Basic relations valid for the Bernstein space B^2_σ and their extensions to functions from larger spaces in terms of their distances from B^2_σ

Part 1: Shannon's sampling theorem and further fundamental theorems of mathematical analysis in the bandlimited and non-bandlimited case

P. L. Butzer, G. Schmeisser, R. L. Stens

New Trends and Directions in Harmonic Analysis, Fractional Operator Theory, and Image Analysis

Summer School, Inzell, September 17–21, 2012

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   - Reproducing kernel formula
   - General Parseval formula
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   - General reproducing kernel formula
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3. The equivalence of the bandlimited and non-bandlimited case

Bernstein spaces

Definition

B_σ^p for $\sigma > 0$, $1 \le p \le \infty$, is the Bernstein space of all entire functions $f : \mathbb{C} \to \mathbb{C}$ that belong to $L^p(\mathbb{R})$ when restricted to the real axis as well as are of exponential type σ, i. e.,

$$f(z) = \mathcal{O}_f\big(\exp(\sigma|\,\mathfrak{Im}\,z|)\big) \qquad (|z| \to \infty).$$

There holds

$$B_\sigma^1 \subset B_\sigma^{p_1} \subset B_\sigma^{p_2} \subset B_\sigma^\infty \qquad (1 \le p_1 \le p_2 \le \infty).$$

Whittaker-Kotel'nikov-Shannon sampling theorem (CST)

Theorem (Whittaker 1915, Kotel'nikov 1933, Shannon 1950)

For $f \in B_\sigma^2$ with $\sigma > 0$ we have

$$f(z) = \sum_{k \in \mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \operatorname{sinc} \frac{\sigma}{\pi}\left(z - \frac{k\pi}{\sigma}\right) \quad (z \in \mathbb{C}),$$

convergence being absolute and uniform on compact subsets of $\mathbb{C}$, and with respect to $L^2(\mathbb{R})$-norm.

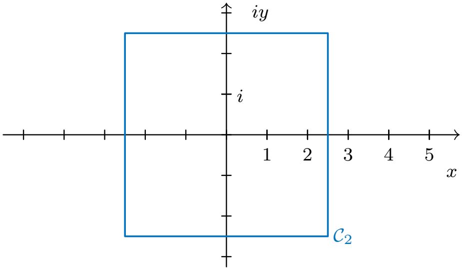
$$\operatorname{sinc} z := \begin{cases} \frac{\sin \pi z}{\pi z}, & z \neq 0 \\ 1, & z = 0. \end{cases}$$

Proof of the classical sampling theorem

Assume $\sigma = \pi$, and $f \in B^2_\tau$ with $0 < \tau < \pi$ rather than to $f \in B^2_\pi$. Consider the contour integral

$$I_m(z) := \frac{\sin \pi z}{2\pi i} \int_{\mathcal{C}_m} \frac{f(\xi)}{(\xi - z)\sin \pi\xi}\, d\xi \qquad (z \notin \mathbb{Z}),$$

where $\mathcal{C}_m$ is the square of side length $2m+1$, centered at the origin, and $m \in \mathbb{N}$ is chosen so large that $z \in \operatorname{int} \mathcal{C}_m$.



Proof of the classical sampling theorem, continued

The integral can be evaluated by the residue theorem to give

$$I_m(z) = \frac{\sin \pi z}{2\pi i} \int_{\mathcal{C}_m} \frac{f(\xi)}{(\xi - z)\sin \pi\xi}\, d\xi = f(z) - \sum_{k=-m}^{m} f(k)\operatorname{sinc}\pi(z-k),$$

noting that

$$\operatorname{res}\left(\frac{f(\xi)}{(\xi - z)\sin \pi\xi}, z\right) = \frac{1}{\sin \pi z} f(z),$$

$$\operatorname{res}\left(\frac{f(\xi)}{(\xi - k)\sin \pi\xi}, k\right) = \frac{1}{\sin \pi z} f(k)\operatorname{sinc}\pi(z-k).$$

