Basic relations valid for the Bernstein space B_σ^2 and their extensions to functions from larger spaces in terms of their distances from B_σ^2

Part 1: Shannon's sampling theorem and further fundamental theorems of mathematical analysis in the bandlimited and non-bandlimited case

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Outline

Bernstein spaces

Definition

B_σ^p for $\sigma > 0$, $1 \le p \le \infty$, is the Bernstein space of all entire functions $f\colon \mathbb{C} \to \mathbb{C}$ that belong to $L^p(\mathbb{R})$ when restricted to the real axis as well as are of exponential type σ, i. e.,

$$f(z) = \mathcal{O}_f\big(\exp(\sigma|\mathfrak{Im}\, z|)\big) \qquad (|z| \to \infty).$$

There holds

$$B_\sigma^1 \subset B_\sigma^{p_1} \subset B_\sigma^{p_2} \subset B_\sigma^\infty \qquad (1 \le p_1 \le p_2 \le \infty).$$

Whittaker-Kotel'nikov-Shannon sampling theorem (CST)

Theorem (Whittaker 1915, Kotel'nikov 1933, Shannon 1950)

For $f \in B_\sigma^2$ with $\sigma > 0$ we have

$$f(z) = \sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big)\, \operatorname{sinc} \frac{\sigma}{\pi}\Big(z - \frac{k\pi}{\sigma}\Big) \qquad (z \in \mathbb{C}),$$

convergence being absolute and uniform on compact subsets of $\mathbb{C}$, and with respect to $L^2(\mathbb{R})$-norm.

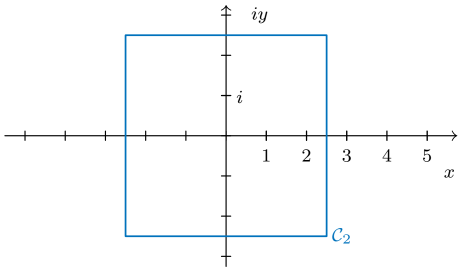
$$\operatorname{sinc} z := \begin{cases} \frac{\sin \pi z}{\pi z}, & z \neq 0 \\ 1, & z = 0. \end{cases}$$

Proof of the classical sampling theorem

Assume $\sigma = \pi$, and $f \in B^2_\tau$ with $0 < \tau < \pi$ rather than to $f \in B^2_\pi$.
Consider the contour integral

$$I_m(z) := \frac{\sin \pi z}{2\pi i} \int_{\mathcal{C}_m} \frac{f(\xi)}{(\xi - z)\sin \pi\xi}\, d\xi \qquad (z \notin \mathbb{Z}),$$

where $\mathcal{C}_m$ is the square of side length $2m+1$, centered at the origin, and $m \in \mathbb{N}$ is chosen so large that $z \in \operatorname{int} \mathcal{C}_m$.



Proof of the classical sampling theorem, continued

The integral can be evaluated by the residue theorem to give

$$I_m(z) = \frac{\sin \pi z}{2\pi i} \int_{\mathcal{C}_m} \frac{f(\xi)}{(\xi - z)\sin \pi\xi}\, d\xi = f(z) - \sum_{k=-m}^{m} f(k) \operatorname{sinc} \pi(z-k),$$

noting that

$$\operatorname{res}\left(\frac{f(\xi)}{(\xi - z)\sin \pi\xi}, z\right) = \frac{1}{\sin \pi z} f(z),$$

$$\operatorname{res}\left(\frac{f(\xi)}{(\xi - k)\sin \pi\xi}, k\right) = \frac{1}{\sin \pi z} f(k) \operatorname{sinc} \pi(z-k).$$

Proof of the classical sampling theorem, continued

Using the estimate

$$\left|\frac{f(\xi)}{\sin \pi\xi}\right| \le c\, \frac{\exp(\tau |\mathfrak{Im}\, \xi|)}{\exp(\pi |\mathfrak{Im}\, \xi|)} = c \exp\left((\tau - \pi)|\mathfrak{Im}\, \xi|\right) \quad (|\xi| \in \mathcal{C}_m),$$

one can show that

$$0 = \lim_{m\to\infty} I_m(z) = f(z) - \lim_{m\to\infty} \sum_{k=-m}^{m} f(k) \operatorname{sinc} \pi(z-k).$$

This is the sampling theorem for $f \in B^2_\tau$ with $0 < \tau < \pi$. By a density argument, the same formula holds in the limiting case $\tau = \pi$.

