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The crystallographic Radon transform and diffusive wavelets

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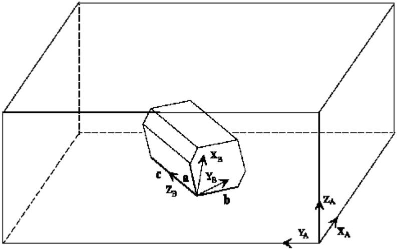
Motivation

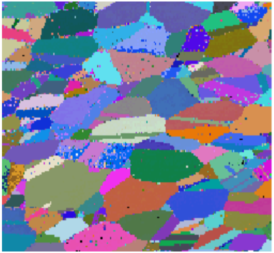
Texture analysis is the analysis of the **statistical distribution of orientations** of crystals within a specimen of a polycrystalline material, which could be metals or rocks. The **crystallographic orientation** g of an individual crystal is the active rotation $g \in SO(3)$ that maps a co-ordinate system fixed to the specimen onto another co-ordinate system fixed to the crystal.

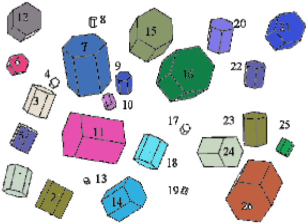


Orientation density function

The orientation distribution by volume ΔV_g requires a measure of the volume portion $\frac{\Delta V_g}{V}$ of total volume V carrying crystal gains with orientations within a volume element $\Delta G \subset G$ of the subgroup G of all feasible $G \in SO(3)$.

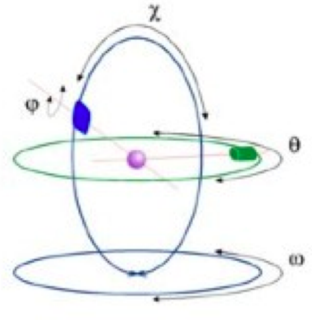
$$\frac{\Delta V_g}{V} \to f(g)\, dg$$





Goniometer

classical goniometer



4-circle-goniometer

Pole density function

- odf cannot be directly measured,
- only pole density functions (pdf) $P(h,r)$ can be sampled,
- let be

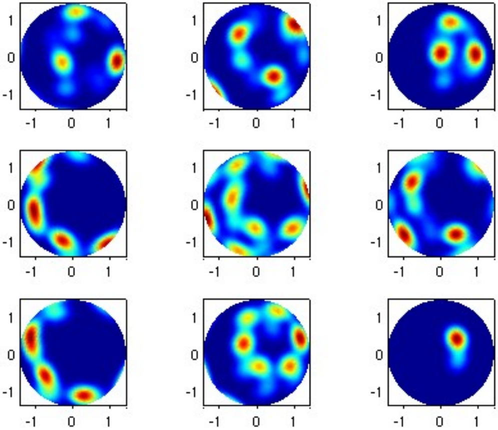
$$(\mathscr{R}f)(h,r) = 4\pi \int_{\{g\in SO(3):h=gr\}} f(g)\,dg$$
$$= 4\pi \int_{SO(3)} f(g)\delta_r(g^{-1h})dg = (f * \delta_r),$$

it represents that a fixed crystal direction h statistically coincides with the specimen direction r

- Due to Friedel's law which says that the X-ray cannot distinguish between the top and the bottom of the lattice planes, we are only able to measure a mean value which correspondence to a negligence of the orientation on $SO(3)$, i.e. the pdf

$$P(h,r) = \frac{1}{2}\left((\mathscr{R}f)(h,r) + (\mathscr{R}f)(-h,r)\right).$$

PDF



Crystallographic Radon transform

Problem (Analytic reconstruction problem)

Reconstruct the ODF $f(g)$, $g \in SO(3)$, *from all pole figures* $P(h,r)$, $h, r \in S^2$. *Because* $f(g)$ *is an ODF we have two additional conditions:*

1. $f(g) \geq 0$, *i.e.* f *is non-negative,*
2. $\int_{SO(3)} f(g)dg = 1$.

