Operators of Harmonic Analysis
of Variable Order
in Variable Exponent Analysis

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New Trends and Directions in Harmonic Analysis, Fractional Operator Theory, and Image Analysis

The plan of these lectures is the following:

1 Introduction

Last decade there was a strong increase of interest to studies of fractional type operators and function spaces in the "variable setting", when the parameters defining the operator or the space (which usually are constant), may vary from point to point.

The area which is now called variable exponent analysis, last decade became a rather branched field with many interesting results obtained in
1) Harmonic Analysis,
2) Approximation Theory,
3)Operator Theory,
4) Pseudo-Differential Operators.

We dwell on some results on the classical operators of harmonic analysis.
We do not give exact references on every occasion. The interested listeners can find the references in the recent book

L. Diening et al. Lebesgue and Sobolev Spaces with Variable Exponents, 2011] Springer-Verlag, Lecture Notes in Mathematics and surveying papers

[L. Diening, P. Hästö, A. Nekvinda. Open problems in variable exponent Lebesgue and Sobolev spaces, 2005],
[V. Kokilashvili. On a progress in the theory of integral operators in weighted Banach function spaces, 2005],
[S. Samko. On a progress in the theory of Lebesgue spaces with variable exponent, 2005],
[V. Kokilashvili, S. Samko. Weighted Boundedness of the Maximal, Singular and Potential Operators in Variable Exponent Spaces, 2008]

2 Typical examples of operators

The Riesz fractional integration operator of order α of functions on $\mathbb{R}^n$ has the form

$$I^\alpha f(x) = \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \qquad 0 < \alpha < n,$$

where the normalizing constant

$$\gamma_n(\alpha) = \frac{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}$$

is chosen so that the semigroup property $I^\alpha I^\beta = I^{\alpha+\beta}$ holds.

We can admit variable order $\alpha(x)$ (Why not?):

$$I^{\alpha(\cdot)} f(x) = \frac{1}{\gamma_n[\alpha(x)]} \int\limits_{\mathbb{R}^n} \frac{f(y)\, dy}{|x-y|^{n-\alpha(x)}},$$

$$0 < \alpha(x) < n.$$

Usually we will omit the normalizing factor, since we have no hope for the semigroup property, but on the other hand there is a sense to keep it when we allow $\alpha(x)$ to approach singular value

$$\alpha(x) = 0$$

at some points.

As is known, in the case of constant α, the operator (left)-inverse to the Riesz potential operator (formally the fractional power $(-\Delta)^{\frac{\alpha}{2}}$) is given by the hypersingular integral

$$\mathbb{D}^\alpha f(x) = \frac{1}{d_n(\alpha)} \int\limits_{\mathbb{R}^n} \frac{f(x) - f(x-y)}{|y|^{n+\alpha}} dy,$$

$$0 < \alpha < 1$$

The corresponding variable order construction is:

$$\mathbb{D}^{\alpha(\cdot)} f(x) = \frac{1}{d_n[\alpha(x)]} \int\limits_{\mathbb{R}^n} \frac{f(x) - f(x-y)}{|y|^{n+\alpha(x)}} dy,$$

$$0 < \alpha(x) < 1.$$

But!! No hope for the inversion formula

$$\mathbb{D}^{\alpha(\cdot)} I^{\alpha(\cdot)} f \equiv f.$$

In the one-dimensional case we can also deal with the Riemann-Liouville form of the fractional differentiation

$$I^{\alpha(\cdot)} f(x) = \frac{1}{\Gamma[\alpha(x)]} \int\limits_a^x \frac{f(y)}{(x-y)^{1-\alpha(x)}} dy,$$

$$0 < \alpha(x) < 1,$$

$$D^{\alpha(\cdot)} f(x) = \frac{1}{\Gamma[1-\alpha(x)]} \frac{d}{dx} \int\limits_a^x \frac{f(y)}{(x-y)^{\alpha(x)}} dy,$$

$$0 < \alpha(x) < 1.$$

No hope for the inversion formula $D^{\alpha(\cdot)} I^{\alpha(\cdot)} f \equiv f$ **.**

There are also known investigations of spherical Riesz type potentials of variable order

$$I^{\alpha(\cdot)} f(x) = \int\limits_{\mathbb{S}^{n-1}} \frac{f(y)}{|x-\sigma|^{n-1-\alpha(x)}} d\sigma,$$

$$0 < \alpha(x) < n-1.$$

In general, potential type operators (fractional integration operators) may be considered on arbitrary domains in $\mathbb{R}^n$**, surfaces, manifolds, fractal sets, and more generally, in the setting of quasimetric measure spaces**

$$(X, d, \mu)$$

with a quasimetric d **and positive Borel measure** μ**:**

$$\mathfrak{I}^{\alpha(\cdot)} f(x) = \int\limits_X \frac{f(y)\, d\mu(y)}{[d(x,y)]^{N-\alpha(x)}},$$

$$0 < \alpha(x) < N. \tag{1}$$

However, what is N? In the general setting, the space X may have no "dimension", but may have the so calledlower and upper dimensions. In the case where the measure satisfies the growth condition

$$\mu B(x,r) \le Cr^N$$

with some $N > 0$, this exponent N (not necessarily an integer), may be used in (1).

An important example of quasimetric measure space is a Carleson curve, i.e. a curve with an arc-length measure on which

$$\mu\Gamma(t,r) \le Cr.$$

Very often fractional operators over an arbitrary quasimetrric measure space are defined as

$$\mathfrak{I}^{\alpha(\cdot)} f(x) = \frac{1}{\alpha(x)} \int\limits_X \frac{[d(x,y)]^{\alpha(x)}}{\mu B(x, d(x,y))} f(y)\, d\mu(y), \tag{2}$$

$$\alpha(x) > 0.$$

Another example of an operator of variable order is the *fractional maximal function*

$$M^{\alpha(\cdot)} f(x) = \sup_{r>0} \frac{1}{r^{\alpha(x)}} \int\limits_{|y-x|<r} |f(y)|\, dy$$

and its corresponding version for an arbitrary quasimetric measure space.

In general, one may also consider

fractional powers $A^{\alpha(x)}$

of this or other operator A; however, different definitions of such powers, which coincide in the case $\alpha = const$, now may lead do quite different objects.

3 Typical examples of spaces

a). *Generalized Lebesgue spaces* $L^{p(\cdot)}(\Omega)$ *with variable exponent* defined by the modular

$$\int\limits_\Omega |f(x)|^{p(x)}\, dx < \infty.$$

b). More generally,*Musielak-Orlicz spaces* $L^{\Phi(\cdot)}(\Omega)$ *with the Young function also varying from point to point*:

$$\int\limits_\Omega \Phi[x, f(x)]\, dx < \infty.$$

The study of classical operators of harmonic analysis (maximal, singular operators and potential type operators) in the generalized Lebesgue spaces $L^{p(\cdot)}$ with variable exponent, weighted or non-weighted, undertaken last decade, continues to attract a strong interest of researchers. It is influenced not only by mathematical curiosity, but also by the fact that these spaces proved to be well adjusted for various applications as revealed in the book

[M. Ružička. *Electroreological Fluids: Modeling and Mathematical Theory*, 2000].

There are also known applications in the problems of image restoration, based on variable exponents.

c). *Variable exponent Morrey spaces* $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ defined by

$$\sup_{x\in\Omega,\ r>0} r^{-\lambda(x)} \int\limits_{B(x,r)\cap\Omega} |f(y)|^{p(y)}dy < \infty. \tag{3.1}$$

d). *Hölder spaces* $H^{\lambda(\cdot)}(\Omega)$ *of variable order*, defined by the condition

$$\sup_{|h|<t} |f(x+h) - f(x)| \le Ct^{\lambda(x)},\ \ x \in \Omega.$$

d1). More generally, *generalized Hölder spaces with variable characteristic* $\omega(h) = \omega(x,h)$ depending on x:

$$\sup_{|h|<t} |f(x+h) - f(x)| \le C\omega(x,t),$$

(the spaces of continuous functions with a given dominant of their continuity modulus, which may vary from point to point).

In these lectures we touch only Lebesgue and Hölder spaces with variable exponent.

4 Basics on variable exponent Lebesgue spaces

In future we abbreviate:

Variable exponent Lebesgue spaces =
= VELS

Let Ω be an open set in $\mathbb{R}^n$. By

$$L^{p(\cdot)}(\Omega)$$

we denote the space of functions $f(x)$ on Ω such that

$$I_p(f) = \int_\Omega |f(x)|^{p(x)}dx < \infty,$$

where $p(x)$ is a measurable function on Ω with values in $[1,\infty)$.

We will always use the notation

$$p_- = \inf_{x\in\Omega} p(x), \qquad p_+ = \sup_{x\in\Omega} p(x).$$

The functional $I_p(f)$ is called ***modular***. This is a linear space if and only if

$$\sup_{x\in\Omega} p(x) < \infty$$

and then it is a Banach space with respect to

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\}. \tag{4.1}$$

We will often abbreviate this to

$$\|f\|_{p(\cdot)}.$$

From the definition of the norm it follows that

$$I_p\left(\frac{f}{\|f\|_{p)\cdot)}}\right) \;=\; 1.$$

The following comparison of the norm with the modular is straightforward:

$$\|f\|^{p_+}_{p(\cdot)} \le I_p(f) \le \|f\|^{p_-}_{p(\cdot)}\ , \quad \text{if} \quad \|f\|_{p(\cdot)} \le 1, \tag{4.2}$$

$$\|f\|^{p_-}_{p(\cdot)} \le I_p(f) \le \ \|f\|^{p_+}_{p(\cdot)}, \quad \text{if} \quad \|f\|_{p(\cdot)} \ge 1. \tag{4.3}$$

NOTE. Very often - but not always - we will assume that the exponent is a little bit better than just continuous. Namely, We will often suppose that it satisfies the following weak Lipschitz condition

$$|p(x) - p(y)| \le \frac{A}{\ln \frac{1}{|x-y|}}\ , \quad |x-y| \le \frac{1}{2}, \quad x,\, y \in \Omega.$$

For brevity, the class of functions satisfying the above condition, will be denoted by

$$\mathcal{P}^{\mathbf{log}}(\Omega).$$

EXERCISE. Let

$$\chi_r(y) = \{y \in \mathbb{R}^n : \ |y| > r\}$$

be the characteristic function of the exterior of a ball. A simple calculation in the case of constant p yields

$$\left\|\frac{\chi_r(y)}{|y|^\beta}\right\|_{L^p} = \frac{const}{r^{\beta - \frac{n}{p}}},$$

if $\beta > \frac{n}{p}$.