Proof of the classical sampling theorem, continued

Using the estimate

$$\left|\frac{f(\xi)}{\sin \pi\xi}\right| \leq c\,\frac{\exp(\tau|\,\mathfrak{Im}\,\xi|)}{\exp(\pi|\,\mathfrak{Im}\,\xi|)} = c\exp\left((\tau-\pi)|\,\mathfrak{Im}\,\xi|\right) \quad (|\xi| \in \mathcal{C}_m),$$

one can show that

$$0 = \lim_{m\to\infty} I_m(z) = f(z) - \lim_{m\to\infty} \sum_{k=-m}^{m} f(k)\,\mathrm{sinc}\,\pi(z-k).$$

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$$0 = \lim_{m\to\infty} I_m(z) = f(z) - \lim_{m\to\infty} \sum_{k=-m}^{m} f(k)\,\mathrm{sinc}\,\pi(z-k).$$

This is the sampling theorem for $f \in B^2_\tau$ with $0 < \tau < \pi$. By a density argument, the same formula holds in the limiting case $\tau = \pi$.

Finally, for $f \in B^2_\sigma$ with arbitrary $\sigma > 0$ the assertion follows by a linear transformation. □

Reproducing kernel formula (RKF)

Theorem

For $f \in B^2_\sigma$ we have

$$f(z) = \frac{\sigma}{\pi} \int_{\mathbb{R}} f(u) \operatorname{sinc}\left(\frac{\sigma}{\pi}(z-u)\right) du \qquad (z \in \mathbb{C}).$$

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This means that B_σ^2 is a reproducing kernel Hilbert space, i. e., there exists a kernel function $k(\cdot, z)$ which belongs to B_σ^2 for each $z \in \mathbb{C}$, such that

$$f(z) = \langle f(\cdot), k(\cdot, z)\rangle \qquad (z \in \mathbb{C}).$$

General Parseval formula (GPF)

Theorem

For $f, g \in B^2_\sigma$ we have

$$\int_{\mathbb{R}} f(u)\overline{g(u)}\,du = \frac{\pi}{\sigma}\sum_{k\in\mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right)\overline{g\left(\frac{k\pi}{\sigma}\right)}.$$

Corollary

For $f \in B^2_\sigma$ there holds

$$\|f\|^2_{L^2(\mathbb{R})} = \frac{\pi}{\sigma}\sum_{k\in\mathbb{Z}}\left|f\left(\frac{k\pi}{\sigma}\right)\right|^2.$$

Poisson's summation formula (particular case) (PSF)

Theorem

For $f \in B^1_\sigma$ we have

$$\int_{\mathbb{R}} f(t)\,dt = \frac{2\pi}{\sigma} \sum_{k\in\mathbb{Z}} f\Big(\frac{2k\pi}{\sigma}\Big).$$

In B^1_σ the trapezoidal rule for integration over $\mathbb{R}$ with step size $2\pi/\sigma$ is exact .

Paley-Wiener theorem (PWT)

Theorem

$$f \in B_\sigma^2 \implies \widehat{f}(v) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) e^{-ivu}\, du = 0 \quad (|v| > \sigma).$$

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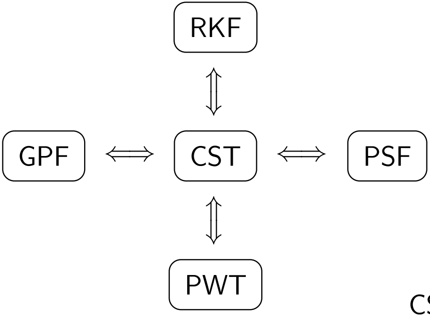
The converse is also true. It follows immediately from the Fourier inversion formula:

$$f(z) := \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} \widehat{f}(v) e^{izv}\, dv \qquad (z \in \mathbb{C}).$$

The equivalences between the theorems mentioned



CST = Classical sampling theorem

RKF = Reproducing kernel formula

GPF = General Parseval formula

PSF = Poisson's summation formula

PWT = Paley-Wiener theorem

Proof of CST $\Longrightarrow$ PWT

We restrict the matter to $\sigma = \pi$, the general case follows by a linear transformation.

We have to show: CST $\Longrightarrow$ $\widehat{f}(v) = 0$, $|v| > \pi$, for all $f \in B^2_\pi$.

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We show: $\widehat{f}(v) = 0$ outside $[-\pi, \pi]$ for all $f \in B^2_\pi \implies$ CST.