Problem (Totally geodesic Radon transform on $SO(3)$)

Reconstruct $f(g)$, $g \in SO(3)$, *from all* $\mathscr{R}(h,r)$, $h, r \in S^2$.

Crystallographic Radon transform

Radon-Transformation

$$(\mathscr{R}f)(h, r) := \int_{\{g \in SO(3) : gh = r\}} f(g)\, dg$$
$$= \int_{SO(3)} f(g)\, \delta_h(g^{-1} r)\, dg = f * \delta_{h,\, r}$$
$$= \int_{SO(2)} f(rlh^{-1}) dl,$$

because a great circle $C_{h,r}$ can be described that way.

Let $\hat{\mathscr{G}}$ denote the set of all equivalence classes of irreducible representations. Then this set parametrerizes an orthogonal decomposition of $L^2(\mathscr{G})$.

Theorem (Peter-Weyl)

Let $\mathscr{G}$ be a compact Lie group. Then the following statements are true.

- *Denote $H_\pi = \{g \mapsto \mathrm{trace}(\pi(g)M) : M \in \mathbb{C}^{d_\pi \times d_\pi}\}$. Then the Hilbert space $L^2(\mathscr{G})$ decomposes into the orthogonal direct sum*

$$L^2(\mathscr{G}) = \bigoplus_{\pi \in \hat{\mathscr{G}}} H_\pi$$

- *For each irreducible representation $\pi \in \hat{\mathscr{G}}$ the orthogonal projection $L^2(\mathscr{G}) \to H_\pi$ is given by*

$$f \mapsto d_\pi \int_{\mathscr{G}} f(h)\chi_\pi(h^{-1}g)\, dh = d_\pi\, f * \chi_\pi,$$

in terms of the character $\chi_\pi(g) = \mathrm{trace}(\pi(g))$ of the representation and dh is the normalized Haar measure.

- the matrix M in the equation $f * \chi_\pi = \mathrm{trace}(\pi(g)M)$ are the Fourier coefficient $\hat{f}(\pi)$ of f at the irreducible representation π.
- $\hat{f}(\pi) = \int_{\mathscr{G}} f(g)\pi^*(g)\,dg$
- inversion formula (the Fourier expansion) $f(g) = \sum_{\pi\in\hat{\mathscr{G}}} d_\pi \,\mathrm{trace}(\pi(g)\hat{f}(\pi))$

If we denote by $||M||_{HS}^2 = \mathrm{trace}(M^*M)$ the Frobenius or Hilbert-Schmidt norm of a matrix M, then the following Parseval identity is true.

Lemma (Parseval identity)

Let $f \in L^2(\mathscr{G})$. Then the matrix-valued Fourier coefficients $\hat{f} \in \mathbb{C}^{d_\pi \times d_\pi}$ satisfy

$$||f||^2 = \sum_{\pi\in\hat{\mathscr{G}}} d_\pi \,||\hat{f}(\pi)||_{HS}^2.$$

Definition

Let $\mathscr{H}$ be a subgroup of the compact Lie group $\mathscr{G}$. The *Radon transform $\mathscr{R}$ of an integrable function f on $\mathscr{G}$ is defined by*

$$\mathscr{R}f(x,y) = \int_{\mathscr{H}} f(xhy^{-1})\, dh, \quad x, y \in \mathscr{G}, \tag{1}$$

where dh denotes the normalized Haar measure on $\mathscr{H}$.

Lemma (B./Ebert/Pesenson - 2012)

The Radon transform (1) is invariant under right shifts of x and y, hence the range is a subset of $\mathscr{G}/\mathscr{H} \times \mathscr{G}/\mathscr{H}$.