This gives an estimation of the growth of this norm when $r \to 0$. What about the case of variable $p(x)$?

In the case of integration in this example over a bounded domain Ω, the following estimate holds.

Lemma 4.1. *Let Ω be a bounded open set in $\mathbb{R}^n$,*

$$p_- \geq 1, \quad p_+ < \infty,$$

and

$$p, \beta \in \mathcal{P}^{log}(\Omega).$$

If

$$\sup_{x\in\Omega} \beta(x)p(x) > n,$$

then

$$\left\| \frac{\chi_r(y-x)}{|y-x|^{\beta(x)}} \right\|_{L^{p(\cdot)}} \leq \frac{C}{r^{\beta(x)-\frac{n}{p(x)}}} \tag{4.4}$$

where C does not depend on x.

We omit the proof, since it is technically rather complicated, the proof may be found in [S.Samko, Integr. Transf. Spec. Funct. 1998]

Remark: the above statement is given for a more general case of a ball centered at an arbitrary point x, keeping in mind that now the space is not invariant with respect to translations.

Sometimes, (4.1) is called Luxemburg norm because of a similar norm for Orlich spaces (Luxemburg, 1955). However, just in form (4.1), this norm for $L^{p(\cdot)}$ was introduced before W.Luxemburg by H.Nakano (1951).

Meanwhile, the norm of type (4.1), as well as a similar norm for the Orlicz spaces is nothing else but the realization of a general norm for"normalizable" topological spacesprovided by the famous Kolmogorov theorem (A.N.Kolmogorov, Zur Normierbarkeit eines allgemeinen topologischen linearen Räumes, *Studia Math.*, Vol. 5, 29-33, 1934.)

This theorem runs as follows.

A Hausdorff linear topological space X admits a norm iff it has a convex bounded neighbourhood of the null-element and in this case Minkowsky functional of this neighbourhood is a norm.

The Minkowsky functional of a set $U \subset X$ is the functional $M_U(x)$, $x \in X$, defined as

$$M_U(x) = \inf\{\lambda : \lambda > 0, \ \frac{1}{\lambda}x \in U\}$$

so that the above norm is nothing else but the Minkowsky functional of the unit ball in $X = L^{p(\cdot)}$.

Therefore, (4.1) = *the Kolmogorov-Minkowsky norm* .

The pioneer paper where the space $L^{p(\cdot)}$ was studied as a special object and as a Banach space, was I.I. Sharapudinov. On a topology of the space $L^{p(t)}([0,1])$. ***Matem. Zametki*** 26(1979), no 4, 613-632,
(in the one-dimensional case, but most of the results are automatically rewritten for the multidimensional case), although this space appeared earlier as an example illustrating the modular spaces in the investigations by H.Nakano (1951).

Later, basic important facts for the spaces $L^{p(\cdot)}$ were developed in the paper O.Kovacik and I.Rakosnik
On spaces $L^{p(x)}$ ***and*** $W^{k,p(x)}$. Czech. Math. J., 1991, vol. 41(116), 592-618, including variable order Soboleb spaces

In general, the spaces $L^{p(\cdot)}$ are particular cases of the generalized Orlicz spaces introduced and investigated earlier by J. Musielak. However, that were namely the specifics of the spaces $L^{p(\cdot)}$ which attracted many researchers and allowed to develop a rich theory of these spaces, this interest being also roused by applications revealed in various areas.

Two main trends in research:

I). Mapping problems for operators in VELS, especially in the situation when the variable exponent may approach some critical values.

II). The corresponding Operator Theory: Fredholmness and invertibility problems.

We mention, in particular, the case of fractional type operators.

Let $I^{\alpha(\cdot)}$ be some fractional integration operator (potential operator), and $D^{\alpha(\cdot)}$ the corresponding differentiation operator, such that we have

$$D^{\alpha} I^{\alpha} f \equiv f$$

in the case of constant orders α. Now we have

$$D^{\alpha(\cdot)} I^{\alpha(\cdot)} = I + \text{something}.$$

What can be said about this "something"?

Depending on the assumptions on the behaviour of $\alpha(x)$ and parameters of the space, sometimes it may be shown that this something is a compact operator.

But in general, this is an open problem.

A more essential question is: even if this "something" is compact, what about the spectrum problem?

Also: Fredholmness (normal solvability) of Wiener-Hopf and singular integral equations with piece-wise continuous coefficients in $L^{p(\cdot)}(\Gamma)$

Various problems in both the trends are mainly solved or are under intensive investigation for the time being. Solution of many problems proved to be a very strong fortress. Nowadays this fortress is worldwide attacked.

5 The principal difficulties (BAD news about VELS

1) no invariance with respect to translations and dilations;

$$f \in L^{p(\cdot)}(\mathbb{R}^n) \quad \not\Rightarrow \quad f(\cdot - h) \in L^{p(\cdot)}(\mathbb{R}^n)$$

(in general), since the latter would mean the integrability

$$\int_{\mathbb{R}^n} |f(x)|^{p(x+h)} dx < \infty$$

with the wrong exponent.Similarly,

$$f \in L^{p(\cdot)}(\mathbb{R}^n) \quad \not\Rightarrow \quad f(\lambda \cdot) \in L^{p(\cdot)}(\mathbb{R}^n)$$

2) As a consequence, Young theorem is no more valid; i.e. let

$$Kf(x) = \int_{\mathbb{R}^n} k(x-y) f(y)\, dy,$$

then the statement

$$\|Kf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|k\|_{L^1(\mathbb{R}^n)} \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

is no more valid.

3) Minkowsky integral inequality

$$\left\| \int_{\Omega} F(\cdot, y)\, dy \right\|_{p(\cdot)} \leq \int_{\Omega} \|F(\cdot, y)\|_{p(\cdot)}\, dy,$$

although valid, is a very rough mean, no help of it ...

For instance, how we prove the Young theorem in the case p is constant:

$$\|Kf\|_p = \left\| \int_{\mathbb{R}^n} k(y) f(x-y) dy \right\|_p \leq \int_{\mathbb{R}^n} |k(y)|\, \|f(\cdot - y)\|_p\, dy$$

which does not work when p is variable.

6 Some hopes (GOOD news)

1) The Hölder inequality still holds, but in the form

$$\left| \int_{\Omega} f(x) g(x)\, dx \right| \leq C \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}$$

with the constant

$$C = \frac{1}{p_-} + \frac{1}{q_-} \leq 2$$

where

$$p_- = \inf_{x\in\Omega} p(x), \quad q_- = \inf_{x\in\Omega} q(x) \quad \text{and} \quad \frac{1}{p(x)} + \frac{1}{q(x)} \equiv 1.$$

Proof.

This Hölder inequality is proved in the the standard way via the numerical inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{6.1}$$

with $a > 0, b > 0, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p < \infty$, so that

$$\left|\frac{f(x)g(x)}{\|f\|_p\|g\|_q}\right| \leq \frac{1}{p(x)}\left|\frac{f(x)}{\|f\|_p}\right|^{p(x)} + \frac{1}{q(x)}\left|\frac{g(x)}{\|g\|_q}\right|^{q(x)},$$

Integrating over Ω and estimating $p(x)$ and $q(x)$,we are done. □

2) The norm in the space $L^{p(\cdot)}(\Omega)$ seems to be complicated in a sense, to be calculated or estimated. So the straightforward estimation of the boundedness of an operator:

$$\|Af\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)} \tag{6.2}$$

is not easy. However, in the case of linear operators, the above inequalities between the norm and the modular and the homogeneity property

$$\|A\|_{X\to X} = \sup_{f\in X} \frac{\|Af\|_X}{\|f\|_X} = \sup_{\|f\|_X=1} \|Af\|_X$$

allow us to replace checking of (6.2) by a work with a modular:

$$\int_\Omega |Af(x)|^{p(x)}\,dx \leq C \quad \text{for all} \quad f \quad \text{with} \quad \|f\|_{p(\cdot)} \leq 1, \tag{6.3}$$

which is certainly easier.

3) Pointwise inequalities are useful.

We have

$$|f(x)| \leq |g(x)| \quad \Longrightarrow \quad \|f\|_{p(\cdot)} \leq \|g\|_{p(\cdot)}.$$

Therefore, if one operator is pointwise dominated by another one:

$$|Af(x)| \leq |Bf(x)|,$$

and we know that the operator B is bounded, then the boundedness of the operator A immediately follows.

EXAMPLE. Stein's inequality states that convolution operators with "sufficiently good" kernels are dominated by the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)\cap\Omega} |f(y)|\, dy$$

where

$$B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}.$$

The Stein inequality is:

$$\boxed{\left| \int\limits_{\mathbb{R}^n} k(x-y) f(y)\, dy \right| \le c\, Mf(x)}$$

if the kernel k is dominated by a radial decreasing integrable function:

$$|k(x)| \le K(|x|), \qquad \int\limits_{\mathbb{R}^n} K(|x|)\, dx < \infty.$$

and $c = 2\|K\|_{L^1(\mathbb{R}^n)}$ in this case.

This inequality holds even in a stronger form

$$\boxed{\sup_{\varepsilon>0} \frac{1}{\varepsilon^n} \left| \int\limits_{\mathbb{R}^n} k\left(\frac{x-y}{\varepsilon}\right) f(y)\, dy \right| \le c\, Mf(x)}$$

i.e. uniformly with respect to dilations.

Therefore, one should be interested in the boundedness of the maximal operator in our spaces: it will involve the boundedness of many and many convolution operators in applications.

7 Sobolev embedding

Another reason which heated this interest was that the famous Sobolev theorem on the boundedness of the Riesz potential from L^p to L^q with the Sobolev exponent

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$$

may be derived from the boundedness of the maximal operator by the so called

Hedberg's pointwise trick

Let us reproduce this Hedberg's trick in the variable exponent setting. We wish to prove that- if the maximal operator M is bounded in our space $L^{p(\cdot)}(\mathbb{R}^n)$ - then the potential operator

$$I^{\alpha(\cdot)} f(x) = \int\limits_{\Omega} \frac{f(y)}{|x-y|^{n-\alpha(x)}} dy, \qquad x \in \Omega$$

where

$$\inf_{x\in\Omega}\alpha(x)>0, \qquad \sup_{x\in\Omega}\alpha(x)<n$$

is bounded from the space $L^{p(\cdot)}(\Omega)$ to the space $L^{q(\cdot)}(\Omega)$ with the limiting variable Sobolev exponent

$$\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha(x)}{n}.$$

For simplicity we consider only the case of bounded sets Ω in $\mathbb{R}^n$.

About the variable exponent $p(x)$ we assume that $p\in\mathcal{P}^{\log}(\Omega)$.

