Basic relations valid for the Bernstein space B_σ^2 and their extensions to functions from larger spaces in terms of their distances from B_σ^2

Part 3: Distance functional approach of Part 2 applied to fundamental theorems of Part 1

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New Trends and Directions in Harmonic Analysis, Fractional Operator Theory, and Image Analysis

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Outline

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Approximate sampling theorem (AST) — recall Lecture 1

Theorem (Weiss 1963, Brown 1967, Butzer-Splettstößer 1977)

Let $f \in F^2 \cap S^2_{\pi/\sigma}$ *with* $\sigma > 0$. *Then*

$$f(t) = \sum_{k\in\mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \operatorname{sinc}\left(\frac{\sigma}{\pi}t - k\right) + (R_\sigma^{\mathrm{WKS}} f)(t) \quad (t \in \mathbb{R}).$$

We have

$$\left|(R_\sigma^{\mathrm{WKS}} f)(t)\right| \le \sqrt{\frac{2}{\pi}} \int_{|v|\ge\sigma} |\widehat{f}(v)|\, dv = \sqrt{\frac{2}{\pi}} \operatorname{dist}_1(f, B_\sigma^2) = o(1).$$

Corollary

If $f \in F^2$, *then one has for any* $r \in \mathbb{N}$, $\sigma > 0$ *and* $t \in \mathbb{R}$ *the* ***derivative free*** *error estimate*

$$\left|(R_\sigma^{\mathrm{WKS}} f)(t)\right| \leq \sqrt{\frac{2}{\pi}}\,\mathrm{dist}_1(f, B_\sigma^2) \leq c \int_\sigma^\infty \omega_r\big(f, v^{-1}, L^2(\mathbb{R})\big) \frac{dv}{\sqrt{v}}.$$

If in addition $f \in \mathrm{Lip}_r(\alpha, L^2(\mathbb{R}))$ *for* $1/2 < \alpha \leq r$, *then*

$$(R_\sigma^{\mathrm{WKS}} f)(t) = \mathcal{O}\big(\sigma^{-\alpha+1/2}\big) \qquad (\sigma \to \infty)$$

uniformly in $t \in \mathbb{R}$.

Corollary

Let $f \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$ *for some* $r \in \mathbb{N}$. *Then for* $t \in \mathbb{R}$,

$$\left|(R_\sigma^{\mathrm{WKS}} f)(t)\right| \le c\,\sigma^{-r+1/2} \|f^{(r)}\|_{L^2(\mathbb{R})} \qquad (\sigma > 0).$$

If moreover $f^{(r)} \in \mathrm{Lip}_1(\alpha, L^2(\mathbb{R}))$, $0 < \alpha \le 1$, *then*

$$(R_\sigma^{\mathrm{WKS}} f)(t) = \mathcal{O}\left(\sigma^{-r-\alpha+1/2}\right) \qquad (\sigma \to \infty).$$

Or, if $f^{(r)} \in M_*^{2,1}$, *then*

$$(R_\sigma^{\mathrm{WKS}} f)(t) = \mathcal{O}\left(\sigma^{-r}\right) \qquad (\sigma \to \infty).$$

Corollary

If $f \in H^2(\mathcal{S}_d)$, *then for* $t \in \mathbb{R}$,

$$\left|(R_\sigma^{\mathrm{WKS}} f)(t)\right| \le c_d\, e^{-d\sigma} \|f\|_{H^2(\mathcal{S}_d)} \qquad (\sigma > 0).$$

Function spaces

$$F^2 := \left\{ f : \mathbb{R} \to \mathbb{C} \;:\; f \in L^2(\mathbb{R}) \cap C(\mathbb{R}),\ \widehat{f} \in L^1(\mathbb{R}) \right\},$$

$$S_h^p := \left\{ f \;:\; \mathbb{R} \to \mathbb{C} \;:\; \big(f(hk)\big)_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z}) \right\}.$$