Theorem (B./Ebert/Pesenson - 2012)

Let $\mathscr{H}$ be a subgroup of $\mathscr{G}$ which determines the Radon transform on $\mathscr{G}$ and let $\hat{\mathscr{G}}_1 \subset \hat{G}$ be the set of irreducible representations with respect to $\mathscr{H}$. Then for $f \in C^\infty(\mathscr{G})$ we have

$$||\mathscr{R}||^2_{L^2(\mathscr{G}/\mathscr{H}\times\mathscr{G}/\mathscr{H})} = \sum_{\pi\in\hat{\mathscr{G}}_1} \mathrm{rank}\,(\pi_{\mathscr{H}})||\hat{f}||^2_{HS}.$$

Hilbert space structure

We want to find a Hilbert space structure such that the Radon transform is an isometry, which means that

$$||f||^2_{L^2(\mathscr{G})} = \sum_{\pi\in\hat{\mathscr{G}}} d_\pi ||\hat{f}(\pi)||^2_{HS}$$

$$\sum_{\pi\in\hat{\mathscr{G}}} d_\pi ||\hat{f}(\pi)||^2_{L^2(\mathscr{G}/\mathscr{H}\times\mathscr{G}/\mathscr{H})} = |||\mathscr{R}f|||^2_{L^2(\mathscr{G}/\mathscr{H}\times\mathscr{G}/\mathscr{H})}$$

Lemma (Taylor, 1986)

If $\mathcal{M}$ is a compact rank one symmetric space, then $\mathscr{G}$ acts irreducibly on each eigenspace V_λ of Δ on $\mathcal{M}$.

Examples of such compact rank one symmetric spaces are

$$\mathscr{G} = SO(n+1), \quad \mathcal{M} = S^n,$$

$$\mathscr{G} = SU(n+1), \quad \mathcal{M} = \mathbb{C}P^n \text{(complex projective plane)}.$$

Definition

A representation $l(g)$ is called a representation of class-1 relative to $\mathscr{H}$ if in its space there are nonzero vectors invariant relative to $\mathscr{H}$ and the restriction of $l(g)$ to $\mathscr{H}$ is unitary.

Lemma

If $\mathcal{M} = \mathscr{G}/\mathscr{H}$ is a rank one symmetric space, with $\mathscr{H}$ connected, than $L^2(\mathcal{M})$ contains each class-1 representation, exactly once, as an eigenspace of Δ.

Spherical harmonics and Wigner polynomials

- orthonormal system of spherical harmonics $\mathcal{Y}_k^i \in C^\infty(S^n),\ k \in \mathbb{N}_0,\ i = 1, \ldots, d_k(n)$ normalized with respect to the Lebesgue measure on S^n.
- Then the Wigner polynomials on $SO(n+1)$ $\mathcal{T}_k^{ij}(g),\ g \in SO(n+1)$ are given by

$$\mathcal{T}_k^{ij}(g) = \int_{S^n} \mathcal{Y}_k^i(g^{-1}x)\overline{\mathcal{Y}_k^j(x)}\,dx$$

- $$\mathcal{Y}_k^i(g^{-1}x) = \sum_{j=1}^{d_k(n)} \mathcal{T}_k^{ij}(g)\mathcal{Y}_k^j(x).$$

- Wigner polynomials build an orthonormal system in $L^2(SO(n+1))$.
- Unfortunately, Wigner polynomials do not give all irreducible unitary representations of $SO(n+1)$ if $n \geq 2$.

Projections

We look for all representations of $SO(n+1)$ which do not have vanishing coefficients under the projection with respect to $SO(n)$, these are the class-1 representations of $SO(n+1)$:

$$\int_{SO(n)} \mathcal{T}_k^{ij}(g)\,dg = \int_{S^n}\int_{SO(n)} \mathcal{Y}_k^i(g^{-1}x)\,dg\,\mathcal{Y}_k^j(x)\,dx$$

$$= \frac{\mathcal{Y}_k^i(x_0)}{\mathcal{C}_k^{(n-1)/2}} \int_{S^n} \mathcal{C}_k^{(n-1)/2}(x_0^T x)\mathcal{Y}_k^i(x)\,dx \quad \text{(zonal averaging)}$$

$$= \frac{\mathcal{Y}_k^i(x_0)\mathcal{Y}_k^j(x_0)}{(\mathcal{C}_k^{(n-1)/2}(1))^2}|S^n| \int_{-1}^{1} (\mathcal{C}_k^{(n-1)/2}(t))^2(1-t^2)^{n/2-1}\,dt$$