Proof. We split the fractional integral:

$$I^{\alpha(x)}f=\int_{|x-y|<r}|x-y|^{\alpha(x)-n}f(y)dy$$

$$+\int_{|x-y|>r}|x-y|^{\alpha(x)-n}f(y)dy \;=:A_r(x)+B_r(x) \tag{7.1}$$

so that $B_r(x)\equiv 0$ for $r\geq D=diam\ \Omega$.

We shall take use of the inequality

$$|A_r(x)| \;\leq\; \frac{2^n r^{\alpha(x)}}{2^{\alpha(x)}-1}\;\; Mf(x) \tag{7.2}$$

which is known in case of $\alpha(x)=const$ and remains valid in case it is variable. Indeed, to check it, we use the *dyadic decomposition*:

$$|A_r(x)|\leq\sum_{k=1}^{\infty}\;\int\limits_{\frac{r}{2^k}<|x-y|<\frac{r}{2^{k-1}}}|f(y)||x-y|^{\alpha(x)-n}dy$$

$$\leq\sum_{k=1}^{\infty}\left(\frac{r}{2^k}\right)^{\alpha(x)-n}\int_{|x-y|<\frac{r}{2^{k-1}}}|f(y)|dy$$

$$\leq\sum_{k=1}^{\infty}\left(\frac{r}{2^k}\right)^{\alpha(x)-n}\left(\frac{r}{2^{k-1}}\right)^{n}Mf(x)$$

$$=2^n r^{\alpha(x)}\sum_{k=1}^{\infty}2^{-k\alpha(x)}Mf(x)\ .$$

By the assumption $\inf\limits_{x\in\Omega}\alpha(x)>0$ we then get

$$|A_r(x)|\leq c_1 r^{\alpha(x)}Mf(x) \tag{7.3}$$

with some absolute constant $c_1>0$.

We assume that $\|f\|_{p(\cdot)}\leq 1$.

Applying the Hölder inequality for variable exponents to the integral $B_r(x)$, we obtain

$$|B_r(x)|\leq 2\|f\|_{p(\cdot)}\left\||x-y|^{\alpha(x)-n}\chi_r\right\|_{p'(\cdot)}$$

$$\leq 2 \left\| |x-y|^{\alpha(x)-n} \chi_r(y) \right\|_{p'(\cdot)},$$

where $\chi_r(y)$ is the characteristic function of the exterior

$$\{y \in \Omega : |x-y| > r\}$$

of the ball.

We can apply the estimate written in Exercise above, which yields:

$$\left\| |x-y|^{\alpha(x)-n} \chi_r(y) \right\|_{p'(\cdot)} \leq c_2 r^{-\frac{n}{q(x)}}.$$

Then

$$I^{\alpha(x)} f \leq A_r(x) + B_r(x)$$
$$\leq c \left[r^{\alpha(x)} Mf(x) + r^{-\frac{n}{q(x)}} \right].$$

Minimizing the right-hand side with respect to r we see that its minimum is reached at

$$r_{\min} = \left[\frac{1}{n} q(x) \alpha(x) Mf(x) \right]^{\frac{p(x)}{n}}$$

and easy evaluations give

$$\left| I^{\alpha(x)} f(x) \right| \leq \frac{C}{p(x)} \left[\frac{q(x)}{n} Mf(x) \right]^{\frac{p(x)}{q(x)}} \left[\frac{1}{\alpha(x)} \right]^{1-\frac{p(x)}{q(x)}}.$$

Hence,

$$\left| I^{\alpha(x)} f(x) \right| \leq C \left[Mf(x) \right]^{\frac{p(x)}{q(x)}}$$

so that

$$\int_\Omega \left| I^{\alpha(x)} f(x) \right|^{q(x)} dx \leq C \int_\Omega \left[Mf(x) \right]^{p(x)} dx.$$

Recall that we assume that the maximal operator M is bounded in the space $L^{p(\cdot)}(\Omega)$, whence the boundedness of the fractional operator $I^{\alpha(\cdot)}$ in the space $L^{p(\cdot)}(\Omega)$ follows. □

CONCLUSION: two key moments in the proof, one is the preassumed boundedness of the maximal operator M, another is the estimate

$$\left\| |x-y|^{\alpha(x)-n} \chi_r(y) \right\|_{p'(\cdot)} \leq c_2 r^{-\frac{n}{q(x)}}.$$

So various efforts were spend to prove the boundedness of the maximal operator . The breakthrough is of 2002 and is due to Lars Diening (Freiburg University till 2010, München University after 2010).

He proved the poundedness of the maximal operator in VELS $L^{p(\cdot)}(\Omega)$ in the case of bounded domains in $\mathbb{R}^n$ under the following natural conditions

$$1 < p_- \leq p(x) \leq p^+ < \infty, \quad x \in \Omega \tag{7.4}$$

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{|x-y|}}, \quad |x-y| \leq \frac{1}{2}, \quad x, y \in \Omega. \tag{7.5}$$

The latter plays in general a very important role in the theory of VELS. It is often referred to as the "*local log-condition*". For the boundedness of the maximal operator this condition is necessary in a sense (in terms of the continuity modulus), A.Lerner, 2005.

The following statement has been proved: Let

$$\min_{x\in\Omega} p(x) > 1, \quad \max_{x\in\Omega} p(x) < \infty$$

$$\text{and} \qquad \min_{x\in\Omega} \alpha(x) > 0, \quad \max_{x\in\Omega} \alpha(x)p(x) < n.$$

Then the fractional operator

$$I^{\alpha(\cdot)} f(x) = \int_\Omega \frac{f(y)\, dy}{|x-y|^{n-\alpha(x)}}$$

is bounded from the variable exponent space $L^{p(\cdot)}(\Omega)$ into another such space $L^{q(\cdot)}(\Omega)$, where

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}.$$

REMARK on Sobolev embedding theorem.

If we wish to deal also with the variable structure of the underlying space X, then we will also have to work with *local variable dimension* $n = n(x)$, and even more, we have to deal with the notion of the lower and upper dimensions

$$\underline{\mathfrak{dim}}(X,x) \quad \text{and} \quad \overline{\mathfrak{dim}}(X,x)$$

at every point $x \in X$.

8 On variable dimensions

Let X be any quasimetric measure space with quasidistance $d(x, y)$.

In the particular case: for every point $x \in X$ there exists a positive number $s = s(x)$ such that

$$C_1 r^{s(x)+\varepsilon} \leq \mu B(x,r) \leq C_2 r^{s(x)-\varepsilon}, \tag{8.1}$$

for every positive $\varepsilon > 0$, where the constants $C_1 > 0, C_2 > 0$ in general depend on x and ε:

$$C_1 = C_1(x,\varepsilon), \qquad C_2 = C_2(x,\varepsilon),$$

then the space X may be said to

have a local dimension

at a point x, equal to $s(x)$. It may be calculated by the formula

$$dim_X(x) := s(x) = \lim_{r\to 0} \frac{\ln \mu B(x,r)}{\ln r}.$$

In the general case, this limit may not exist, but the lower and upper limits always exist and these lower and upper limits

$$\underline{dim}_X(x) = \underline{\lim}_{r\to 0} \frac{\ln \mu B(x,r)}{\ln r},$$

$$\overline{dim}_X(x) = \overline{\lim}_{r\to 0} \frac{\ln \mu B(x,r)}{\ln r}$$

are called upper and lower local dimensions attributed to the point x. This approach is well known in the theory of fractal sets, see for instance the book [Falconer, 1997]

Meanwhile in the first studies of variable exponent operators in variable exponent spaces, it proved to be that more refined definition is necessary. Namely, the dimensions introduced as

$$\underline{\mathfrak{dim}}_X(x) := \sup_{r>1} \frac{\ln \left(\underline{\lim}_{t\to 0} \frac{\mu B(x,rt)}{\mu B(x,t)} \right)}{\ln r},$$

$$\overline{\mathfrak{dim}}_X(x) := \inf_{r>1} \frac{\ln \left(\overline{\lim}_{t\to 0} \frac{\mu B(x,rt)}{\mu B(x,t)} \right)}{\ln r}.$$

Such limits when applied to Young functions in the theory of Orlicz spaces are known as Matuszewska-Orlicz indices.

REMARK . Introduction of a new notion of dimensions

$\underline{\mathfrak{dim}}_X(x)$ and $\overline{\mathfrak{dim}}_X(x)$

in the above form is caused by the fact that they arise naturally when dealing with Muckenhoupt type condition for radial type weights on metric measure spaces. They seem may not coincide with dimensions

$$\underline{dim}\, X(x), \qquad \overline{dim}\, X(x).$$

There is an impression that probably for different goals different notions of dimensions may be useful.

The Sobolev embedding theorem in its general form

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{?(x)}$$

is not proved, up to now, but some weaker versions have been obtained.

9 Convolution operators

Convolution operators

$$Af(x) = \int\limits_{\mathbb{R}^n} k(y)f(x-y)dy$$

with rather "nice" kernels for which the local log-condition is not needed,

The Young theorem in its natural form is not valid in the case of variable exponent, even if $p(x)$ is very smooth.

As is known, the Young theorem is valid under the log-condition on $p(x)$ if the kernel is dominated by a radial integrable non-increasing function.

However, a natural expectation was that the Young theorem may be valid in the case of rather "nice" kernels without the local log-condition, which was proved inDiening,Samko, 2007.

Let

$$\mathcal{P}_\infty(\mathbb{R}^n)$$

be the set of measurable bounded functions on $\mathbb{R}^n$ such that: 1) $\quad 1 \le p_- \le p(x) \le p_+ < \infty, \quad x \in \mathbb{R}^n$,2) there exists $p(\infty) = \lim\limits_{x\to\infty} p(x)$3) and

$$|p(x) - p(\infty)| \le \frac{A}{\ln(2+|x|)}, \quad x \in \mathbb{R}^n. \tag{9.1}$$

THEOREM. Let

$$|k(y)| \le C(1+|y|)^{-\lambda}, y \in \mathbb{R}^n$$

for some

$$\lambda > n\left(1 - \frac{1}{p(\infty)} + \frac{1}{q(\infty)}\right).$$

Then the convolution operator

$$\int_{\mathbb{R}^n} k(x-y)f(y)\,dy$$

is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$ under the only assumption that

$$p, q \in \mathcal{P}_\infty(\mathbb{R}^n) \quad \text{and} \quad q(\infty) \ge p(\infty).$$

10 Characterization of the range of the fractional operator

Recall that the Riesz fractional integration operator is given by

$$I^\alpha f(x) = \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} \frac{f(y)\,dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n,$$

where the normalizing constant $\gamma(\alpha) = 2^{\alpha}\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right) \Big/ \Gamma\left(\frac{n-\alpha}{2}\right)$ is chosen so that the semigroup property $I^{\alpha}I^{\beta} = I^{\alpha+\beta}$ holds.