Lipschitz spaces:

$$\omega_r(f;\delta;L^2(\mathbb{R})) := \sup_{|h|\le\delta} \left\| \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(u+jh) \right\|_{L^2(\mathbb{R})},$$

$$\mathrm{Lip}_r(\alpha;L^2(\mathbb{R})) := \{ f \in L^2(\mathbb{R}) \;:\; \omega_r(f;\delta;L^2(\mathbb{R})) = \mathcal{O}(\delta^\alpha),\ \delta \to 0+ \}.$$

Sobolev spaces:

$$W^{r,p}(\mathbb{R}) := \left\{ f \;:\; \mathbb{R} \to \mathbb{C} \;:\; f \in \mathrm{AC}_{\mathrm{loc}}^{r-1}(\mathbb{R}),\ f^{(k)} \in L^p(\mathbb{R}), 0 \le k \le r \right\}.$$

Function spaces, continued

Hardy spaces:

$$H^p(\mathcal{S}_d) := \Big\{ f \ : \ f \text{ analytic on } \{z \in \mathbb{C} \ : \ |\Im z| < d\},$$

$$\|f\|_{H^p(\mathcal{S}_d)} := \left\{ \sup_{0<y<d} \int_{\mathbb{R}} \frac{|f(t-iy)|^p + |f(t+iy)|^p}{2}\, dt \right\}^{1/p} < \infty \Big\}.$$

Wiener amalgam space — modulation space:

$$M^{2,1} := \left\{ f : \mathbb{R} \to \mathbb{C} \ : \ \sum_{n\in\mathbb{Z}} \frac{1}{h} \left\{ \int_n^{n+1} \left| f\left(\frac{v}{h}\right) \right|^2 dv \right\}^{1/2} < \infty,\ h > 0 \right\},$$

$$M^{2,1}_* := \Big\{ f \in M^{2,1} \ : \ \text{series converges uniformly in } h \text{ on bounded subintervals of } (0,\infty) \Big\}.$$

Reproducing kernel formula (RKF)

Theorem (extended, Butzer et al. 2011)

Let $f \in F^2 \cap S^2_{\sigma/\pi}$, $\sigma > 0$. *Then*

$$f(t) = \frac{\sigma}{\pi} \int_{\mathbb{R}} f(u) \operatorname{sinc}\left(\frac{\sigma}{\pi}(t-u)\right) du + (R_\sigma^{\mathrm{RKF}} f)(t) \qquad (t \in \mathbb{R}),$$

$$(R_\sigma^{\mathrm{RKF}} f)(t) := \frac{1}{\sqrt{2\pi}} \int_{|v|>\sigma} \widehat{f}(v) e^{itv}\, dv\,.$$

Furthermore,

$$\left|(R_\sigma^{\mathrm{RKF}} f(t)\right| \le \frac{1}{\sqrt{2\pi}} \int_{|v|>\sigma} \left|\widehat{f}(v)\right| dv \;=\; \frac{1}{\sqrt{2\pi}} \operatorname{dist}_1(f, B^2_\sigma) = o(1).$$

Three corollaries above are also valid here.

General Parseval decomposition formula (GDPF)

Theorem (Butzer–Gessinger 1995/97)

Let $f \in F^2 \cap S^1_{\pi/\sigma}$ *and* $g \in F^2$. *Then for* $\sigma > 0$

$$\int_{\mathbb{R}} f(u)\overline{g(u)}\,du = \frac{\pi}{\sigma}\sum_{k\in\mathbb{Z}} f\Big(\frac{\pi}{\sigma}k\Big)\overline{g\Big(\frac{\pi}{\sigma}k\Big)} + R_\sigma(f,g),$$

$$R_\sigma(f,g) = \int_{\mathbb{R}} (R_\sigma^{\mathrm{WKS}} f)(u)\overline{g(u)}\,du$$

$$- \sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big)\int_{|v|\geq\sigma} \widehat{g}(v)\,e^{ik\pi v/\sigma}\,dv.$$

$$\big|R_\sigma(f,g)\big| \leq \|R_\sigma^{\mathrm{WKS}} f\|_{L^2}\|g\|_{L^2} + \sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{k\in\mathbb{Z}}\Big|f\Big(\frac{k\pi}{\sigma}\Big)\Big|\int_{|v|>\sigma}\big|\widehat{g}(v)\big|\,dv.$$