Finally, for $f \in B^2_\sigma$ with arbitrary $\sigma > 0$ the assertion follows by a linear transformation. □

Reproducing kernel formula (RKF)

Theorem

For $f \in B^2_\sigma$ we have

$$f(z) = \frac{\sigma}{\pi} \int_{\mathbb{R}} f(u) \operatorname{sinc}\left(\frac{\sigma}{\pi}(z-u)\right) du \qquad (z \in \mathbb{C}).$$

This means that B^2_σ is a reproducing kernel Hilbert space, i. e., there exists a kernel function $k(\cdot, z)$ which belongs to B^2_σ for each $z \in \mathbb{C}$, such that

$$f(z) = \langle f(\cdot), k(\cdot, z)\rangle \qquad (z \in \mathbb{C}).$$

General Parseval formula (GPF)

Theorem

For $f, g \in B^2_\sigma$ we have

$$\int_{\mathbb{R}} f(u)\overline{g(u)}\,du = \frac{\pi}{\sigma}\sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big)\overline{g\Big(\frac{k\pi}{\sigma}\Big)}.$$

Corollary

For $f \in B^2_\sigma$ there holds

$$\|f\|^2_{L^2(\mathbb{R})} = \frac{\pi}{\sigma}\sum_{k\in\mathbb{Z}}\Big|f\Big(\frac{k\pi}{\sigma}\Big)\Big|^2.$$

Poisson's summation formula (particular case) (PSF)

Theorem

For $f \in B^1_\sigma$ we have

$$\int_{\mathbb{R}} f(t)\,dt = \frac{2\pi}{\sigma}\sum_{k\in\mathbb{Z}} f\Big(\frac{2k\pi}{\sigma}\Big).$$

In B^1_σ the trapezoidal rule for integration over $\mathbb{R}$ with step size $2\pi/\sigma$ is exact .

Paley-Wiener theorem (PWT)

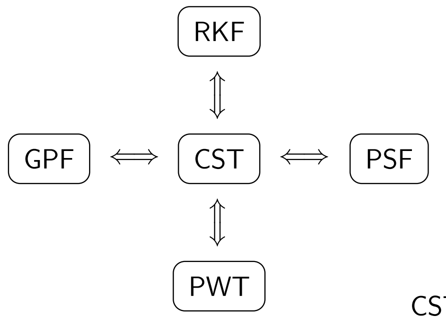
Theorem

$$f \in B^2_\sigma \implies \widehat{f}(v) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(u)e^{-ivu}\,du = 0 \quad (|v| > \sigma).$$

The converse is also true. It follows immediately from the Fourier inversion formula:

$$f(z) := \frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma}\widehat{f}(v)e^{izv}\,dv \qquad (z \in \mathbb{C}).$$

The equivalences between the theorems mentioned



CST = Classical sampling theorem
RKF = Reproducing kernel formula
GPF = General Parseval formula
PSF = Poisson's summation formula
PWT = Paley-Wiener theorem

Proof of CST $\Longrightarrow$ PWT

We restrict the matter to $\sigma = \pi$, the general case follows by a linear transformation.

We have to show: CST $\Longrightarrow$ $\widehat{f}(v) = 0$, $|v| > \pi$, for all $f \in B_\pi^2$.

$$f(t) = \sum_{k\in\mathbb{Z}} f(k)\,\mathrm{sinc}(t-k) \quad \text{in } L^2(\mathbb{R})$$

$$\Longrightarrow \widehat{f}(v) = \sum_{k\in\mathbb{Z}} f(k)[\mathrm{sinc}(\cdot - k)]^\frown(v)$$

$$= \Big(\frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} f(k) e^{-ikv}\Big)\chi_{[-\pi,\pi]}(v)$$

$$\Longrightarrow \widehat{f}(v) = 0 \text{ for } |v| > \pi. \qquad \square$$

Proof of PWT $\Longrightarrow$ CST

We show: $\widehat{f}(v) = 0$ outside $[-\pi,\pi]$ for all $f \in B_\pi^2$ $\Longrightarrow$ CST.

$$\Big\| f(t) - \sum_{|k|\le n} f(k)\,\mathrm{sinc}(t-k)\Big\|_{L^2(\mathbb{R})}$$

$$= \Big\| \widehat{f}(v) - \Big(\frac{1}{\sqrt{2\pi}} \sum_{|k|\le n} f(k) e^{-ikv}\Big)\chi_{[-\pi,\pi]}(v)\Big\|_{L^2(\mathbb{R})}$$

$$= \Big\| \widehat{f}(v) - \frac{1}{\sqrt{2\pi}} \sum_{|k|\le n} f(k) e^{-ikv}\Big\|_{L^2(-\pi,\pi)}$$

$$= \Big\| \widehat{f}(v) - \frac{1}{\sqrt{2\pi}} \sum_{|k|\le n} f(-k) e^{ikv}\Big\|_{L^2(-\pi,\pi)}.$$