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$$= \left\| \widehat{f}(v) - \frac{1}{\sqrt{2\pi}} \sum_{|k| \le n} f(-k) e^{ikv} \right\|_{L^2(-\pi,\pi)}.$$

Proof of PWT $\Longrightarrow$ CST, continued

$$f(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(v) e^{-ikv}\, dv \quad \text{(Fourier inversion formula)}$$

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$$\Longrightarrow \quad \left\| f(t) - \sum_{|k| \le n} f(k)\, \mathrm{sinc}(t-k) \right\|_{L^2(\mathbb{R})}$$

$$= \quad \left\| \widehat{f}(v) - \frac{1}{\sqrt{2\pi}} \sum_{|k| \le n} f(-k) e^{ikv} \right\|_{L^2(-\pi,\pi)} \quad .$$

Proof of PWT $\Longrightarrow$ CST, continued

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$$\Longrightarrow \quad \lim_{n\to\infty} \left\| f(t) - \sum_{|k|\le n} f(k)\,\mathrm{sinc}(t-k) \right\|_{L^2(\mathbb{R})}$$

$$= \lim_{n\to\infty} \left\| \widehat{f}(v) - \frac{1}{\sqrt{2\pi}} \sum_{|k|\le n} f(-k) e^{ikv} \right\|_{L^2(-\pi,\pi)} = 0. \qquad \square$$

Equivalent assertions in the instance of non-bandlimited functions

Instead of the Bernstein spaces B_σ^p we now consider the following function spaces:

$$F^p := \left\{ f : \mathbb{R} \to \mathbb{C}\,;\ f \in L^p(\mathbb{R}) \cap C(\mathbb{R}),\ \widehat{f} \in L^1(\mathbb{R}) \right\}$$

$$S_\lambda^p := \left\{ f : \mathbb{R} \to \mathbb{C}\,;\ \left\{ f(\lambda k) \right\}_{k \in \mathbb{Z}} \in \ell^p(\mathbb{Z}) \right\} \quad (\lambda > 0).$$

There holds $B_\sigma^p \subset F^p \cap S_\lambda^p$ for all $\sigma, \lambda > 0$ in view of Nikol'skiĭ's inequality:

$$\left\{ \lambda \sum_{k \in \mathbb{Z}} |f(\lambda k)|^p \right\}^{1/p} \leq (1 + \lambda\sigma) \|f\|_{L^p(\mathbb{R})} \qquad (f \in B_\sigma^p).$$

More details in Part 2.

Theorem (Weiss 1963, Brown 1967, Butzer-Splettstößer 1977)

Let $f \in F^2 \cap S^2_{\pi/\sigma}$ *with* $\sigma > 0$. *Then*

$$f(t) = \sum_{k \in \mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \operatorname{sinc}\left(\frac{\sigma}{\pi}t - k\right) + (R_\sigma^{\mathrm{WKS}} f)(t) \qquad (t \in \mathbb{R}).$$

The series converges absolutely and uniformly on $\mathbb{R}$.
We have

$$(R_\sigma^{\mathrm{WKS}} f)(t) := \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \left(1 - e^{-i2kt\sigma}\right) \int_{(2k-1)\sigma}^{(2k+1)\sigma} \widehat{f}(v)\, e^{ivt}\, dv$$

$$\left|(R_\sigma^{\mathrm{WKS}} f)(t)\right| \leq \sqrt{\frac{2}{\pi}} \int_{|v| \geq \sigma} |\widehat{f}(v)|\, dv = o(1) \qquad (\sigma \to \infty).$$

General reproducing kernel formula (GRKF)

Theorem (extended, Butzer et al. 2011)

Let $f \in F^2 \cap S^2_{\sigma/\pi}$, $\sigma > 0$. Then

$$f(t) = \frac{\sigma}{\pi} \int_{\mathbb{R}} f(u) \operatorname{sinc}\left(\frac{\sigma}{\pi}(t-u)\right) du + (R_\sigma^{\mathrm{RKF}} f)(t) \qquad (t \in \mathbb{R}),$$

$$(R_\sigma^{\mathrm{RKF}} f)(t) := \frac{1}{\sqrt{2\pi}} \int_{|v|>\sigma} \widehat{f}(v) e^{itv}\, dv\,.$$