(Funk-Hecke formula)

$$= \frac{|S^n|}{d_k(n)}\mathcal{Y}_k^i(x_0)\mathcal{Y}_k^j(x_0).$$

x_0 is the base point of $SO(n+1)/SO(n) \sim S^n$, $\mathcal{C}_k^{(n-1)/2}$ are the Gegenbauer polynomials.

Radon transform from $span(\mathcal{T}_k)$ into $S^n \times S^n$

Let be

$$f(g) = \sum_{k=0}^{\infty} \sum_{i,j=1}^{d_k(n)} \hat{f}(k)_{ij} \mathcal{T}_k^{ij}.$$

then

$$\begin{aligned}
(\mathscr{R} f)(h,r) &= \sum_{k=0}^{\infty} d_k(n) \mathrm{trace}\left(\hat{f}(k)\mathcal{T}_k(h)\pi_{SO(n)}\mathcal{T}_k^*(r)\right) \\
&= \sum_{k=0}^{\infty} d_k(n) \sum_{i,j=1}^{d_k(n)} \hat{f}(k)_{ij} \mathcal{T}_k^{i1}(h)\overline{\mathcal{T}_k^{1j}(r)} \\
&= \sum_{k=0}^{\infty} \frac{|S^n|}{d_k(n)} d_k(n) \sum_{i,j=1}^{d_k(n)} \hat{f}(k)_{ij} \mathcal{Y}_k^i(h)\overline{\mathcal{Y}_k^j(r)} \\
&= |S^n| \sum_{k=0}^{\infty} \sum_{i,j=1}^{d_k(n)} \hat{f}(k)_{ij} \mathcal{Y}_k^i(h)\overline{\mathcal{Y}_k^j(r)}
\end{aligned}$$

Mapping properties

- $\Delta_h(\mathscr{R}f) = \Delta_r(\mathscr{R}f)$
- For $g = Rf$ the Fourier coefficients fulfill $\hat{g}(k)_{ij} = |S^n|\hat{f}(k)_{ij}$

The crystallographic Radon transform maps Wigner poynomials, i.e. $\operatorname{span}(\mathcal{T}_k)$, onto $\operatorname{span}(\mathcal{Y}_k^i\overline{\mathcal{Y}_k^j})$.

The case $SO(3)$

Lemma (Taylor, 1986)

The decomposition

$$L^2(S^2) = \bigoplus_k V_k$$

contains each irreducible unitary representation of $SO(3)$, exactly once.

Choosing $\mathscr{G} = SO(3)$, $\mathscr{H} = SO(2)$ and thus $\mathscr{G}/\mathscr{H} = SO(3)/SO(2) = S^2$, all irreducible representations are equivalent to an irreducible component of the left regular representation

$$T(g) : \, f(x) \mapsto f(g^{-1} \cdot x),$$

where $\cdot$ denotes the canonical action of $SO(3)$ on S^2. The T invariant subspaces of $L^2(S^2)$ are $\mathcal{H}_k = \{\mathcal{Y}_k^i, \, i = 1, \ldots, 2k+1\}$, which are spanned by all spherical harmonics of degree k. The dimension of the representations space is $d_k = 2k+1$ and $-\lambda_k^2 = -k(k+1)$ we get $d_k = \sqrt{1 + 4\lambda_k^2}$ and $\sqrt{d_k} = \sqrt[4]{(2(\lambda_k^2 + \lambda_k^2) + 1)}$.

