estimate signature by orientation in complex plane of $\overline{w}_b$

Discretization (II)

In practice: finite number of scale samples $\{a_j\}_{j=1}^{N}$

Idea

Consider the finite sum

$$\overline{w}_b := \frac{1}{N} \sum_{j=1}^{N} \operatorname{sgn} \langle f, \kappa_{a_j,b} \rangle,$$

as N-th element of Cesàro-sequence

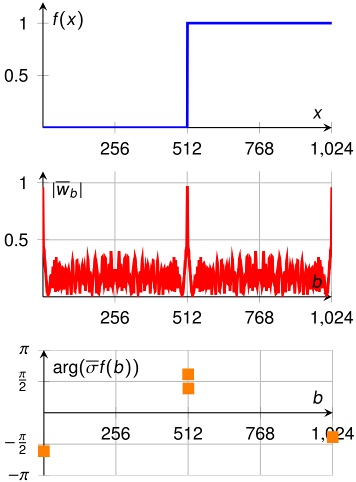
- $|\overline{w}_b| \ll 1 \rightarrow$ estimate signature as 0
- $|\overline{w}_b| \approx 1 \rightarrow$ estimate signature by orientation in complex plane of $\overline{w}_b$

Discrete signature

$$\overline{\sigma} f(b) := \begin{cases} 0, & \text{if } |\overline{w}_b| < \tau, \\ \operatorname{sgn} \overline{w}_b, & \text{if } |\overline{w}_b| \geq \tau, \end{cases}$$

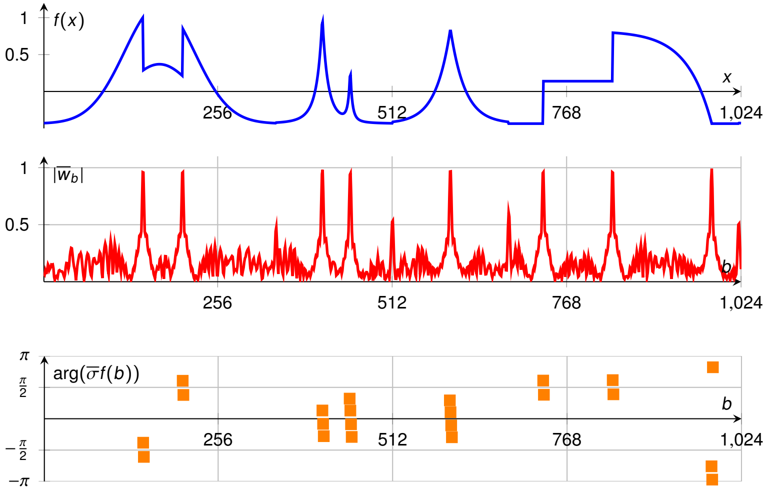
where $\tau \in [0, 1]$ is an empirical threshold parameter

Experiment - Jump and cusp



Threshold $\tau = 0.7$

Experiment - Piecewise polynomial



Threshold $\tau = 0.7$

Summary

Summary

- Signature as new rigorous approach to sign-based signal analysis
- Detection of salient points in signals and determination of local symmetry
- Discrete signature for numerical estimation

Main reference

- L. Demaret, P. Massopust, and M. Storath. Signal analysis based on complex wavelet signs.
  Submitted, 2012.
  Preprint arXiv:1208.4578v1

References I

L. Demaret, P. Massopust, and M. Storath.
Signal analysis based on complex wavelet signs.
Submitted, 2012.
Preprint arXiv:1208.4578v1.

M. Holschneider.
Wavelets – An Analysis Tool.
Oxford Science Publications, 1995.

R. Kronland-Martinet, J. Morlet, and A. Grossmann.
Analysis of sound patterns through wavelet transforms.
International Journal of Pattern Recognition and Artificial Intelligence, 1(2):273–302, 1987.

P. Kovesi.
Image features from phase congruency.
Videre: Journal of Computer Vision Research, 1(3):1–26, 1999.

BF Logan Jr.
Information in the zero crossings of bandpass signals.
AT & T Technical Journal, 56:487–510, 1977.

References II

SG Mallat.
A wavelet tour of signal processing: The sparse way. 2009.
Academic, Burlington.

M.C. Morrone and R.A. Owens.
Feature detection from local energy.
Pattern Recognition Letters, 6(5):303–313, 1987.

A.V. Oppenheim and J.S. Lim.
The importance of phase in signals.
Proceedings of the IEEE, 69(5):529–541, 1981.

M. Storath.
Multiscale amplitude and sign decompositions in image analysis.
PhD thesis, Technische Universität München, Germany, to appear at the end of 2012.

Difference to phase congruency

Phase congruency (Kovesi 1999)

$$\widetilde{\mathrm{PC}}(b) = \frac{\sum_{j=1}^{N} \langle f, \kappa_{a_j,b} \rangle}{\sum_{j=1}^{N} |\langle f, \kappa_{a_j,b} \rangle|}.$$

Discrete signature

$$\overline{\sigma} f(b) := \begin{cases} 0, & \text{if } |\overline{w}_b| < \tau, \\ \operatorname{sgn} \overline{w}_b, & \text{if } |\overline{w}_b| \geq \tau, \end{cases}$$

where

$$\overline{w}_b := \frac{1}{N} \sum_{j=1}^{N} \operatorname{sgn} \langle f, \kappa_{a_j,b} \rangle,$$