Theorem

Let $f \in F^2 \cap S^1_{\pi/\sigma}$ *and* $g \in F^2$. *Then for* $\sigma > 0$

$$\int_{\mathbb{R}} f(u)\overline{g(u)}\,du = \frac{\pi}{\sigma}\sum_{k\in\mathbb{Z}} f\left(\frac{\pi}{\sigma}k\right)\overline{g\left(\frac{\pi}{\sigma}k\right)} + R_\sigma(f,g)\,,$$

$$\left|R_\sigma(f,g)\right| \le \|R_\sigma^{\mathrm{WKS}} f\|_{L^2}\|g\|_{L^2} + \sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{k\in\mathbb{Z}}\left|f\left(\frac{k\pi}{\sigma}\right)\right|\int_{|v|>\sigma}\left|\widehat{g}(v)\right|dv.$$

If $f, g \in W^{1,2}(\mathbb{R}) \cap C(\mathbb{R})$, *then*

$$\left|R_\sigma(f,g)\right| \le \frac{C}{\sigma}\Big\{\operatorname{dist}_2(f', B^2_\sigma) + \operatorname{dist}_2(g', B^2_\sigma) + \frac{\pi}{\sigma}\operatorname{dist}_2(f', B^2_\sigma)\operatorname{dist}_2(g', B^2_\sigma)\Big\}.$$

Recall Lecture 2

If $f \in W^{1,2}(\mathbb{R}) \cap C(\mathbb{R})$ with $v\widehat{f}(v) \in L^1(\mathbb{R})$, then for each $r \in \mathbb{N}$,

$$\text{dist}_2(f', B^2_\sigma) \le c\left\{\int_\sigma^\infty \left[\omega_r\left(f', v^{-1}, L^2(\mathbb{R})\right)\right]^2 dv\right\}^{1/2}.$$

If in addition $f' \in \text{Lip}_r(\alpha, L^2(\mathbb{R}))$, $0 < \alpha \le r$, then

$$\text{dist}_2(f', B^2_\sigma) = \mathcal{O}\left(\sigma^{-1-\alpha}\right) \qquad (\sigma \to \infty).$$

Corollary

If $f, g \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$, $r \geq 2$, *and* $f^{(r)}, g^{(r)} \in \mathrm{Lip}_1(\alpha, L^2(\mathbb{R}))$, $0 < \alpha \leq 1$, *then*

$$R_\sigma(f,g) = \mathcal{O}\left(\sigma^{-r-\alpha}\right) \qquad (\sigma \to \infty).$$

If instead $f^{(r)}, g^{(r)} \in M_*^{2,1}$, *then*

$$R_\sigma(f,g) = \mathcal{O}\left(\sigma^{-r-1/2}\right) \qquad (\sigma \to \infty).$$

Corollary

If $f, g \in H^2(\mathcal{S}_d)$, *then*

$$R_\sigma(f,g) \leq c \exp(-\sigma d) \|f\|_{H^2(\mathcal{S}_d)} \|g\|_{H^2(\mathcal{S}_d)} \qquad (\sigma > 0).$$

Approximate sampling theorem for derivatives

Theorem

For $f \in F^2 \cap S^2_{\pi/\sigma}$, $\sigma > 0$, *with* $v^s \widehat{f}(v) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *for some* $s \in \mathbb{N}_0$. *Then*

$$f^{(s)}(t) = \sum_{k \in \mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big) \operatorname{sinc}^{(s)} \frac{\sigma}{\pi}\Big(t - \frac{k\pi}{\sigma}\Big) + (R^{\mathrm{WKS}}_{s,\sigma} f)(t) \qquad (t \in \mathbb{R}),$$

$$(R^{\mathrm{WKS}}_{s,\sigma} f)(t) := \frac{1}{\sqrt{2\pi}} \int_{|v|>\sigma} \widehat{f}(v) \Big[(iv)^r e^{ivt} - [(iv)^r e^{ivt}]^* \Big] \, dv.$$

$[(iv)^s e^{ivt}]^*$ *is the* 2π*-periodic extens. of* $(iv)^s e^{ivt}$ *from* $(-\pi, \pi)$ *to* $\mathbb{R}$. *Furthermore,*

$$|(R^{\mathrm{WKS}}_{s,\sigma} f)(t)| \leq \sqrt{\frac{2}{\pi}} \int_{|v|\geq\sigma} |v^s \widehat{f}(v)| \, dv = \sqrt{\frac{2}{\pi}} \operatorname{dist}_1(f^{(s)}, B^2_\sigma) = o(1)$$

uniformly for $t \in \mathbb{R}$ *for* $\sigma \to \infty$.