Proof of PWT $\Longrightarrow$ CST, continued

$$f(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(v) e^{-ikv}\,dv \quad \text{(Fourier inversion formula)}$$

$\Longrightarrow$ $\displaystyle\frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} f(-k) e^{ikv}$ is the trigonometric Fourier series of $\widehat{f} \in L^2(-\pi,\pi)$, which converges in $L^2(-\pi,\pi)$ towards $\widehat{f}$

$$\Longrightarrow \quad \lim_{n\to\infty} \Big\| f(t) - \sum_{|k|\le n} f(k)\,\mathrm{sinc}(t-k)\Big\|_{L^2(\mathbb{R})}$$

$$= \lim_{n\to\infty} \Big\| \widehat{f}(v) - \frac{1}{\sqrt{2\pi}} \sum_{|k|\le n} f(-k) e^{ikv}\Big\|_{L^2(-\pi,\pi)} = 0. \qquad \square$$

Equivalent assertions in the instance of non-bandlimited functions

Instead of the Bernstein spaces B^p_σ we now consider the following function spaces:

$$F^p := \{f : \mathbb{R} \to \mathbb{C};\, f \in L^p(\mathbb{R}) \cap C(\mathbb{R}),\, \widehat{f} \in L^1(\mathbb{R})\}$$

$$S^p_\lambda := \left\{f : \mathbb{R} \to \mathbb{C};\, \{f(\lambda k)\}_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z})\right\} \quad (\lambda > 0).$$

There holds $B^p_\sigma \subset F^p \cap S^p_\lambda$ for all $\sigma, \lambda > 0$ in view of Nikol'skiĭ's inequality:

$$\left\{\lambda \sum_{k\in\mathbb{Z}} |f(\lambda k)|^p\right\}^{1/p} \le (1+\lambda\sigma)\|f\|_{L^p(\mathbb{R})} \qquad (f \in B^p_\sigma).$$

More details in Part 2.

Theorem (Weiss 1963, Brown 1967, Butzer-Splettstößer 1977)

Let $f \in F^2 \cap S^2_{\pi/\sigma}$ with $\sigma > 0$. Then

$$f(t) = \sum_{k\in\mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \operatorname{sinc}\left(\frac{\sigma}{\pi}t - k\right) + (R^{\mathrm{WKS}}_\sigma f)(t) \qquad (t \in \mathbb{R}).$$

The series converges absolutely and uniformly on $\mathbb{R}$.
We have

$$(R^{\mathrm{WKS}}_\sigma f)(t) := \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left(1 - e^{-i2kt\sigma}\right) \int_{(2k-1)\sigma}^{(2k+1)\sigma} \widehat{f}(v)\, e^{ivt}\, dv$$

$$\left|(R^{\mathrm{WKS}}_\sigma f)(t)\right| \le \sqrt{\frac{2}{\pi}} \int_{|v|\ge\sigma} |\widehat{f}(v)|\, dv = o(1) \qquad (\sigma \to \infty).$$

General reproducing kernel formula (GRKF)

Theorem (extended, Butzer et al. 2011)

Let $f \in F^2 \cap S^2_{\sigma/\pi}$, $\sigma > 0$. Then

$$f(t) = \frac{\sigma}{\pi} \int_{\mathbb{R}} f(u) \operatorname{sinc}\left(\frac{\sigma}{\pi}(t-u)\right) du + (R^{\mathrm{RKF}}_\sigma f)(t) \qquad (t \in \mathbb{R}),$$

$$(R^{\mathrm{RKF}}_\sigma f)(t) := \frac{1}{\sqrt{2\pi}} \int_{|v|>\sigma} \widehat{f}(v) e^{itv}\, dv\,.$$

Furthermore,

$$\left|(R^{\mathrm{RKF}}_\sigma f(t)\right| \le \frac{1}{\sqrt{2\pi}} \int_{|v|>\sigma} |\widehat{f}(v)|\, dv = o(1) \qquad (\sigma \to \infty).$$

General Parseval decomposition formula (GPDF)

Theorem (Butzer–Gessinger 1995/97)