Furthermore,

$$\left|(R_\sigma^{\mathrm{RKF}} f(t)\right| \leq \frac{1}{\sqrt{2\pi}} \int_{|v|>\sigma} \left|\widehat{f}(v)\right| dv = o(1) \qquad (\sigma \to \infty).$$

General Parseval decomposition formula (GPDF)

Theorem (Butzer–Gessinger 1995/97)

Let $f \in F^2 \cap S^1_{\pi/\sigma}$ *and* $g \in F^2$. *Then for* $\sigma > 0$

$$\int_{\mathbb{R}} f(u)\overline{g(u)}\,du = \frac{\pi}{\sigma}\sum_{k\in\mathbb{Z}} f\Big(\frac{\pi}{\sigma}k\Big)\overline{g\Big(\frac{\pi}{\sigma}k\Big)} + R_\sigma(f,g),$$

$$R_\sigma(f,g) = \int_{\mathbb{R}} (R_\sigma^{\mathrm{WKS}} f)(u)\overline{g(u)}\,du$$

$$-\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big)\int_{|v|\geq\sigma} \widehat{\overline{g}}(v)\,e^{ik\pi v/\sigma}\,dv.$$

$$\big|R_\sigma(f,g)\big| \leq \|R_\sigma^{\mathrm{WKS}} f\|_{L^2}\|g\|_{L^2} + \sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{k\in\mathbb{Z}}\Big|f\Big(\frac{k\pi}{\sigma}\Big)\Big|\int_{|v|>\sigma}\big|\widehat{g}(v)\big|\,dv.$$

Poisson's summation formula (general case) (PSF)

Theorem

Let $f \in F^1$ such that $\widehat{f} \in S^1_\sigma$, then

$$\frac{\sqrt{2\pi}}{\sigma} \sum_{k\in\mathbb{Z}} f\left(x + \frac{2k\pi}{\sigma}\right) = \sum_{k\in\mathbb{Z}} \widehat{f}(k\sigma)e^{ik\sigma x} \quad (a.\,e.).$$

Euler-Maclaurin summation formula (EMSF)

Theorem

For $n, r \in \mathbb{N}$ and $f \in C^{(2r)}[0, n]$, we have

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx +$$

$$+ \frac{1}{2}\left[f(0) + f(n)\right] + \sum_{k=1}^{r} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right] +$$

$$+ (-1)^r \sum_{k=1}^{\infty} \int_0^n \frac{e^{i2\pi kt} + e^{-i2\pi kt}}{(2\pi k)^{2r}}\, f^{(2r)}(t)\,dt\,,$$

where B_{2k} are the Bernoulli numbers.

Functional equation for Riemann's zeta-function (FERZ)

Definition

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s} \qquad (s \in \mathbb{C}, \mathfrak{Re}\, s > 1).$$

ζ has a meromorphic extension to $\mathbb{C} \setminus \{1\}$. At $s = 1$ it has a simple pole with residue 1.

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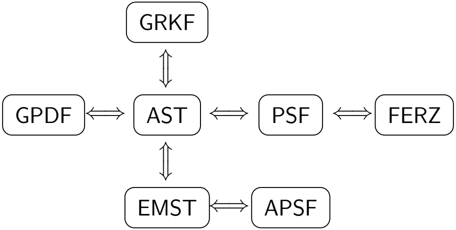
Theorem

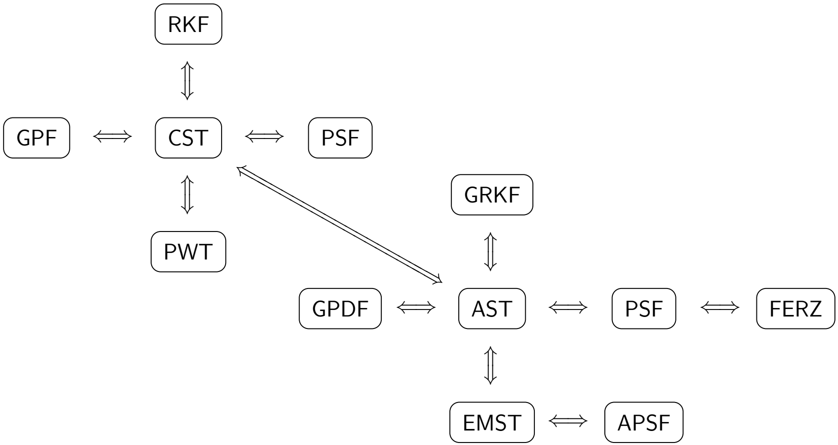
$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \quad (s \in \mathbb{C}).$$