Corollary

If $f \in F^2$, $s \in \mathbb{N}$, *and* $v^s \widehat{f}(v) \in L^1(\mathbb{R})$, *then for any* $r \in \mathbb{N}$,

$$\left|(R_{s,\sigma}^{\mathrm{WKS}} f)(t)\right| \leq \sqrt{\frac{2}{\pi}}\, \mathrm{dist}_1(f^{(s)}, B_\sigma^2) \leq c \int_\sigma^\infty \omega_r\big(f^{(s)}, v^{-1}, L^2(\mathbb{R})\big) \frac{dv}{\sqrt{v}}.$$

If in addition $f^{(s)} \in \mathrm{Lip}_r(\alpha, L^2(\mathbb{R}))$ *for* $1/2 < \alpha \leq r$, *then*

$$(R_{s,\sigma}^{\mathrm{WKS}} f)(t) = \mathcal{O}\big(\sigma^{-\alpha+1/2}\big) \qquad (\sigma \to \infty).$$

Corollary

Let $s \in \mathbb{N}$, $f \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$ *for some* $r \geq s+1$. *Then for* $t \in \mathbb{R}$,

$$\left|(R_{s,\sigma}^{\mathrm{WKS}} f)(t)\right| \leq c\, \sigma^{-r+s+1/2} \|f^{(r)}\|_{L^2(\mathbb{R})} \qquad (\sigma > 0).$$

If moreover $f^{(r)} \in \mathrm{Lip}_1(\alpha, L^2(\mathbb{R}))$, $0 < \alpha \leq 1$, *then*

$$(R_{s,\sigma}^{\mathrm{WKS}} f)(t) = \mathcal{O}\left(\sigma^{-r-\alpha+s+1/2}\right) \qquad (\sigma \to \infty).$$

Corollary

If $f \in H^2(\mathcal{S}_d)$, *then for* $t \in \mathbb{R}$,

$$\left|(R_{s,\sigma}^{\mathrm{WKS}} f)(t)\right| \leq c_d\, \sigma^s e^{-d\sigma} \|f\|_{H^2(\mathcal{S}_d)} \qquad (\sigma > 0).$$

Boas formula for the first derivative

In 1937 Boas established a differentiation formula which may be presented as follows:

Let $f \in B_\sigma^\infty$, *where* $\sigma > 0$. *Then, for* $h = \pi/\sigma$, *we have*

$$f'(t) \ = \frac{1}{h} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k+1}}{\pi (k - \frac{1}{2})^2} \, f\Big(t + h\Big(k - \frac{1}{2}\Big)\Big) \, .$$

Boas-type formulae for higher derivatives (Schmeisser 2009)

The Boas-type formulae to be established will be deduced as applications of the Whittaker-Kotel'nikov-Shannon sampling theorem for higher order derivatives.

Theorem (Boas-type formulae)

Let $f \in B_\sigma^\infty$ *for some* $\sigma > 0$, *and* $s \in \mathbb{N}$. *Then*

$$f^{(2s-1)}(t) = \frac{1}{h^{2s-1}} \sum_{k=-\infty}^{\infty} (-1)^{k+1} A_{s,k}\, f\Big(t + h\Big(k - \frac{1}{2}\Big)\Big) \quad (t \in \mathbb{R}),$$

$$A_{s,k} := \frac{(2s-1)!}{\pi(k-\frac{1}{2})^{2s}} \sum_{j=0}^{s-1} \frac{(-1)^j}{(2j)!} \left[\pi\left(k - \tfrac{1}{2}\right)\right]^{2j} \qquad (k \in \mathbb{Z}),$$

the series being absolutely and uniformly convergent.