Let $f \in F^2 \cap S^1_{\pi/\sigma}$ and $g \in F^2$. Then for $\sigma > 0$

$$\int_{\mathbb{R}} f(u)\overline{g(u)}\, du = \frac{\pi}{\sigma} \sum_{k\in\mathbb{Z}} f\left(\frac{\pi}{\sigma}k\right) \overline{g\left(\frac{\pi}{\sigma}k\right)} + R_\sigma(f,g)\,,$$

$$R_\sigma(f,g) = \int_{\mathbb{R}} (R^{\mathrm{WKS}}_\sigma f)(u)\overline{g(u)}\, du$$

$$- \sqrt{\frac{\pi}{2}} \frac{1}{\sigma} \sum_{k\in\mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \int_{|v|\ge\sigma} \widehat{\overline{g}}(v)\, e^{ik\pi v/\sigma}\, dv.$$

$$\left|R_\sigma(f,g)\right| \le \|R^{\mathrm{WKS}}_\sigma f\|_{L^2} \|g\|_{L^2} + \sqrt{\frac{\pi}{2}} \frac{1}{\sigma} \sum_{k\in\mathbb{Z}} \left|f\left(\frac{k\pi}{\sigma}\right)\right| \int_{|v|>\sigma} |\widehat{g}(v)|\, dv.$$

Poisson's summation formula (general case) (PSF)

Theorem

Let $f \in F^1$ such that $\widehat{f} \in S^1_\sigma$, then

$$\frac{\sqrt{2\pi}}{\sigma} \sum_{k\in\mathbb{Z}} f\left(x + \frac{2k\pi}{\sigma}\right) = \sum_{k\in\mathbb{Z}} \widehat{f}(k\sigma) e^{ik\sigma x} \quad (a.\,e.).$$

Euler-Maclaurin summation formula (EMSF)

Theorem

For $n, r \in \mathbb{N}$ and $f \in C^{(2r)}[0,n]$, we have

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx +$$

$$+ \frac{1}{2}[f(0) + f(n)] + \sum_{k=1}^{r} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right] +$$

$$+ (-1)^r \sum_{k=1}^{\infty} \int_0^n \frac{e^{i2\pi kt} + e^{-i2\pi kt}}{(2\pi k)^{2r}} f^{(2r)}(t)\,dt\,,$$

where B_{2k} are the Bernoulli numbers.

Functional equation for Riemann's zeta-function (FERZ)

Definition

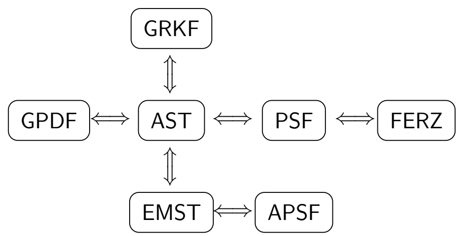
$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s} \qquad (s \in \mathbb{C}, \mathfrak{Re}\, s > 1).$$

ζ has a meromorphic extension to $\mathbb{C} \setminus \{1\}$. At $s = 1$ it has a simple pole with residue 1.

Theorem

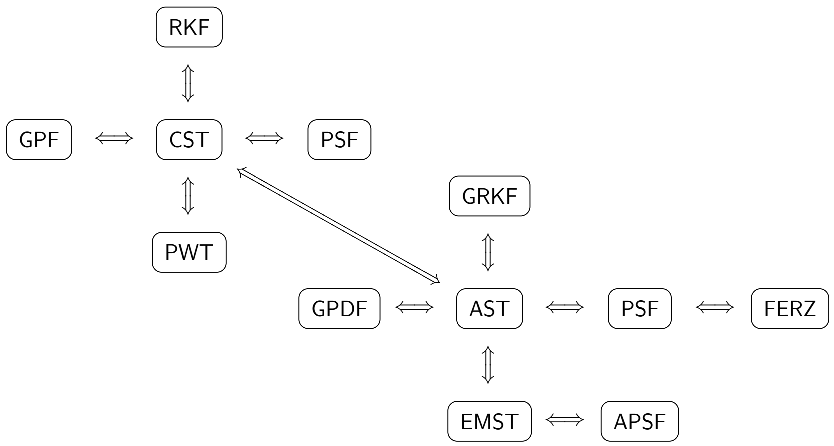
$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \quad (s \in \mathbb{C}).$$

The equivalences in the non-bandlimited case



AST = Approximate sampling theorem
ARKF = Approximate reproducing kernel formula
GPDF = General Parseval decomposition formula
PSF = Poisson's summation formula
FERZ = Functional equation for Riemann's zeta-function
EMSF = Euler-Maclaurin summation formula
APSF = Abel-Plana summation formula

Equivalence of the bandlimited and non-bandlimited case



Proof of AST $\implies$ CST

We restrict the matter to $\sigma = \pi$, the general case follows by a linear transformation.