How can one construct the inverse operator which may be called Riesz fractional differentiation or Riesz fractional derivative? This inverse operator will be denoted by

$$\mathbb{D}^{\alpha} f.$$

Formally,

$$I^{\alpha} f = F^{-1}|\xi|^{-\alpha} F f \qquad \Longrightarrow \qquad \mathbb{D}^{\alpha} f = F^{-1}|\xi|^{\alpha} F f.$$

It is known that

$$|\xi|^{\alpha}$$

is the Fourier transform (in the distributional sense) of the distribution

$$\frac{const}{|x|^{n+\alpha}},$$

so that the operator $\mathbb{D}^{\alpha}$ is the convolution with the above distribution.

One of the known ways to construct a realization of this convolution (*regularization*) is via subtracting Taylor terms in a neighborhood of the singular point. Another way avoids the usage of derivatives and is more direct: it uses finite differences.

We start with the case

$$0 < \alpha < 1.$$

In this case Taylor formula approach and finite differences approach coincide and provide the hypersingular construction

$$\mathbb{D}^{\alpha} f = \frac{1}{d_n(\alpha)} \int\limits_{\mathbb{R}^n} \frac{f(x) - f(y)}{|y|^{n+\alpha}}\, dy,$$

where

$$d_n(\alpha) = -\gamma_n(-\alpha).$$

This is what is usually called Riesz fractional derivative of order α. It exists for sufficiently nice functions. For *not so nice functions* it is interpreted as the limit of the corresponding truncated integrals:

$$\mathbb{D}^{\alpha} f(x) = \frac{1}{d_n(\alpha)} \lim_{\varepsilon \to 0} \int\limits_{|y|>\varepsilon} \frac{f(x) - f(y)}{|y|^{n+\alpha}}\, dy,$$

where the limit is interpreted in this or other sense, depending on the class of functions involved, for instance in the norm of L^p :

$$\mathbb{D}^{\alpha} f(x) = \frac{1}{d_n(\alpha)} \lim_{\substack{\varepsilon \to 0 \\ L^p(\mathbb{R}^n)}} \int\limits_{|y|>\varepsilon} \frac{f(x) - f(y)}{|y|^{n+\alpha}}\, dy. \tag{10.1}$$

The construction (10.1) first appeared in a paper by E.Stein (1961), where it was used to characterize the space

$$B^{\alpha}(L^p) := \{f : \ f = B^{\alpha}\varphi, \quad \varphi \in L^p(\mathbb{R}^n)\}$$

of Bessel potentials

$$B^{\alpha}f := \int\limits_{\mathbb{R}^n} G_{\alpha}(x-y)f(y)\,dy,$$

where

$$\widehat{G_{\alpha}}(\xi) = \frac{1}{(1+|\xi|^2)^{\frac{\alpha}{2}}}.$$

It was shown that a function $f \in L^p(\mathbb{R}^n)$ belongs to $B^{\alpha}(L^p)$ if and only if the limit (10.1) exists in $L^p(\mathbb{R}^n)$.

Let now

$$0 < \alpha < \infty.$$

We define the finite difference

$$\left(\Delta_h^{\ell} f\right)(x) \;=\; \sum_{k=0}^{\ell}(-1)^k \binom{\ell}{k} f(x-kh)$$

with the integer order ℓ,

$$\ell > \alpha$$

(fractional orders ℓ may be also used, but we do not touch this case) and introduce the hypersingular integral of higher order by

$$\mathbb{D}^{\alpha} f = \lim_{\varepsilon\to 0} \mathbb{D}^{\alpha}_{\varepsilon} f,$$

where the truncated hypersingular integral $\mathbb{D}^{\alpha}_{\varepsilon} f$ has the form

$$\mathbb{D}^{\alpha}_{\varepsilon} f := \frac{1}{d_{n,\ell}(\alpha)} \int_{|y|>\varepsilon} \frac{\left(\Delta_y^{\ell} f\right)(x)}{|y|^{n+\alpha}}\,dy$$

and the normalizing constant

$$d_{n,\ell}(\alpha) = \int\limits_{\mathbb{R}^n} \left(1-e^{iy_1}\right)^{\ell} \frac{dy}{|y|^{n+\alpha}}$$

may be explicitly calculated by the formula

$$d_{n,\ell}(\alpha) = \frac{\pi^{1+\frac{n}{2}}}{2^{\alpha}\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{n+\alpha}{2}\right)} \frac{A_{\ell}(\alpha)}{\sin\frac{\alpha\pi}{2}},$$

where

$$A_{\ell}(\alpha) = \sum_{k=0}^{\ell}(-1)^{k-1}\binom{\ell}{k} k^{\alpha}.$$

The extension, from $0 < \alpha < 1$ to the general case $0 < \alpha < \infty$, of above Stein's characterization of the Bessel potential space $B^\alpha(L^p(\mathbb{R}^n))$ is due to P.Lizorkin (1970).

But what about the Riesz potential case?

The first arising question is about the inversion of the Riesz potential operator. We expect that $\mathbb{D}^\alpha$ is the left inverse to I^α:

$$\mathbb{D}^\alpha I^\alpha f \equiv f, \qquad 0 < \alpha < n$$

within the frameworks of the $L^p(\mathbb{R}^n)$-spaces, $\quad 1 < p < \frac{n}{\alpha}$.

A justification if this inversion for constant p was given in [S.Samko] (1976). In this justification we have to show that the limit

$$\lim_{\substack{\varepsilon \to 0 \\ L^p}} \mathbb{D}^\alpha_\varepsilon I^\alpha f$$

exists in the norm of the space $L^p(\mathbb{R}^n)$ for $f \in L^p(\mathbb{R}^n)$.

Since both $\mathbb{D}^\alpha_\varepsilon$ and I^α are convolution operators, their composition $\mathbb{D}^\alpha_\varepsilon I^\alpha$ is the same:

$$\mathbb{D}^\alpha_\varepsilon I^\alpha = \int\limits_{\mathbb{R}^n} K_\varepsilon(x-y) f(y)\, dy.$$

The key moment:the kernel $K_\varepsilon(x)$ has the dilation structure

$$K_\varepsilon(x) = \frac{1}{\varepsilon^n} \mathcal{K}_{\ell,\alpha}\left(\frac{x}{\varepsilon}\right).$$

It may be shown that

$$\mathcal{K}_{\ell,\alpha} \in L^1(\mathbb{R}^n) \qquad \text{and} \qquad \int\limits_{\mathbb{R}^n} \mathcal{K}_{\ell,\alpha}(y)\, dy = 1.$$

Then it suffices to make use of the well known identity approximation theorem.

Now, what about variable $L^{p(\cdot)}$-spaces?

The fact that we just have $\mathcal{K}_{\ell,\alpha} \in L^1(\mathbb{R}^n)$ is helpless: no Young theorem.

Fortunately, there may be obtained an additional information: the kernel $\mathcal{K}_{\ell,\alpha}$ is a bounded function beyound the origin and admits the bounds

$$|\mathcal{K}_{\ell,\alpha}(x) \le \frac{C}{|x|^{n-\alpha}} \qquad \text{for} \qquad |x| \le 1$$

and

$$|\mathcal{K}_{\ell,\alpha}(x) \le \frac{C}{|x|^{n+\ell-\alpha}} \qquad \text{for} \qquad |x| \ge 1.$$

(recall that ℓ is always chosen so that $\quad \ell > \alpha$.)

Consequently, this kernel is dominated by a radial decreasing integrable function. Then Stein's pointwise uniform inequality

$$\boxed{\frac{1}{\varepsilon^n}\left|\int\limits_{\mathbb{R}^n} k\left(\frac{x-y}{\varepsilon}\right) f(y)\,dy\right| \le c\,Mf(x)}$$

is applicable, so that

$$\boxed{|\mathbb{D}^\alpha_\varepsilon I^\alpha f| \le c\,Mf(x)}$$

where c does not depend on ε.

We assume that

$$1 < \alpha < \frac{n}{p_+}$$

when apply the above to functions $f \in L^{p(\cdot)(\mathbb{R}^n)}$.

Thus we need the boundedness of the maximal operator M in the VELS $L^{p(\cdot)}(\mathbb{R}^n)$. We already know that in the case of bounded set Ω, the maximal operator M is bounded in the space $L^{p(\cdot)}(\Omega)$ under the condition

$$p \in \mathcal{P}^{\mathbf{log}}(\Omega).$$

In the case of unbounded sets, there appears an additional condition (the decay condition): there exists

$$p_\infty := \lim_{|x|\to\infty} p(x)$$

and

$$|p(x) - p_\infty| \le \frac{C}{\ln(e+|x|)}.$$

Both these conditions and the standard assumption

$$1 < p_- \le p(x) \le p_+ < \infty,$$

guarantee the boundedness of the operator M in $L^{p(\cdot)}(\mathbb{R}^n)$ (D. Cruz-Uribe, A.Fiorenza, C.J. Neugebauer, 2003)

Then under these conditions we obtain that the composition

$$\mathbb{D}^\alpha_\varepsilon I^\alpha$$

admits the uniform boundedness

$$\|\mathbb{D}^\alpha_\varepsilon I^\alpha f\|_{p(\cdot)} \le C\|f\|_{p(\cdot)}.$$

Then (by Banach-Steinhaus theorem) it suffices to check the convergence

$$\mathbb{D}^\alpha_\varepsilon I^\alpha f(x) \quad \to \quad f(x)$$

in $L^{p(\cdot)}(\mathbb{R}^n)$-norm on a dense set.

Since $p_- \le p(x) \le p_+$, by standard arguments it may be proved that

$$\|f\|_{p(\cdot)} \le C \max\{\|f\|_{p_-}, \|f\|_{p_+}\}$$

Consequently,

$$\|\mathbb{D}^\alpha_\varepsilon I^\alpha f - f\|_{p(\cdot)}$$
$$\le C \max\left\{\|\mathbb{D}^\alpha_\varepsilon I^\alpha f - f\|_{p_-}, \|\mathbb{D}^\alpha_\varepsilon I^\alpha f - f\|_{p_+}\right\}.$$

Thus the convergence

$$\|\mathbb{D}^\alpha_\varepsilon I^\alpha f - f\|_{p(\cdot)} \quad \to \quad 0$$

reduces to similar convergence with constant exponents, which is known. It remains to note that the set

$$L^{p_-}(\mathbb{R}^n) \bigcap L^{p_+}(\mathbb{R}^n)$$

is dense in $L^{p(\cdot)}(\mathbb{R}^n)$.