A similar expansion holds for even order derivatives.

Proof

Assume $\sigma = \pi$. Setting $t = 1/2$ in derivative sampling theorem yields

$$f^{(2s-1)}\left(\frac{1}{2}\right) = \sum_{k=-\infty}^{\infty} f(k)\,\mathrm{sinc}^{(2s-1)}\left(\frac{1}{2} - k\right).$$

The sinc-terms can be evaluated by Leibnitz' rule, namely,

$$\mathrm{sinc}^{(2s-1)}\left(\frac{1}{2} - k\right) = \underbrace{\frac{(-1)^{k+1}(2s-1)!}{\pi(k-\frac{1}{2})^{2s}}\left\{\sum_{j=0}^{s-1}\frac{(-1)^j\left[\pi(k-\frac{1}{2})\right]^{2j}}{(2j)!}\right\}}_{=(-1)^{k+1}A_{s,k}}.$$

$$f^{(2s-1)}\left(\frac{1}{2}\right) = \sum_{k=-\infty}^{\infty} f(k)(-1)^{k+1}A_{s,k}.$$

For arbitrary $\sigma > 0$, $t \in \mathbb{R}$ apply this to $u \mapsto f(hu + t - h)$. □

Theorem (Boas-type formulae, extended to non-bandlimited funct.)

Let $s \in \mathbb{N}$, $f \in F^2$ and let $v^{2s-1}\widehat{f}(v) \in L^1(\mathbb{R})$. Then $f^{(2s-1)}$ exists and for $h > 0$, $\sigma := \pi/h$ there holds

$$f^{(2s-1)}(t) = \frac{1}{h^{2s-1}} \sum_{k\in\mathbb{Z}} (-1)^{k+1} A_{s,k} f\Big(t+h\Big(k-\frac{1}{2}\Big)\Big) + (R_{2s-1,\sigma}f)(t),$$

$$\big|(R_{2s-1,\sigma}f)(t)\big| \le \sqrt{\frac{2}{\pi}} \int_{|v|\ge\sigma} \big|v^{2s-1}\widehat{f}(v)\big|\, dv$$

$$= \sqrt{\frac{2}{\pi}}\, \mathrm{dist}_1(f^{(2s-1)}, B_\sigma^2).$$

The corollaries stated above for the remainder of the approximate sampling theorem are also valid for $R_{2s-1,\sigma}f$.

Theorem (Bernstein inequality)

For $f \in B_\sigma^p$, $1 \le p \le \infty$, $\sigma > 0$, there holds

$$\|f^{(s)}\|_{L^p(\mathbb{R})} \le \sigma^s \|f\|_{L^p(\mathbb{R})} \qquad (s \in \mathbb{N}).$$

Corollary (Bernstein inequality for trigonometric polynomials)

If $f(t) = t_n(t) = \sum_{k=-n}^{n} c_k e^{ikt}$, a trigonometric polynomial of degree n, then $t_n \in B_n^\infty$, and

$$\|t_n^{(s)}\|_{L^\infty(\mathbb{R})} \le n^s \|t_n\|_{L^\infty(\mathbb{R})} \qquad (s \in \mathbb{N}).$$

Theorem (Extended Bernstein inequality)

Let $s \in \mathbb{N}$, $f \in F^2$ and suppose that $v^s \widehat{f}(v) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ as a function of v. Then, for any $\sigma > 0$, we have

$$\|f^{(s)}\|_{L^2(\mathbb{R})} \le \sigma^s \, \|f\|_{L^2(\mathbb{R})} + \mathrm{dist}_2(f^{(s)}, B_\sigma^2)\,.$$

The proofs follow from the Boas-type formulae above.