Identity theorem for bandlimited functions:

If $f \in B^p_\pi$, $1 \le p < \infty$, *with* $f(j) = 0$, $j \in \mathbb{Z}$, *then* $f = 0$.

Idea of proof: Use

$$f(z) = \mathcal{O}_f\big(\exp(\pi|\mathfrak{Im}\, z|)\big) \qquad (|z| \to \infty)$$

to show that the entire function $f(z)/sin(\pi z)$ is bounded. By Liouville it follows that $f(z)/sin(\pi z) = const$. Hence

$$f(z) = c\, sin(\pi z) \qquad (z \in \mathbb{C}).$$

Since $f \in B^p_\pi$, $1 \le p < \infty$, the constant must be zero.

Proof of AST $\implies$ CST, continued

Now assume that $f \in B^2_\pi$. Then ASF applies:

$$f(t) = \sum_{k\in\mathbb{Z}} f(k)\,\mathrm{sinc}(t-k) + (R^{\mathrm{WKS}}_\pi f)(t) \qquad (t \in \mathbb{R}),$$

$$(R^{\mathrm{WKS}}_\pi f)(t) := \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left(1 - e^{-i2k t\pi}\right) \int_{(2k-1)\pi}^{(2k+1)\pi} \widehat{f}(v)\, e^{ivt}\, dv\,.$$

Here f, the infinite series and hence the remainder belong to B^2_π. Moreover, the remainder vanishes for $t = j \in \mathbb{Z}$. In view of the identity theorem, it follows that the remainder vanishes for all $t \in \mathbb{R}$, i. e., AST reduces to CST. □

Proof of CST $\implies$ AST

Let $Sg(t) := \sum_{k\in\mathbb{Z}} g(k)\,\mathrm{sinc}(t-k)$. We have to prove:

$$f(t) = Sf(t) + \underbrace{\frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left(1 - e^{-i2kt\pi}\right) \int_{(2k-1)\pi}^{(2k+1)\pi} \widehat{f}(v)\, e^{ivt}\, dv}_{=(R^{\mathrm{WKS}}_\pi f)(t)}\,.$$

By the Fourier inversion formula

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(v) e^{itv}\, dv + \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) e^{itv}\, dv =: f_1(t) + f_2(t).$$

Now $f_1 \in B^2_\pi$ and hence $f_1 = Sf_1$ by CST. It follows that

$$f = Sf_1 + f_2 = S(f_1 + f_2) + \{f_2 - Sf_2\} = Sf + \{f_2 - Sf_2\}$$

We have to show that

$$R^{\mathrm{WKS}}_\pi f = f_2 - Sf_2.$$

Proof of CST $\implies$ AST, continued

$$(R_\pi^{\mathrm{WKS}} f)(t) := \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left(1 - e^{-i2kt\pi}\right) \int_{(2k-1)\pi}^{(2k+1)\pi} \widehat{f}(v)\, e^{ivt}\, dv$$

$$= \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) e^{ikv}\, dv - \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \int_{(2k-1)\pi}^{(2k+1)\pi} \widehat{f}(v)\, e^{it(v-2k\pi)}\, dv$$

$$= f_2(t) - \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) \left[e^{ivt}\right]^* dv,$$

where $[e^{ivt}]^*$ denotes the 2π-periodic extension of $v \mapsto e^{ivt}$ from $(-\pi, \pi)$ to $\mathbb{R}$.

Proof of CST $\implies$ AST, continued

Expanding $[e^{ivt}]^*$ in its Fourier series gives

$$(R_\pi^{\mathrm{WKS}} f)(t) = f_2(t) - \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) \left[e^{ivt}\right]^* dv$$

$$= f_2(t) - \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) \Big\{ \sum_{k\in\mathbb{Z}} \mathrm{sinc}(t-k) e^{ikv} \Big\} dv$$

$$= f_2(t) - \sum_{k\in\mathbb{Z}} \mathrm{sinc}(t-k) \underbrace{\frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) e^{ikv}\, dv}_{f_2(k)}$$

$$= f_2(t) - Sf_2(t).$$

Interchange of summation and integration is valid, since the partial sums $\sum_{-N}^{N} \mathrm{sinc}(t-k)e^{ikv}$ are uniformly bounded with respect to $v \in \mathbb{R}$ and $N \in \mathbb{N}$. □

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