The equivalences in the non-bandlimited case



AST = Approximate sampling theorem
ARKF = Approximate reproducing kernel formula
GPDF = General Parseval decomposition formula
PSF = Poisson's summation formula
FERZ = Functional equation for Riemann's zeta-function
EMSF = Euler-Maclaurin summation formula
APSF = Abel-Plana summation formula

Equivalence of the bandlimited and non-bandlimited case



Proof of AST $\implies$ CST

We restrict the matter to $\sigma = \pi$, the general case follows by a linear transformation.

Identity theorem for bandlimited functions:

If $f \in B_\pi^p$, $1 \leq p < \infty$, *with* $f(j) = 0$, $j \in \mathbb{Z}$, *then* $f = 0$.

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Idea of proof: Use

$$f(z) = \mathcal{O}_f\big(\exp(\pi|\,\mathfrak{Im}\,z|)\big) \qquad (|z| \to \infty)$$

to show that the entire function $f(z)/sin(\pi z)$ is bounded. By Liouville it follows that $f(z)/sin(\pi z) = const$. Hence

$$f(z) = c\,sin(\pi z) \qquad (z \in \mathbb{C}).$$

Since $f \in B^p_\pi$, $1 \leq p < \infty$, the constant must be zero.

Proof of AST $\implies$ CST, continued

Now assume that $f \in B_\pi^2$. Then ASF applies:

$$f(t) = \sum_{k\in\mathbb{Z}} f(k)\,\mathrm{sinc}(t-k) + (R_\pi^{\mathrm{WKS}} f)(t) \qquad (t \in \mathbb{R}),$$

$$(R_\pi^{\mathrm{WKS}} f)(t) := \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left(1 - e^{-i2kt\pi}\right) \int_{(2k-1)\pi}^{(2k+1)\pi} \widehat{f}(v)\, e^{ivt}\, dv\,.$$

Proof of AST $\implies$ CST, continued

Now assume that $f \in B^2_\pi$. Then ASF applies:

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Here f, the infinite series and hence the remainder belong to B^2_π. Moreover, the remainder vanishes for $t = j \in \mathbb{Z}$. In view of the identity theorem, it follows that the remainder vanishes for all $t \in \mathbb{R}$, i. e., AST reduces to CST. □

Proof of CST $\implies$ AST

Let $Sg(t) := \sum_{k\in\mathbb{Z}} g(k)\,\mathrm{sinc}(t-k)$. We have to prove:

$$f(t) = Sf(t) + \underbrace{\frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left(1 - e^{-i2kt\pi}\right) \int_{(2k-1)\pi}^{(2k+1)\pi} \widehat{f}(v)\, e^{ivt}\, dv}_{=(R_\pi^{\mathrm{WKS}} f)(t)} .$$

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$$f(t) = Sf(t) + \underbrace{\frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left(1 - e^{-i2kt\pi}\right) \int_{(2k-1)\pi}^{(2k+1)\pi} \widehat{f}(v)\, e^{ivt}\, dv}_{=(R_\pi^{\mathrm{WKS}} f)(t)} .$$

By the Fourier inversion formula

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(v) e^{itv}\, dv + \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) e^{itv}\, dv =: f_1(t) + f_2(t).$$

Proof of CST $\implies$ AST

Let $Sg(t) := \sum_{k\in\mathbb{Z}} g(k)\,\mathrm{sinc}(t-k)$. We have to prove:

$$f(t) = Sf(t) + \underbrace{\frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left(1 - e^{-i2kt\pi}\right) \int_{(2k-1)\pi}^{(2k+1)\pi} \widehat{f}(v)\, e^{ivt}\, dv}_{=(R_\pi^{\mathrm{WKS}} f)(t)} .$$

By the Fourier inversion formula

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(v) e^{itv}\, dv + \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) e^{itv}\, dv =: f_1(t) + f_2(t).$$