The Hilbert transform

Hilbert transform or conjugate function of $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$, is defined by Cauchy principal value

$$\widetilde{f}(t) := \lim_{\delta \to 0+} \int_{|u|>\delta} \frac{f(t-u)}{u}\, du = \mathrm{PV} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t-u)}{u}\, du,$$

It defines a bounded linear operator from $L^2(\mathbb{R})$ into itself, and $[\widetilde{f}\,]\widehat{\ }(v) = (-i \operatorname{sgn} v)\widehat{f}(v)$ a. e. If $v^s \widehat{f}(v) \in L^1(\mathbb{R})$, then by Fourier inversion formula,

$$\left[\widetilde{f}\,\right]^{(s)}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(v)(-i \operatorname{sgn} v)(iv)^s e^{ivt}\, dv \qquad (t \in \mathbb{R}).$$

Thus $\left[\widetilde{f}\,\right]^{(s)} = \widetilde{f^{(s)}}$; i. e. derivation and taking Hilbert transform are commutative operations.

Since $\operatorname{sinc}\widehat{\ }(v) = 1/\sqrt{2\pi}$ for $|v| \le \pi$ and $= 0$ otherwise,

$$\operatorname{sinc}\widetilde{\ }(t) = \frac{1 - \cos \pi t}{\pi t} = \frac{\sin^2\left(\frac{\pi t}{2}\right)}{\frac{\pi t}{2}}.$$

Theorem (Boas-type formulae for the Hilbert transform)

Let $f \in B^2_\sigma$, $\sigma > 0$ *and* $h := \pi/\sigma$. *Then for* $s \in \mathbb{N}$,

$$\widetilde{f}^{(2s-1)}(t) = \frac{1}{h^{2s-1}} \sum_{k=-\infty}^{\infty} (-1)^{k+1} \widetilde{A}_{s,k}\, f(t+hk) \qquad (t \in \mathbb{R}).$$

$$\widetilde{A}_{s,0} := (-1)^s \frac{\pi^{2s-1}}{2s},$$

$$\widetilde{A}_{s,k} := \frac{(2s-1)!}{\pi k^{2s}} \left\{ (-1)^k - \sum_{j=0}^{s-1} \frac{(-1)^j}{(2j)!} (\pi k)^{2j} \right\} \quad (k \neq 0).$$

The proof is based on the sampling theorem for the Hilbert transform,

$$\widetilde{f}^{(2s-1)}(0) = \sum_{k=-\infty}^{\infty} f(k)\, \operatorname{sinc}^{\sim (2s-1)}(-k).$$

Theorem (Boas formulae for the Hilbert transform, extended vers.)

Let $s \in \mathbb{N}$, $f \in F^2$ and let $v^{2s-1}\widehat{f}(v) \in L^1(\mathbb{R})$. Then $f^{(2s-1)}$ exists and for $h > 0$, $\sigma := \pi/h$,

$$\widetilde{f}^{(2s-1)}(t) = \frac{1}{h^{2s-1}} \sum_{k\in\mathbb{Z}} (-1)^{k+1} \widetilde{A}_{s,k} f(t+hk) + (\widetilde{R}_{2s-1,\sigma} f)(t),$$

where

$$\left|(\widetilde{R}_{2s-1,\sigma} f)(t)\right| \le \frac{1}{\sqrt{2\pi}} \int_{|v|\ge\sigma} \left|v^{2s-1}\widehat{f}(v)\right| dv$$

$$= \frac{1}{\sqrt{2\pi}} \operatorname{dist}_1(f^{(2s-1)}, B_\sigma^2).$$

Proof

First assume $h = 1$, i. e., $\sigma = \pi$. Let

$$f_1(t) := \frac{1}{\sqrt{2\pi}} \int_{|v| \ge \pi} \widehat{f}(v) e^{itv} \, dv \, .$$

Then $f - f_1 \in B_\pi^\infty$ and so the bandlimited case applies to this difference, i. e.,

$$(\widetilde{R}_{2s-1,\pi} f)(t) = \big(\widetilde{R}_{2s-1,\pi}(f - f_1)\big)(t) + (\widetilde{R}_{2s-1,\pi} f_1)(t) = (\widetilde{R}_{2s-1,\pi} f_1)(t),$$

and we find that for $t = 0$,

$$(\widetilde{R}_{2s-1,\pi} f)(0) = \widetilde{f}_1^{(2s-1)}(0) - \sum_{k=-\infty}^{\infty} (-1)^{k+1} \widetilde{A}_{s,k} \, f_1(k) \, .$$