Hilbert space structure

Parseval's identity

$$||f||^2_{L^2(SO(3))} = \sum_{k=1}^{\infty} (2k+1) ||\hat{f}(k)||^2_{HS}.$$

Because of Δ on S^2 is equal to $-k(k+1)$ on the representation space $=$ eigenspace $\mathcal{H}_k$ of the Laplacian we obtain

$$= \sum_{k=1}^{\infty} (2k{+}1) ||4\pi \hat{f}(k)||^2_{L^2(S^2 \times S^2)} = ||4\pi(-2(\Delta_1{+}\Delta_2){+}1)^{1/4} \mathscr{R} f||^2_{L^2(S^2 \times S^2)},$$

where $\Delta_1 + \Delta_2$ is a Laplace operator on $S^2 \times S^2$. Thus we define the following norm for $u \in C^\infty(S^2 \times S^2)$

$$|||u|||^2 = (4\pi)^2 ((-2\Delta_{S^2 \times S^2} + 1)^{1/2} u,\, u)_{L^2(S^2 \times S^2)},$$

where $\Delta_{S^2 \times S^2} = \Delta_1 + \Delta_2$.

Sobolev spaces

Definition

The Sobolev space $H_t(S^2 \times S^2)$, $t \in \mathbb{R}$, *is defined as the domain of the operator* $(1 - 2\Delta_{S^2 \times S^2})^{\frac{t}{2}}$ *with graph norm*

$$||f||_t = ||(1 - 2\Delta_{S^2 \times S^2})^{\frac{t}{2}} f||_{L^2(S^2 \times S^2)}, \;\; f \in L^2(S^2 \times S^2).$$

The Sobolev space $H_t^{\Delta}(S^2 \times S^2)$, $t \in \mathbb{R}$, *is defined as the subspace of all functions* $f \in H_t(S^2 \times S^2)$ *such* $\Delta_1 f = \Delta_2 f$.

Definition

The Sobolev space $H_t(SO(3))$, $t \in \mathbb{R}$, *is defined as the domain of the operator* $(1 - 4\Delta_{SO(3)})^{\frac{t}{2}}$ *with graph norm*

$$|||f|||_t = ||(1 - 4\Delta_{SO(3)})^{\frac{t}{2}} f||_{L^2(SO(3))}, \;\; f \in L^2(SO(3)).$$

Theorem (Range description, B./Ebert/Pesenson, 2012)

For any $t \geq 0$ the Radon transform on $SO(3)$ is an invertible mapping

$$\mathscr{R} : H_t(SO(3)) \to H^{\Delta}_{t+\frac{1}{2}}(S^2 \times S^2). \tag{2}$$

Proof: It is sufficient to consider case $t = 0$.

Grouptheoretical approach

Wavelets are coherent states.
Consider the affine group of translations and dilations acting on the real line. Let $\mathcal{M}$ be a Riemannian manifold. A wavelet transform in $L^2(\mathcal{M})$ is defined in terms of an unitary representation U of Lie group $\mathcal{G}$

$$U : \mathcal{G} \to \mathcal{L}(L^2(\mathcal{M})).$$

A non-zero vector $\Psi \in L^2(\mathcal{M})$ is an admissible wavelet if

$$\int_{\mathcal{G}} |\langle f, U(g)\Psi\rangle_{L^2(\mathcal{M})}|^2 \, dg < \infty$$

for all $f \in L^2(\mathcal{M})$. The associated wavelet transform is

$$\mathcal{W}f(g) = \langle f, U(g)\Psi\rangle_{L^2(\mathcal{M})}$$

bounded and invertible on its range.

Grouptheretical approach – drawbacks

- Because $L^2(\mathcal{M})$ is infinite dimensional, no compact group admits an irreducible unitary representation of this form.
- However, compact groups seem natural at least in the situation where $\mathcal{M}$ itself is a homogeneous space of a compact group. For example $\mathbb{S}^2 = SO(3)/SO(2)$.
- Spheres: Irreducible representation of that form are not square integrable and hence one cannot find an admissible wavelet.