We have proved the following

THEOREM.*Let* $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ *satisfy the decay condition and* $1 < p_- \le p(x) \le p_+ < \frac{n}{\alpha}$. If

$$f(x) = (I^\alpha \varphi)(x)$$

where $\varphi \in L^{p(\cdot)}(\mathbb{R}^n)$, then

$$\varphi(x) = \lim_{\varepsilon \to 0} \left(\mathbb{D}^\alpha_\varepsilon f\right)(x)$$

where the limit is understood in the $L^{p(\cdot)}(\mathbb{R}^n)$*-norm.*

The above theorem was proved in [A.Almeida, 2003].

A development of this result, obtained in [Almeida, Samko, 2006] concerns the description of functions representable by the fractional integral of functions in $L^{p(\cdot)}(\mathbb{R}^n)$.

THEOREM Let p satisfy the decay condition and

$$p \in \mathcal{P}^{\log}(\mathbb{R}^n), \quad 1 < p_- \le p^+ < \frac{n}{\alpha}$$

and f be a locally integrable function. Then

$$f \in I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)],$$

if and only if

$$f \in L^{q(\cdot)}(\mathbb{R}^n), \qquad \frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n},$$

and

$$\mathbb{D}^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n).$$

A study of the range $I^\alpha[L^{p(\cdot)}(\Omega)]$ for domains $\Omega \subset \mathbb{R}^n$ is an open question; in a form similar to the previous Theorem, it is open even in the case of constant p.

One of the reasons is in the absence of the corresponding tool of hypersingular integrals adjusted to domains in $\mathbb{R}^n$;

11 Variable order Hölder spaces

In application often not Lebesgue spaces, but Hölder (Lipschitz) spaces are of more use. In the Euclidean case Hölder spaces are defined by the condition

$$f \in H\lambda(\Omega): \quad \sup_{|h|<t} |f(x+h) - f(x)| \leq Ct^\lambda, \ x \in \Omega,$$

where $0 < \lambda \leq 1$.

We now are going to consider fractional type operators $I^{\alpha(\cdot)}$ inHölder spaces $H^{\lambda(\cdot)}$ of variable order.

First, what we know in the case of constant orders λ and α?

In the one-dimensional case, for the Riemann-Liouville fractional integrals

$$\left(I^\alpha_{a+}\varphi\right)(x) := \frac{1}{\Gamma(\alpha)} \int\limits_a^x \frac{\varphi(t)\,dt}{(x-t)^{1-\alpha}}, \qquad x > a$$

we have

$$\varphi \in H^\lambda([a,b]), \qquad -\infty < a < b < \infty,$$

$$\Longrightarrow \qquad f(x) = \frac{f(a)}{\Gamma(1+\alpha)}(x-a)^\alpha \ + \ g(x)$$

where

$$g \in H^{\lambda+\alpha}([a,b])$$

under the condition

$$\lambda + \alpha < 1$$

(G.Hardy, J.Littlewood, 1928).

Note that

$$f(a) \neq 0 \quad \Longleftrightarrow \quad \frac{f(a)}{\Gamma(1+\alpha)}(x-a)^\alpha \notin H^{\lambda+\alpha}([a,b])$$

(the influence of the boundary is expected in the multidimensional case). Multidimensional case: first there were obtained statements on such napping properties for the spherical fractional integrals

$$\mathbb{I}^\alpha f(\xi) = \int\limits_{\mathbb{S}^{n-1}} \frac{f(\sigma)\,d\sigma}{|\xi-\sigma|^{n-1-\alpha}}, \qquad \xi \in \mathbb{S}^{n-1},$$

on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. It was shown (B.Vakulov, 1978-1980) that

$$\mathbb{I}^\alpha(H^\lambda(\mathbb{S}^{n-1})) = H^{\lambda+\alpha}(\mathbb{S}^{n-1}) \quad !!$$

From these spherical versions there were derived weighted statements for spatial Riesz fractional integrals viathe stereographical projection of $\mathbb{R}^{n-1}$ onto $\mathbb{S}^{n-1}$ in the space $\mathbb{R}^n$:

$$\xi = s(x) = \{s_1(x), s_2(x), ..., s_n(x)\}, \quad \xi \in \mathbb{S}^{n-1}, \quad x \in \mathbb{R}^n$$

where

$$s_k(x) = \frac{2x_k}{1+|x|^2}, \;\; k=1,2,...,n-1 \quad \text{and} \quad s_n(x) = \frac{|x|^2-1}{|x|^2+1}.$$

This projection transforms spherical potential into the space potential and vice versa. Namely, the formula is valid:

$$\int_{\mathbb{S}^{n-1}} \frac{f(\sigma)\, d\sigma}{|\xi-\sigma|^{n-\alpha}}$$

$$= 2^\alpha (1+|x|^2)^{\frac{n-1-\alpha}{2}} \int_{\mathbb{R}^{n-1}} \frac{f[s(y)]\, dy}{|x-y|^{n-1-\alpha}(1+|y|^2)^{\frac{n-1+\alpha}{2}}}$$

No results known for domains in $\mathbb{R}^n$ till 2011.

Why only $\mathbb{S}^{n-1}$ and $\mathbb{R}^n$?

Answer: cancelation property

$$\int\limits_{\mathbb{R}^N} \left[\frac{1}{|z-x|^{N-\alpha}} - \frac{1}{|z-y|^{N-\alpha}}\right] dz \equiv 0.$$

and similarly for the sphere.

In cases where the potential of a constant function on X is well defined, the cancelation property means: the potential of a constant is constant.

The cancelation property is very restrictive in applications:it fails for domains Ω in $\mathbb{R}^n$.

Results on mapping properties of fractional integrals were proved in the general setting of quasimetric measure spaces assuming that the cancelation property holds (A.Gatto, 1996-2006)

Now we pass to recent results:A) we admit variable orders $\lambda(x)$ and $\alpha(x)$;B) for functions vanishing on the boundary we avoid the cancelation property

We deal with the general setting of quasimetric measure spaces

$$(X, d, \mu)$$

with quasidistance $d(x,y)$ and measure μ which satisfy the growth condition

$$\boxed{\mu B(x,r) \le K r^N \quad \text{as} \quad r \to 0, \quad K>0,}$$

where $N>0$ need not be an integer, and for the fractional operators

$$I^{\alpha(\cdot)} f(x) = \int\limits_X \frac{f(y)\, d\mu(y)}{[d(x,y)]^{N-\alpha(x)}}$$

we admit variable exponent $\alpha(x)$ with

$$0 \le \alpha(x) < 1,$$

and assume that Ω is an open bounded set in X.

We also consider the corresponding hypersingular operators

$$(D^\alpha f)(x) = \lim_{\varepsilon\to 0} \int\limits_{y\in\Omega:\varrho(x,y)>\varepsilon} \frac{f(y)-f(x)}{\varrho(x,y)^{N+\alpha(x)}} d\mu(y), \tag{11.1}$$

within the frameworks of the Hölder spaces $H^{\lambda(\cdot)}(\Omega)$.

In the case of constant α and λ a study in the general setting of quasimetric measure spaces (X,ϱ,μ) with growth condition and cancelation property, is known, see [Gatto, 1990, 1996, 2004, 2006].

Avoiding cancelation property via vanishing $\alpha(x)$ on the boundary

The estimate we present here reveal the mapping properties of the operators I^α and D^α in dependence on local values of $\alpha(x)$ and $\lambda(x)$. Note that estimations with variable $\lambda(x)$ and $\alpha(x)$ were known in the special case

$$X = \mathbb{S}^{n-1}$$

for spherical potential operators and related hypersingular integrals, and even in a more general setting of generalized Hölder spaces defined by a given (variable) dominant $w(x,h)$ of continuity modulus (Vakulov, 2005-06, N.Samko, Vakulov, 2009).

The estimates we present here are related to a general quasimetric measure spaces and admit the situation when $\alpha(x)$ may be degenerate on Ω. We denote

$$\Pi_\alpha = \{x\in\Omega : \alpha(x) = 0\}$$

and suppose that

$$\mu(\Pi_\alpha) = 0.$$

To obtain results stating that the range of the potential operator

over this or that Hölder space is imbedded into a better space of a similar nature, the usual mean is Zygmund type estimates for the continuity modulus. In the case we study, these estimates are local, depending on points x. By means of Zygmund type estimates of such a kind, it is possible to prove theorems on the mapping properties

$$\boxed{I^{\alpha(\cdot)} : \quad H_0^{\lambda(\cdot)}(\Omega) \to H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega),}$$

and similar results for the operator $D^{\alpha(\cdot)}$, $0<\alpha(x)<1$.

12 Preliminaries on quasimetric measure spaces

Recall some facts for the quasimetric measure spaces (QMMS) (X, d, μ).

Given a set X, a function

$$d : X \times X \to [0, \infty)$$

is called ***quasimetric***, if

$$d(x, y) \le K[d(x, z) + d(z, y)], \quad K \ge 1 \tag{12.1}$$

where $x, y, z \in X$. We assume that

$$d(x, y) = d(y, x).$$

Other assumptions:1) $C_0(\Omega)$ is dense in $L^1(X, \mu)$;2) X is closed with respect to the quasimetric d;3) the measure μ satisfies ***the growth condition*** = upper Ahlfors N-regularity), if

$$\mu B(x, r) \le cr^N, \qquad N > 0; \tag{12.2}$$

5) $\mu(\partial\Omega) = 0$.

By

$$\delta(x) = \delta(x, \partial\Omega) := \inf_{y \in \partial\Omega} d(x, y)$$

we denote the distance of x to the boundary.

We do not assume the measure μ to be doubling, but base ourselves on the growth condition (12.2).

An important fact, proved in R. A. Macías and C. Segovia.Lipshitz functions on spaces of homogeneous type. ***Adv. Math.***, 33:257–270, 1979.

Every quasidistance d ***on a quasimetric space*** (X, d) ***admits an equivalent quasimetric*** d_1 ***for which there exists an exponent*** $\theta \in (0, 1]$ ***such that***

$$|d_1(x, z) - d_1(y, z)| \tag{12.3}$$

$$\le M d_1^\theta(x, y) \{d_1(x, z) + d_1(y, z)\}^{1-\theta}$$

and

$$d_1(x, y) = d(x, y)^{\frac{1}{\theta}} \tag{12.4}$$

where $d(x, y)$ ***is a quasimetric.***

By the elementary inequality

$$|a^\beta - b^\beta| \le |\beta||a - b| \max(a^{\beta-1}, b^{\beta-1}), \qquad a, b \in \mathbb{R}^1_+, \tag{12.5}$$

the property (12.3) is an immediate consequence of (12.4) and it holds with

$$M = \frac{1}{\theta}.$$

Definition ***We say that the quasimetric*** d ***is*** *regular* ***of order*** $\theta \in (0, 1]$***, if it itself satisfies property*** (12.4)***.***

We suppose that our quasimetric d is regular of order $\theta \in (0, 1]$.