Proof, continued

$$(\widetilde{R}_{2s-1,\pi} f)(0) = \widetilde{f}_1^{(2s-1)}(0) - \sum_{k=-\infty}^{\infty} (-1)^{k+1} \widetilde{A}_{s,k}\, f_1(k)\,.$$

$$\widetilde{f}_1^{(2s-1)}(0) = \frac{1}{\sqrt{2\pi}} \int_{|v|\geq\pi} \widehat{f}(v)(-i \operatorname{sgn} v)(iv)^{2s-1}\, dv$$

$$f_1(k) = \frac{1}{\sqrt{2\pi}} \int_{|v|\geq\pi} \widehat{f}(v) e^{ikv}\, dv \qquad (k \in \mathbb{Z}).$$

Inserting the last two equations into the first one and interchanging summation and integration yields

$$(\widetilde{R}_{2s-1,\pi} f)(0) = \frac{1}{\sqrt{2\pi}} \int_{|v|\geq\pi} \widehat{f}(v) \Bigg[(-i \operatorname{sgn} v)(iv)^{2s-1} - \sum_{k=-\infty}^{\infty} (-1)^{k+1} \widetilde{A}_{s,k}\, e^{ikv} \Bigg]\, dv.$$

Proof, continued

The foregoing infinite series is a Fourier series of a 2π-periodic function and can be evaluated to be

$$\sum_{k=-\infty}^{\infty} (-1)^{k+1} \widetilde{A}_{s,k}\, e^{ikv} = (-1)^{s+1} |v|^{2s-1} \qquad (|v| < \pi).$$

$$\begin{aligned} &\left|(\widetilde{R}_{2s-1,\pi} f)(0)\right| \\ &= \left| \frac{1}{\sqrt{2\pi}} \int_{|v|\geq\pi} \widehat{f}(v) \Big[(-i \operatorname{sgn} v)(iv)^{2s-1} - (-1)^{s+1} \left[|v|^{2s-1} \right]^* \Big]\, dv \right| \\ &\leq \sqrt{\frac{2}{\pi}} \int_{|v|\geq\sigma} \left| v^{2s-1} \widehat{f}(v) \right| dv = \sqrt{\frac{2}{\pi}}\, \operatorname{dist}_1(f^{(2s-1)}, B_\sigma^2). \end{aligned}$$

$[|v|^{2s-1}]^*$ is the 2π-periodic extension of $|v|^{2s-1}$ from $(-\pi, \pi)$ to $\mathbb{R}$.

This is the desired inequality for $\sigma = \pi$, $h = 1$ and $t = 0$. For the general case apply this estimate to the function $u \mapsto f(t + hu)$. $\square$

Applications

Consider $g(t) := 1/(1+t^2)$, $t \in \mathbb{R}$, Fourier transform $\sqrt{\pi/2}\,\exp(-|v|)$, Hilbert transform $\widetilde{g}(t) = t/(1+t^2)$. Extended sampling theorem for Hilbert transform takes on concrete form for $\widetilde{g}\,'$,

$$\left| \frac{1-t^2}{(1+t^2)^2} \right.$$

$$\left. - \sum_{k=-\infty}^{\infty} \frac{\sigma^2}{\sigma^2+(k\pi)^2} \, \frac{\pi(\sigma t - k)\sin\big(\pi(\sigma t - k)\big) + \cos(\pi(\sigma t - k)) - 1}{\pi(\sigma t - k)^2} \right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{|v| \geq \sigma} \sqrt{\frac{\pi}{2}} |v| e^{-|v|} \, dv = (1+\sigma)e^{-\sigma} \qquad (\sigma > 0).$$

Truncation error

In practise one has to deal with finite sum, no infinite series. Leads to additional *truncation error*,

$$(T_{\sigma,N}f)(t)$$

$$= \sum_{|k|\geq N+1} \frac{\sigma^2}{\sigma^2+(k\pi)^2} \frac{\pi(\sigma t-k)\sin(\pi(\sigma t-k))+\cos(\pi(\sigma t-k))-1}{\pi(\sigma t-k)^2}$$