Now $f_1 \in B_\pi^2$ and hence $f_1 = Sf_1$ by CST. It follows that

$$f = Sf_1 + f_2 = S(f_1 + f_2) + \{f_2 - Sf_2\} = Sf + \{f_2 - Sf_2\}$$

We have to show that

$$R_\pi^{\mathrm{WKS}} f = f_2 - Sf_2.$$

Proof of CST $\implies$ AST, continued

$$(R_\pi^{\mathrm{WKS}} f)(t) := \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \left(1 - e^{-i2kt\pi}\right) \int_{(2k-1)\pi}^{(2k+1)\pi} \widehat{f}(v)\, e^{ivt}\, dv$$

Proof of CST $\implies$ AST, continued

$$(R_\pi^{\mathrm{WKS}} f)(t) := \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left(1 - e^{-i2kt\pi}\right) \int_{(2k-1)\pi}^{(2k+1)\pi} \widehat{f}(v)\, e^{ivt}\, dv$$

$$= \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) e^{ikv}\, dv - \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \int_{(2k-1)\pi}^{(2k+1)\pi} \widehat{f}(v)\, e^{it(v-2k\pi)}\, dv$$

Proof of CST $\implies$ AST, continued

$$(R_\pi^{\mathrm{WKS}} f)(t) := \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left(1 - e^{-i2kt\pi}\right) \int_{(2k-1)\pi}^{(2k+1)\pi} \widehat{f}(v)\, e^{ivt}\, dv$$

$$= \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) e^{ikv}\, dv - \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \int_{(2k-1)\pi}^{(2k+1)\pi} \widehat{f}(v)\, e^{it(v-2k\pi)}\, dv$$

$$= f_2(t) - \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) \left[e^{ivt}\right]^* dv,$$

where $[e^{ivt}]^*$ denotes the 2π-periodic extension of $v \mapsto e^{ivt}$ from $(-\pi, \pi)$ to $\mathbb{R}$.

Proof of CST $\implies$ AST, continued

Expanding $[e^{ivt}]^*$ in its Fourier series gives

$$(R_\pi^{\mathrm{WKS}} f)(t) = f_2(t) - \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) \left[e^{ivt}\right]^* dv$$

Proof of CST $\implies$ AST, continued

Expanding $[e^{ivt}]^*$ in its Fourier series gives

$$(R_\pi^{\mathrm{WKS}} f)(t) = f_2(t) - \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) \left[e^{ivt}\right]^* dv$$

$$= f_2(t) - \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) \left\{ \sum_{k\in\mathbb{Z}} \mathrm{sinc}(t-k) e^{ikv} \right\} dv$$

Proof of CST $\implies$ AST, continued

Expanding $[e^{ivt}]^*$ in its Fourier series gives

$$
\begin{aligned}
(R_\pi^{\mathrm{WKS}} f)(t) &= f_2(t) - \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) \left[e^{ivt} \right]^* dv \\
&= f_2(t) - \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) \Big\{ \sum_{k\in\mathbb{Z}} \mathrm{sinc}(t-k) e^{ikv} \Big\} dv \\
&= f_2(t) - \sum_{k\in\mathbb{Z}} \mathrm{sinc}(t-k) \underbrace{\frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) e^{ikv}\, dv}_{f_2(k)}
\end{aligned}
$$

Proof of CST $\implies$ AST, continued

Expanding $[e^{ivt}]^*$ in its Fourier series gives

$$
\begin{aligned}
(R_\pi^{\mathrm{WKS}} f)(t) &= f_2(t) - \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) \left[e^{ivt}\right]^* dv \\
&= f_2(t) - \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) \Big\{ \sum_{k\in\mathbb{Z}} \operatorname{sinc}(t-k) e^{ikv} \Big\} dv \\
&= f_2(t) - \sum_{k\in\mathbb{Z}} \operatorname{sinc}(t-k) \underbrace{\frac{1}{\sqrt{2\pi}} \int_{|v|>\pi} \widehat{f}(v) e^{ikv}\, dv}_{f_2(k)} \\
&= f_2(t) - Sf_2(t).
\end{aligned}
$$

Interchange of summation and integration is valid, since the partial sums $\sum_{-N}^{N} \operatorname{sinc}(t-k)e^{ikv}$ are uniformly bounded with respect to $v \in \mathbb{R}$ and $N \in \mathbb{N}$. $\square$

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