13 Mapping properties of fractional integrals; the 1st approach

For fixed $x \in \Omega$ we consider the local continuity modulus

$$\omega(f,x,h) = \sup_{\substack{z\in\Omega:\\ d(x,z)\leq h}} |f(x)-f(z)| \tag{13.1}$$

of a function f at the point x, $|h|<1$.

The following lemma explicitly show us teh role of the log-condition for variable exponents

Lemma 13.1. ***For all*** $x,y\in\Omega$ ***with*** $d(x,y)\leq h$***,***

$$\frac{1}{C}\omega(f,x,h) \leq \omega(f,y,h) \leq C\omega(f,x,h) \tag{13.2}$$

where $C=[2k]+2$***. If*** $a(x)\in\mathcal{P}^{\log}(\Omega)$***, then***

$$\frac{1}{C}h^{a(x)} \leq h^{a(y)} \leq Ch^{a(x)} \tag{13.3}$$

for all x,y ***such that*** $d(x,y)<h$***, where*** $C\geq 1$ ***depends on the function*** a***, but does not depend on*** x,y ***and*** h***.***

The p r o o f of (13.2) is direct. Let us check (13.3). It suffices to consider only small values

$$h \leq \min(1, \mathbf{diam}\,\Omega).$$

the inequality (13.3) is equivalent to

$$\frac{1}{C} \leq h^{a(y)-a(x)} \leq C$$

or

$$|a(y)-a(x)|| \cdot |\ln h|| \leq C_1, \quad C_1 = \ln C.$$

By the definition of the class $\mathcal{P}^{\log}(\Omega)$, the function a satisfies the condition

$$|a(y)-a(x)| \leq \frac{A}{\ln\frac{1}{d(x,y)}}, \quad d(x,y)\leq\frac{1}{2}$$

Then moreover

$$|a(y)-a(x)| \leq \frac{A}{\ln\frac{1}{h}}.$$

For a function $\lambda(x)$ defined on Ω we suppose that

$$\lambda_- := \inf_{x\in X}\lambda(x) > 0 \quad \text{and} \quad \lambda_+ := \sup_{x\in X}\lambda(x) < 1.$$

DEFINITION. ***By*** $H^{\lambda(\cdot)}(\Omega)$ ***we denote the space of functions*** $f\in C(\overline{\Omega})$ ***such that***

$$\omega(f,x,h) \leq Ch^{\lambda(x)},$$

where $C > 0$ ***does not depend on*** $x, y \in \Omega$***. Equipped with the norm***

$$\|f\|_{H^{\lambda(\cdot)}(\Omega)} = \|f\|_{C(\overline{\Omega})} + \sup_{x\in\Omega} \sup_{h\in(0,1)} \frac{\omega(f,x,h)}{h^{\lambda(x)}},$$

this is a Banach space.

In Hölder norm estimations of fractional integrals $I^\alpha f$, the case $f \equiv const$ play an important role, in the case where

$$\mathfrak{I}_\alpha(x) := I^\alpha(1)(x) = \int\limits_\Omega \frac{d\mu(z)}{d(x,z)^{N-\alpha(x)}} \tag{13.4}$$

is well defined.

In the Euclidean case $\Omega = X = \mathbb{R}^N$, this integral although not well directly defined, may be treated as a constant in the case

$$\alpha(x) = \alpha = const$$

in the sense that the cancelation property

$$\int\limits_{\mathbb{R}^N} \left[\frac{1}{|z-x|^{N-\alpha}} - \frac{1}{|z-y|^{N-\alpha}}\right] dz \equiv 0,$$

holds for all $x, y \in \mathbb{R}^N$, when $0 < \alpha < 1$.

But the cancelation property of the type

$$\int\limits_\Omega \left[\frac{1}{|z-x|^{N-\alpha(x)}} - \frac{1}{|z-y|^{N-\alpha(y)}}\right] d\mu(z) \equiv 0,$$

no more holds even for $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{S}^{N-1}$.

When we consider Hölder type spaces $H^{\lambda(\cdot)}(\Omega)$ which contain constants, the condition

$$\mathfrak{I}_\alpha(1) \in H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$$

is necessary for the mapping

$$I^\alpha : \quad H^{\lambda(\cdot)}(\Omega) \to H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$$

to hold.

Remark 13.2. Let $\inf\limits_{x\in\Omega} \alpha(x) \geq 0$ and $x, y \notin \Pi_\alpha$. Then

$$|\mathfrak{I}_\alpha(x) - \mathfrak{I}_\alpha(y)| \leq$$

$$C\frac{|\alpha(x) - \alpha(y)|}{\min(\alpha(x), \alpha(y))} +$$

$$\left| \int\limits_{\Omega} \left[d(x,z)^{\alpha(x)-N} - d(y,z)^{\alpha(x)-N} \right] d\mu(z) \right|$$

and

$$|\alpha(x)\mathfrak{I}_{\alpha}(x) - \alpha(y)\mathfrak{I}_{\alpha}(y)| \le C\,|\alpha(x) - \alpha(y)|$$

$$+ \min(\alpha(x), \alpha(y)) \left| \int\limits_{\Omega} \left[d(x,z)^{\alpha(x)-N} - d(y,z)^{\alpha(x)-N} \right] d\mu(z) \right|$$

where $C > 0$ does not depend on $x, y \in \Omega$.

Remark 13.3. The meaning of the above estimates is in the fact that the second term on the right-hand sides may be subject to the cancelation property: at the least it disappears when $\Omega = X = \mathbb{R}^N$ or $\Omega = X = \mathbb{S}^{N-1}$.

The estimate given in the following theorem clearly shows how the behaviour of the local continuity modulus

$$\omega(I^{\alpha} f, x, h)$$

worsens when x approaches the points where $\alpha(x)$ vanishes. We also give a weighted estimate with the weight $\alpha(x)$.

We use the notation

$$\alpha_h(x) = \min_{d(x,y)<h} \alpha(y).$$

Theorem 13.4. *Let Ω be a bounded open set in X, let*

$$\alpha \in \mathcal{P}^{\log}(\Omega)$$

and

$$0 \le \inf_{x\in\Omega} \alpha(x) \le \sup_{x\in\Omega} \alpha(x) < \min(1, N).$$

Then for all the points $x \in \Omega \backslash \Pi_{\alpha}$ such that

$$\alpha_h(x) \ne 0, 0 < h < \frac{d}{2},$$

the following Zygmund type estimate is valid

$$\omega(I^{\alpha} f, x, h) \le \frac{C}{\alpha_h(x)} h^{\alpha(x)} \omega(f, x, h) +$$

$$C h^{\theta} \int\limits_{h}^{d} \frac{\omega(f, x, t)dt}{t^{1+\theta-\alpha(x)}}$$

$$+ C\omega(\alpha, x, h) \int\limits_{h}^{d} \frac{\omega(f, x, t)dt}{t^{2-\alpha(x)}} + C\omega(\mathfrak{I}_{\alpha}, x, h)\|f\|_{C(\Omega)}.$$

Also, for all the points $x \in \Omega \backslash \Pi_\alpha$ ***the weighted estimate holds***

$$\omega(\alpha I^\alpha f, x, h) \leq Ch^{\alpha(x)}\omega(f, x, h) +$$

$$Ch^\theta \int\limits_h^d \frac{\omega(f, x, t)dt}{t^{1+\theta-\alpha(x)}}$$

$$+C\omega(\alpha, x, h) \int\limits_h^d \frac{\omega(f, x, t)dt}{t^{2-\alpha(x)}} +$$

$$C\omega(\alpha\mathfrak{I}_\alpha, x, h)\|f\|_{C(\Omega)},$$

Zygmund type estimates of hypersingular integrals

Theorem 13.5. ***Let***

$$\alpha \in \mathcal{P}^{\log}(\Omega)$$

and

$$0 \leq \inf_{x\in\Omega} \alpha(x) \leq \max_{x\in\Omega} \alpha(x) < \min\{\theta, N\}.$$

If $f \in C(\Omega)$***, then for all*** $x, y \in \Omega$ ***with*** $d(x, y) < h$ ***such that*** $\alpha(x) \neq 0$ ***and*** $\alpha(y) \neq 0$***, the following estimate is valid***

$$|(D^\alpha f)(x) - (D^\alpha f)(y)| \leq$$

$$\frac{C}{\min(\alpha(x), \alpha(y))} \int\limits_0^h \left[\frac{\omega(f, x, t)}{t^{1+\alpha(x)}} + \frac{\omega(f, y, t)}{t^{1+\alpha(y)}}\right] dt$$

$$+C \int\limits_h^2 \left[\omega(\alpha, x, h) + h^\theta t^{1-\theta}\right] \frac{\omega(f, x, t)dt}{t^{2+\alpha(x)}},$$

where $C > 0$ ***does not depend on*** x, y ***and*** h***.***

Theorems on mapping properties

Recall that for the fractional operator $I^{\alpha(\cdot)}$ **we allow the variable order** $\alpha(x)$ **to be degenerate on a set** Π_α **(of measure zero).**

Consider the weighted space

$$H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega, \alpha) = \{f : \alpha(x)f(x) \in H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega\}.$$

Theorem 13.6. ***Let***

$$\alpha(x) \geq 0, \quad \max_{x\in\Omega} \alpha(x) < \min(\theta, N),$$

$$\alpha \in \mathrm{Lip}(\Omega),$$

and

$$\sup_{x\in\Omega}[\lambda(x)+\alpha(x)]<\theta. \tag{13.5}$$

If

$$\alpha\mathfrak{I}_\alpha \in H^{\lambda(\cdot)+\alpha(\cdot)}, \tag{13.6}$$

then the operator $I^{\alpha(\cdot)}$ ***is bounded from the space*** $H^{\lambda(\cdot)}(\Omega)$ ***into the weighted space*** $H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega,\alpha)$***.***

A "non-degeneracy" version of Theorem 13.6 runs as follows.

Theorem 13.7. ***Let*** $\alpha\in \mathrm{Lip}(\Omega)$ ***and***

$$0<\min_{x\in\Omega}\alpha(x)\le\max_{x\in\Omega}\alpha(x)<\min(\theta,N). \tag{13.7}$$

Under the conditions

$$\sup_{x\in\Omega}[\lambda(x)+\alpha(x)]<\theta.$$

$$\alpha\mathfrak{I}_\alpha \in H^{\lambda(\cdot)+\alpha(\cdot)},$$

the operator $I^{\alpha(\cdot)}$ ***is bounded from the space*** $H^{\lambda(\cdot)}(\Omega)$ ***into the space*** $H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$***.***

The corresponding mapping theorem for the hypersingular operator runs as follows.