This yields for the truncation error

$$\left|(T_{\sigma,N}f)(t)\right| \leq \frac{\sigma^2(2\gamma+1)}{\pi^2(\gamma-1)} \sum_{|k|\geq N+1} \frac{1}{|k|^3}$$

$$\leq \frac{2\sigma^2(2\gamma+1)}{\pi^2(\gamma-1)} \int_N^\infty \frac{1}{u^3}\,du = \frac{\sigma^2(2\gamma+1)}{\pi^2(\gamma-1)} N^{-2}.$$

for $N \geq \gamma\sigma|t|$ and some constant $\gamma > 1$.

Combined error

Combining the aliasing error with truncation error we finally obtain

$$\left| \frac{1-t^2}{(1+t^2)^2} - \sum_{k=-N}^{N} \text{clear} \right| \leq (1+\sigma)e^{-\sigma} + \frac{\sigma^2(2\gamma+1)}{\pi^2(\gamma-1)} N^{-2}$$

$$(\sigma > 0;\, N \geq \gamma\sigma|t|).$$

Similarly, Boas-type theorem for derivative $\widetilde{g}'$ takes the form

$$\left| \frac{1-t^2}{(1+t^2)^2} - \left\{ \frac{\pi}{2h} \frac{1}{1+t^2} - \frac{1}{h} \sum_{k=-\infty}^{\infty} \frac{2}{\pi(2k+1)^2} \frac{1}{1+[t+(2k+1)h]^2} \right\} \right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{|v|\geq\sigma} \sqrt{\frac{\pi}{2}} |v| e^{-|v|}\, dv = (1+\sigma)e^{-\sigma} \qquad (\sigma > 0).$$

For the second order derivative $\widetilde{g}''$ one obtains

$$\left| \frac{2t^3 - 6t}{(1+t^2)^3} - \frac{1}{h^2} \sum_{k=-\infty}^{\infty} \frac{8\big(\pi - 2\pi k + 2(-1)^k\big)}{\pi(2k-1)^3} \, \frac{(-1)^k}{1 + [t + (2k+1)h]^2} \right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{|v| \geq \sigma} \sqrt{\frac{\pi}{2}} v^2 e^{-|v|} \, dv = (2 + 2\sigma + \sigma^2) e^{-\sigma} \qquad (\sigma > 0).$$

These are the aliasing errors for the reconstruction of derivatives of the Hilbert transform in terms of Boas-type formulae. In both cases the truncation errors can be handled in a similar fashion as above.

Some Boas-type formulae for bandlimited functions and their Hilbert transform

$$f'(t) = \frac{1}{h}\sum_{k\in\mathbb{Z}} \frac{(-1)^{k+1}}{\pi(k-\frac{1}{2})^2} f\Big(t+h\Big(k-\frac{1}{2}\Big)\Big),$$

$$f''(t) = -\frac{\pi^2}{3h^2}f(t) + \frac{2}{h^2}\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} f(t+hk)\frac{(-1)^{k+1}}{k^2},$$

$$f^{(3)}(t) = \frac{1}{h^3}\sum_{k=-\infty}^{\infty} \frac{(-1)^{k+1}6}{\pi(\frac{1}{2}-k)^4}\left[1-\frac{\pi^2}{2}\Big(\frac{1}{2}-k\Big)^2\right] f\Big(t+h\Big(k-\frac{1}{2}\Big)\Big),$$

$$\widetilde{f}\,'(t) = \frac{\pi}{2h}f(t) + \frac{1}{h}\sum_{k=-\infty}^{\infty} \frac{-2}{\pi(2k+1)^2} f\big(t+h(2k+1)\big),$$

$$\widetilde{f}\,''(t) = \frac{1}{h^2}\sum_{k=-\infty}^{\infty} (-1)^{k+1}\frac{2\big[(-1)^k + \big(\pi\big(k-\frac{1}{2}\big)\big]}{\pi\big(k-\frac{1}{2}\big)^3} f\Big(t+h\Big(k-\frac{1}{2}\Big)\Big).$$

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