Alternative approaches

Classical wavelet theory (in $\mathbb{R}^n$) is based on the group generated by translations and dilations.

Translations on a sphere (seen as a homogeneous space of rotations) are rotations.

What are dilations?

Key idea: generate dilations from a diffusive semigroup, e.g., from time-evolution of solutions to a heat equation on a homogeneous space.

W. Freeden, T. Gervens, and M. Schreiner, *Constructive Approximation on the Sphere with Applications to Geomathematics*, Oxford Univ. Press, Oxford, 1999.

Discrete wavelet transforms in such a setting:

R. Coifman, M. Maggioni, *Diffusion wavelets*, Appl. Comp. Harm. Anal. 21(1):53-94, 2006.

Diffusive wavelets – General philosophy

Let $p_t \in L^1(G)$ be an *approximate convolution identity*, i.e. $\varphi * p_t \to \varphi$ as $t \to 0$ for all $\varphi \in L^2(G)$.
Assign families $\psi_\rho, \Psi_\rho \in L^1(G)$ to p_t such that

$$p_t = \int_t^\infty \check{\psi}_\rho * \Psi_\rho \, \alpha(\rho) \, d\rho.$$

We assign to φ a two-parameter function $W\varphi$, the Wavelet transform $\varphi(g) \mapsto W\varphi(\rho, g)$,

$$W\varphi(\rho, g) = \varphi * \check{\psi}_\rho = \int_G \varphi(h) \check{\psi}_\rho(h^{-1}g) \, d\mu_G(h) = \langle \varphi, T_g \psi_\rho \rangle,$$

and the inversion formula

$$\varphi = \int_{\to 0}^{\infty} W\varphi(\rho,\cdot) * \Psi_\rho\, \alpha(\rho)\, d\rho = \varphi * \int_{\to 0}^{\infty} \check{\psi}_\rho * \Psi_\rho\, \alpha(\rho)\, d\rho.$$

Of interest are in particular those for which the operator $*\partial_t p_t$ is positive. Then the corresponding Fourier coefficients are positive matrices and the choice $\psi_\rho = \Psi_\rho$ seems reasonable.

Diffusive approximate identity

Definition

Let $\hat{G}_+ \subset \hat{G}$ be cofinite. A family $t \to p_t$ from $C^1(\mathbb{R}_+; L^1(G))$ will be called diffusive approximate identity with respect to $\hat{G}_+$ if it satisfies

- $||\hat{p}_t(\pi)|| \leq C$ *uniform in* $\pi \in \hat{G}_+$ *and* $t \in \mathbb{R}_+$;
- $\lim_{t\to 0} \hat{p}_t(\pi) = I$ *for all* $\pi \in \hat{G}_+$;
- $\lim_{t\to\infty} \hat{p}_t(\pi) = 0$ *for all* $\pi \in \hat{G}_+$;
- $-\partial_t \hat{p}_t(\pi)$ *is a positive matrix for all* $t \in \mathbb{R}_+$.

Diffusive wavelets on a compact Lie group

Definition

Let p_t be a diffusive approximate identity and $\alpha(\rho) > 0$ a given weight function.
A family $\psi_\rho \in L_0^2(G) = \bigoplus_{\pi \in \hat{G}_+} H_\pi$ is called diffusive wavelet family, if it satisfies the admissibility condition

$$p_t|_{\hat{G}_+} = \int_t^\infty \check{\psi}_\rho * \psi_\rho \, \alpha(\rho) \, d\rho.$$

This equation can be solved explicitely. Applying Fourier transform to both sides and by differentiating both sides yields

$$-\partial_t \hat{p}_t(\pi) = \hat{\psi}_\rho(\pi) \hat{\psi}_\rho^*(\pi) \alpha(\rho).$$