Theorem 13.8. ***Under the conditions***

$$\alpha\mathfrak{I}_\alpha \in H^{\lambda(\cdot)+\alpha(\cdot)},$$

and

$$0<\min_{x\in\Omega}\alpha(x)\le\max_{x\in\Omega}\alpha(x)<\min(\theta,N),$$

the operator $D^{\alpha(\cdot)}$ ***is bounded from the space*** $H^{\lambda(\cdot)}(\Omega)$ ***into the space*** $H^{\lambda(\cdot)-\alpha(\cdot)}(\Omega)$***, if***

$$0<\inf_{x\in\Omega}\{\lambda(x)-\alpha(x)\},\qquad \sup_{x\in\Omega}\lambda(x)<1.$$

REMARK.Condition $\alpha\mathfrak{I}_\alpha\in H^{\lambda(\cdot)+\alpha(\cdot)}$ **is rather restrictive. In general, the function** $\mathfrak{I}_\alpha(x)$ **is Lipschitz inside** Ω**, but**

$$\mathfrak{I}_\alpha(x)-\mathfrak{I}_\alpha\Big|_{\partial(\Omega)} \sim\ [\delta(x,\partial\Omega)]^{\alpha(x)}$$

as $x\to\partial\Omega$**. This means that the above condition may hold only when** $\alpha(x)$ **vanishes at the boundary. Thus it may not be valid when** α **is constant.**

14 Mapping properties on sets without cancelation property

Now we return to the case of constant α, but keep $\lambda(x)$ variable, and will show how it is possible to avoid cancelation property in the case of Hölder functions vanishing on the boundary.

In the Euclidean case for instance, statements of the type

$$I_\Omega^\alpha : H^\lambda(\Omega) \to H^{\lambda+\alpha}(\Omega), \qquad \Omega \subset \mathbb{R}^n,$$

for the potential operator

$$I_\Omega^\alpha f(x) := \int\limits_\Omega \frac{f(y)\,dy}{|x-y|^{n-\alpha}}$$

may not be valid for domains,since , as was already noted,

$$\mathfrak{I}_\alpha(x) - \mathfrak{I}_\alpha\Big|_{\partial(\Omega)} \sim \ \delta^\alpha(x)$$

where

$$\delta(x) = \delta(x, \partial\Omega).$$

However, one may expect that

$$I_\Omega^\alpha : H_0^\lambda(\Omega) \to H^{\lambda+\alpha}(\Omega) \tag{14.1}$$

for the subspace

$$H_0^\lambda(\Omega) = \{f \in H^\lambda(\Omega) : \ f\big|_{\partial\Omega} = 0\}.$$

Such a mapping is elementary in the one-dimensional case (G.Hardy and J.Littlewood),

A multi-dimensional result of such a kind was recently proved in the paper L. Diening and S. Samko. On potentials in generalized Hölder spaces over uniform domains in $\mathbb{R}^n$. ***Revista Matematica Complutense***, 24(2):357–373, 2011. where this result was obtained for uniform domains(Jones domains or banana domains).

We present a more general approach (S.Samko, to appear in ***Nonlinear Analysis***) in a general setting of quasimetric measure spaces

$$(X, d, \mu)$$

with the growth condition on the measure. This approach allows us to cover the case of an arbitrary open set Ω in $\mathbb{R}^n$. No restriction on the geometry of Ω !

We show that a mapping of type

$$I_\Omega^\alpha : H_0^\lambda(\Omega) \to H^{\lambda+\alpha}(\Omega)$$

and more generally, for spaces of the type

$$H^\omega(\Omega)$$

holds for measurable bounded sets Ω in (X, d, μ) satisfying the so called α-property.
Roughly speaking: we can state a result on mapping properties of the fractional operator, if we know how the potential operator of the constant, i.e.

$$J_{\Omega,\alpha}(x) = \int\limits_\Omega \frac{d\mu(y)}{d(x,y)^{N-\alpha}}, \qquad x \in \Omega, \tag{14.2}$$

behaves near the boundary of Ω.
NOTE: we deal with the problem in intrinsic terms of the given set $\Omega \subseteq X$. I.e. we do not use any continuation of functions on Ω to X with preservation of the continuity modulus, possible at the least in teh Euclidean case $\Omega = \mathbb{R}^n$.

We study mapping properties of potential operators

$$(I^\alpha f)(x) = \int\limits_\Omega \frac{f(y)\, d\mu(y)}{d(x,y)^{N-\alpha}}, \quad x \in \Omega \subseteq X, \tag{14.3}$$

for functions f defined on an open set Ω of a quasimetric measure space (X, d, μ), where N is the exponent from the growth condition.

The following estimates are known:

$$\int\limits_{B(x,r)} \frac{d\mu(y)}{d(x,y)^{N-\alpha}} \leq cr^\alpha, \tag{14.4}$$

$$\int\limits_{X\backslash B(x,r)} \frac{d\mu(y)}{d(x,y)^{N+\beta}} \leq cr^{-\beta}, \qquad \beta > 0. \tag{14.5}$$

and

$$\int\limits_{\Omega\backslash B(x,r)} \frac{d\mu(y)}{d(x,y)^{N}} \leq c \ln \frac{D}{r}, \qquad D > diam\, \Omega. \tag{14.6}$$

Their standard proof is via the dyadic splitting of the ball or its exterior.

15 Fractional integrals of constant functions on a set

We denote

$$J_{\Omega,\alpha}(x) = \int\limits_\Omega \frac{d\mu(y)}{d(x,y)^{N-\alpha}}, \qquad x \in \Omega, \tag{15.1}$$

which is well defined when Ω is bounded, in view of (14.4). When Ω is not necessarily bounded, we define the *difference of the potential* by

$$J_{\Omega,\alpha}(x,y) :$$

$$= \int\limits_{\Omega} \left(\frac{1}{d(x,z)^{N-\alpha}} - \frac{1}{d(y,z)^{N-\alpha}} \right) d\mu(z).$$

If Ω is bounded, then

$$J_{\Omega,\alpha}(x,y) = J_{\Omega,\alpha}(x) - J_{\Omega,\alpha}(y).$$

Examples of explicitly calculated functions $J_{\Omega,\alpha}(x)$:

1) $\mathbf{X = \mathbb{R}^n,\ \Omega = B(0,R),\quad 0 < \alpha < n}$: $J_{\Omega,\alpha}(x) = c_0 + c_1(R - |x|)^\alpha + g(x), \quad x \in B(0,R)$,where $g \in$ Lip$(\overline{B(0,R)})$ and $g\big|_{|x|=R} = 0$;

2) $\mathbf{X = \mathbb{R}^n,\ \Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n :\ x_n > 0\},\ 0 < \alpha < 1}$:$J_{\Omega,\alpha}(x,y) = c_n(\alpha)(sgn(x_n)|x_n|^\alpha - sgn(y_n)|y_n|^\alpha)$;

3) $\mathbf{X = \mathbb{R}^2,\ \Omega = \mathbb{R}^2_{++} = \{x \in \mathbb{R}^2_+ : x_1 > 0, x_2 > 0\},\ 0 < \alpha < 1}$:
$J_\Omega(x,y) = \frac{c}{\alpha}\left([\delta(x)]^\alpha - [\delta(y)]^\alpha + x_1^\alpha - y_1^\alpha + x_2^\alpha - y_2^\alpha\right) + U(x) - U(y)$, where $c = \frac{\sqrt{\pi}}{2\alpha}\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma^{-1}\left(\frac{2-\alpha}{2}\right)$ and$U(x) = |x|tA(t),\ t = \min\left\{\frac{x_1}{x_2}, \frac{x_2}{x_1}\right\}$, $A(t)$ is analytic in t.

4) $X = \mathbb{S}^{n-1} = \{\sigma = (\sigma_1, ..., \sigma_n) :\ |\sigma| = 1\}$ with the Euclidean distance, and $\Omega = \mathbb{S}^{n-1}_+ := \{\sigma \in \mathbb{S}^{n-1} : \sigma_n > 0\}$; $\delta(\sigma, \partial\Omega) \sim \sigma_n$;in this case $J_{\mathbb{S}^{n-1}_+,\alpha}(\sigma) = c_0 + 2c_1|\sigma_n|^\alpha + k(\sigma)$where c_0 and c_1 are the same as in example 1), and $k \in$ Lip$(\overline{\mathbb{S}^{n-1}_+})$, and $k\big|_{\partial\Omega} = 0$.

The functions $J_{\Omega,\alpha}(x)$ and $J_{\Omega,\alpha}(x,y)$ are continuous. But they are better than just continuous in the inner points of Ω, see Lemma 16.1 below.

16 On the α-property of sets

Is known in the Euclidean case:the fractional integral of order α of a bounded function on a bounded domain is α-Hölder continuous in Ω. This is a particular case of the Sobolev theorem

$$I^\alpha_\Omega : L^p(\Omega) \to H^{\alpha - \frac{n}{p}}(\Omega), \qquad 1 < p \le \infty$$

when

$$\frac{n}{p} < \alpha < \frac{n}{p} + 1.$$

In the following lemma we extend this for sets Ω in

$$(X, d, \mu)$$

in the case $p = \infty$, where Ω may be unbounded and we include all $x, y \in X$ into the Hölder condition, not only $x, y \in \Omega$.

Lemma 16.1. *Let $\Omega \subset X$ be measurable and $\alpha \in (0, \theta)$. Then*

$$|\mathcal{J}_{\Omega,\alpha}(x, y)| \le c\, d(x, y)^{\alpha}, \tag{16.1}$$

where c depends on x and y. If Ω is bounded, the case

$$\alpha = \theta$$

may be also admitted with the estimate

$$|\mathcal{J}_{\Omega,\theta}(x, y)| \le c\, d(x, y)^{\theta} \ln \frac{D}{d(x, y)}, \qquad x, y \in \Omega, \tag{16.2}$$

where $D > diam\,\Omega$.