Heat wavelet family

If p_t is the heat kernel we get

$$\hat{\psi}_\rho(\pi) = \frac{1}{\sqrt{\alpha(\rho)}} \lambda_\pi e^{-\rho\lambda_\pi^2/2} \eta_\pi(\rho)$$

for any (fixed) choice of a family $\eta_\pi(\rho) \in U(d_\pi)$. This implies

$$\psi_\rho = \frac{1}{\sqrt{\alpha(\rho)}} \sum_{\pi \in \hat{G}} d_\pi \lambda_\pi e^{-\rho\lambda_\pi^2/2} \operatorname{trace}\left(\pi(g)\eta_\pi(\rho)\right).$$

The weight function $\alpha(\rho)$ can be used to normalize the family ψ_ρ.

$$\alpha(\rho) = -\Delta_G p_\rho(1),$$

where $p_\rho(1)$ is just the heat trace on G.

We have two options to construct wavelets on homogeneous spaces:

The naive way: We apply the wavelet transform to the lifted function $\tilde{\varphi}(g) = \varphi(g \cdot x_0)$ with base-point $x_0 \in X = G/H$ for some $\varphi \in L^2(X)$. This defines a function on $\mathbb{R}_+ \times G$ via

$$W\tilde{\varphi}(\rho, g) = \int_G \tilde{\varphi}(h)\check{\psi}_\rho(h^{-1}g)\, d\mu_G(h) = \int_G \varphi(h \cdot x_0)\check{\psi}_\rho(h^{-1}g)\, d\mu_G(h)$$

But we would prever to have a transform living on $\mathbb{R}_+ \times X$ instead of $\mathbb{R}_+ \times G$.

Diffusive wavelets on homogeneous spaces $X = G/H$

Let p_t be a diffusive approximate identity and $\alpha(\rho) > 0$ be a given weight function. A family $\psi_\rho \in L^2(X)$ is called a diffusive wavelet family if the admissibility condition

$$p_t^X(x)\Big|_{\hat{G}_+} = \int_t^\infty \psi_\rho \hat{\bullet} \psi_\rho(x)\, \alpha(\rho)\, d\rho$$

is satiesfied. We associate to this family the wavelet transform

$$W_X \varphi(\rho, g) = \varphi \bullet \psi_\rho(g) = \int_X \varphi(x) \overline{\psi(g^{-1} \cdot x)}\, dx$$

with inverse given as

$$\tilde{\varphi} = \int_{\to 0}^\infty W_X \varphi(\rho, \cdot) * \tilde{\psi}_\rho\, \alpha(\rho)\, d\rho \quad \text{for all} \quad \varphi \in L_0^2(X).$$

$$\varphi * \psi(x) = \int_G \varphi(g \cdot x_0) \psi(g^{-1} \cdot x)\, d\mu_g \in L^1(X), \qquad \widetilde{\varphi * \psi} = \tilde{\varphi} * \tilde{\psi}$$

$$\varphi \hat{\bullet} \psi(x) = \int_X \overline{\varphi(g \cdot x_0)} \psi(g \cdot x)\, d\mu_g = \langle \varphi, T_g \psi \rangle \in L^1(G), \qquad \varphi \hat{\bullet} \psi = \check{\tilde{\varphi}} * \tilde{\psi},$$

$$\varphi \bullet \psi(g) = \int_X \varphi(x) \overline{\psi(g^{-1} \cdot x)}\, dx \in L^1(X), \qquad \varphi \bullet \psi = \tilde{\varphi} * \check{\check{\tilde{\psi}}}.$$

Crystallographic Radon transform

Theorem (Wavelets from wavelets)

Let $\{\Psi_\rho,\, \rho > 0\}$ *be a family of class type wavelets (*$\eta_\rho(\pi) = I$*) on* $SO(3)$*, then the family of function* $\{R\Psi_\rho(x, .),\, \rho > 0,\, x \in S^2 fixed\}$ *defines a family of zonal wavelets on* S^2*.*

If we make a non-trivial choice $\eta_\rho(\pi) \neq 0$, we obtain non-zonal wavelets.

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Thank you for your attention!