Proof. **Splitting:**

$$J_{\Omega,\alpha}(x, y)$$

$$= \int\limits_{\Omega\backslash B(x,r)} \left[d(x, z)^{\alpha-N} - d(y, z)^{\alpha-N}\right] d\mu(z)$$

$$+ \int_{\Omega\cap B(x,r)} d(x, z)^{\alpha-N}\, d\mu(z)$$

$$- \int\limits_{\Omega\cap B(x,r)} d(y, z)^{\alpha-N}\, d\mu(z) =: J_1 + J_2 - J_3.$$

□

In the above examples: $J_{\Omega,\alpha}(x, y)$ is even Lipschitz off the boundary $\partial\Omega$.
In the general setting of quasimetric spaces:it is natural to suppose that in many cases the function $J_{\Omega,\alpha}(x, y)$ is Hölderian of order θ off the boundary,the case $\alpha = \theta$ being an analogue of the Lipschitz case.
The next definition is aimed to provide an appropriate language to single out the class of sets $\Omega \subseteq X$, with a prescribed way of how the Lipschitz θ-behaviour worsens to Hölder α-behaviour, $\alpha < \theta$, when x and y approach the boundary.
Definition. *Let $\Omega \subset X$ be a measurable set and $\alpha \in (0, \theta]$. We say that Ω has the α-property, if there exists $c > 0$ such that for $x, y \in \Omega$*

$$|J_{\Omega,\alpha}(x, y)| \le c \frac{d(x, y)^{\theta}}{\max\{\delta(x), \delta(y)\}^{\theta-\alpha}}, \tag{16.3}$$

if

$$d(x, y) \le \frac{1}{2^{\frac{1}{\theta}}} \max\{\delta(x), \delta(y)\}$$

Lemma 16.2. *Let Ω be bounded, the quasidistance d be regular of order $\theta \in (0,1]$ and $\alpha \in (0,\theta]$. Then*

$$d(x,y) \leq \frac{1}{2^{\frac{1}{\theta}}} \max\{\delta(x), \delta(y)\} \quad \Longrightarrow$$

$$|\delta(x)^\alpha - \delta(y)^\alpha| \leq \alpha 2^{\frac{2-\alpha-\theta}{\theta}} \frac{d(x,y)^\theta}{\max\{\delta(x), \delta(y)\}^{\theta-\alpha}}. \tag{16.4}$$

The following corollary provides a sufficient condition for a domain Ω to possess the α-property.

Corollary. *If $J_{\Omega,\alpha}(x)$ has the structure*

$$J_{\Omega,\alpha}(x) = c\delta(x)^\alpha + g(x), \qquad x \in \Omega, \tag{16.5}$$

where c is a constant and $g \in Lip^\theta(\overline{\Omega})$, then Ω possesses the α-property.

17 Mapping properties of the fractional operator I^α in generalized Hölder type spaces

We now define the generalized Hölder spaces on a set Ω, with the continuity modulus

$$\omega(f,h) = \sup_{\substack{x,y\in\Omega: \\ d(x,y)<h}} |f(x) - f(y)|$$

dominated by a given function $\omega(h)$.

Definition. *Given a continuous semi-additive function $\omega(h)$, positive for $h > 0$, with $\omega(0) = 0$, by $H^\omega(\Omega)$ we denote the space of functions $f \in C(\overline{\Omega})$ with the finite norm*

$$\|f\|_{H^\omega} = \|f\|_{C(\bar{\Omega})} + \sup_{0<h<\mathbf{diam}\,\Omega} \frac{\omega(f,h)}{\omega(h)}.$$

By $H_0^\omega(\Omega)$ we denote the subspace in $H^\omega(\Omega)$ of functions f which vanish on the boundary $\partial\Omega$ of Ω.

Lemma 17.1. *Let $0 < \alpha < \theta$ and $\Omega \subseteq X$ have the α-property. Let $f \in H_0^\omega(\Omega)$, where*

$$\omega(h) \quad \textit{is almost increasing} \tag{17.1}$$

and

$$\frac{\omega(h)}{h^{\theta-\alpha}} \quad \textit{is almost decreasing.}$$

Then

$$\sup_{x,y\in\Omega: d(x,y)<h} \big| f(x)[J_{\Omega,\alpha}(x,y)] \big| \tag{17.2}$$

$$\leq C\omega_\alpha(h)\|f\|_{H^\omega(\Omega)},$$

where

$$\omega_\alpha(h) = h^\alpha \omega(h).$$

In particular,

$$\sup_{x,y\in\Omega: d(x,y)<h} \left| f(x)[J_{\Omega,\alpha}(x,y)] \right|$$

$$\leq Ch^{\alpha+\lambda} \|f\|_{H^\lambda(\Omega)},$$

when $f \in H_0^\lambda(\Omega)$ ***and*** $\lambda + \alpha \leq \theta$.

Definition. ω ***belongs to a Zygmund class*** Φ_β, $\beta > 0$, ***if it is continuous, non-negative, almost increasing and***

$$\int_h^d \left(\frac{h}{t}\right)^\beta \frac{w(t)}{t} dt \leq cw(h),$$

The most general result we have for variable $\alpha(x)$ up to now is as follows

Theorem 17.2. ***Let*** $\alpha \in \mathrm{Lip}(\Omega)$ ***and***

$$\alpha(x) \geq 0, \quad \max_{x\in\Omega} \alpha(x) < \min(\theta, N),$$

If $\omega \in \Phi_{\theta-\alpha}$ ***and***

$$\alpha(\cdot)\mathfrak{I}_\alpha \in H^{\lambda(\cdot)+\alpha(\cdot)}.$$

Then the operator

$$\alpha(x) I^\alpha$$

is bounded from the space $H^{\omega(\cdot)}(\Omega)$ ***into the space*** $H^{\omega_\alpha(\cdot)}(\Omega)$.

(This result was given in the preceding part for the case $\omega = h^{\lambda(x)}$.)

Now, making use of the above arguments, we mayavoid the condition

$$\alpha \mathfrak{I}_\alpha \in H^{\lambda(\cdot)+\alpha(\cdot)}$$

on the set Ω, replacing it by the assumption that Ω has the α-property (which often holds, for instance for any domain in $\mathbb{R}^n$.)

Theorem 17.3. ***Let***

$$0 < \alpha < \theta$$

and Ω ***have the*** α***-property. If*** $\omega \in \Phi_{\theta-\alpha}$, ***then***

$$I^\alpha : \quad H_0^\omega(\Omega) \ \to \quad H^{\omega_\alpha}(\Omega).$$

In particular, I^α ***is bounded from*** $H_0^\lambda(\Omega)$ ***to*** $H^{\lambda+\alpha}(\Omega)$ ***if*** $\lambda + \alpha < \theta$.

Proof. **We may adjust the proof from the above cited paper for our goals. The key moment: for $x, y \in \Omega$ with $d(x,y) < h$ we have**

$$I^\alpha f(x) - I^\alpha f(y) = \int_{d(x,z)<2h} [f(z) - f(x)] d(x,z)^{\alpha-N} d\mu(z)$$

$$-\int\limits_{d(x,z)<2h}[f(z)-f(x)]d(y,z)^{\alpha-N}d\mu(z)$$

$$+\int\limits_{d(x,z)>2h}[f(z)-f(x)]\left\{d(x,z)^{\alpha-N}-d(y,z)^{\alpha-N}\right\}d\mu(z)$$

$$+f(x)\int\limits_{\Omega}\left\{d(x,z)^{\alpha-N}-d(y,z)^{\alpha-N}\right\}d\mu(z)$$

$$=:\Delta_1+\Delta_2+\Delta_3+\Delta_4$$

We need to take care of

$$\Delta_4=f(x)[J_{\Omega,\alpha}(x)-J_{\Omega,\alpha}(y)].$$

The term Δ_4 is now estimated by means of Lemma 17.1. Note that assumptions of that lemma follow from the assumption $\omega\in\Phi_{\theta-\alpha}$. This completes the proof. □

18 The case of spatial and spherical fractional integrals in $\mathbb{R}^n$

18.1 Any domain in $\mathbb{R}^n$ possesses the α-property

We improve a result from
L. Diening and S. Samko. On potentials in generalized Hölder spaces over uniform domains in $\mathbb{R}^n$. *Revista Matematica Complutense*, 24(2):357–373, 2011. where it was shown that the α-property holds for uniform domains.

Thus we show that the validity of the α-propertydoes not depend on the structure of the boundary, at the least in the case of the Lebesgue measure.

Lemma 18.1. *Every domain in $\mathbb{R}^n$ has the α-property,* $0<\alpha<1$.

Theorem 17.3 and Lemma 18.1 yield the following statement (Hardy-Littlewood type multidimensional result).

Theorem 18.2. *Let Ω be an arbitrary bounded domain in $\mathbb{R}^n$, let $f\in H^\omega(\Omega)$ and*

$$f\big|_{x\in\partial\Omega}\equiv f_0=const.$$

If $\omega(h)$ satisfies the assumptions of Theorem 17.3, then the fractional integral $I^\alpha_\Omega f,\ 0<\alpha<1,$ has the following structure

$$I^\alpha_\Omega f(x)=f_0a(x)+Af(x),\qquad x\in\Omega,$$

where A ***is an operator bounded from*** $H^{\omega}(\Omega)$ ***to*** $H^{\omega_\alpha}(\Omega)$, ***while the function***

$$a(x) = J_{\Omega,\alpha}(x)$$

is Lipschitz beyond the boundary $\partial\Omega$ ***and its Hölder properties near the boundary are described by the condition***

$$|a(x) - a(y)| \le c\frac{|x-y|}{\max\{\delta(x), \delta(y)\}^{1-\alpha}}$$

18.2 The case of spherical fractional integrals over caps

Now let Ω be an arbitrary surface domain on the unit sphere

$$X = \mathbb{S}^{n-1} = \{\sigma = (\sigma_1, ..., \sigma_n) : \ |\sigma| = 1\}$$

in $\mathbb{R}^n$, we will call it

spherical cap.

The subsequente application of Theorem 17.3 is inspired by some applications of spherical harmonic analysis to a problem of aerodynamics given in N. Plakhov and S. Samko. An inverse problem of Newtonian aerodynamics. *Math. Engin. Sci. and Aerospace*, 1(4):351–369, 2010.

The corresponding fractional integral in that paper was

$$\mathbb{I}^{\alpha}_{\Omega} f(\xi) = \int\limits_{\mathbb{S}^{n-1}_{+}} \frac{f(\sigma)\, d\sigma}{|\xi - \sigma|^{n-1-\alpha}}, \qquad \xi \in \Omega, \tag{18.1}$$

over a semisphere.

Lemma 18.3. ***Every spherical cap has the*** α***-property, with respect to the potential*** (18.1)***,*** $0 < \alpha < 1$.

Proof. Reduce to the case of domains in $\mathbb{R}^n$ via the stereographic projection □

In view of Lemma 18.3, similarly to the previous subsection, from Theorem 17.3 we obtain that the same mapping properties remain validfor the spherical fractional integral over any spherical cap on $\mathbb{S}^{n-1}$.