DISCRETE DIRAC OPERATORS AND HARMONIC ANALYSIS

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Complex and Hypercomplex Analysis

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INTRODUCTION

- Broad theory in complex case: Ferrand, Lovasz, Kenyon, Bobenko, Mercat, Stephenson, Smirnov, ...
- Several variables approach: Bobenko/Mercat/Suris, discrete holomorphic functions on bricks
- First approach in Clifford analysis: Gürlebeck/Sproessig, operator theory/potential theory
- First „complete" theory in Clifford analysis by N. Faustino
- Several works in different directions (P. Leopardi, A. Stern - Dirac operators based on discrete exterior derivative, H. de Ridder, F. Sommen - Dirac operator on (infinite) square lattice)
- Principal problem in discrete Clifford analysis: How to construct a Weyl-Heisenberg (-type) structure?

DISCRETE LAPLACIAN

- Given by Kirchhoff's Law:

$$\Delta H = \sum_{v \text{ neighbour of } u} H(v) - H(u)$$

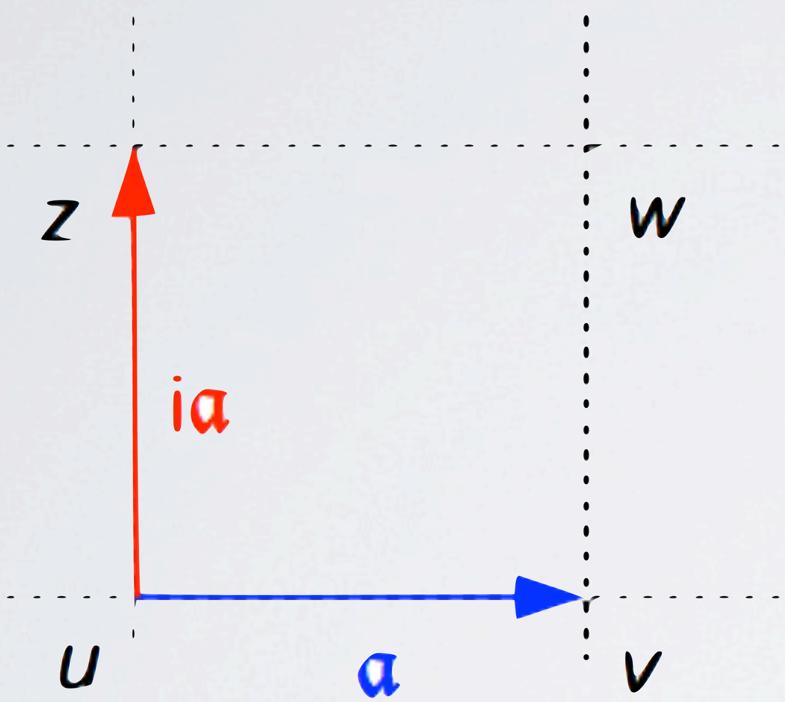
- Weighted case:

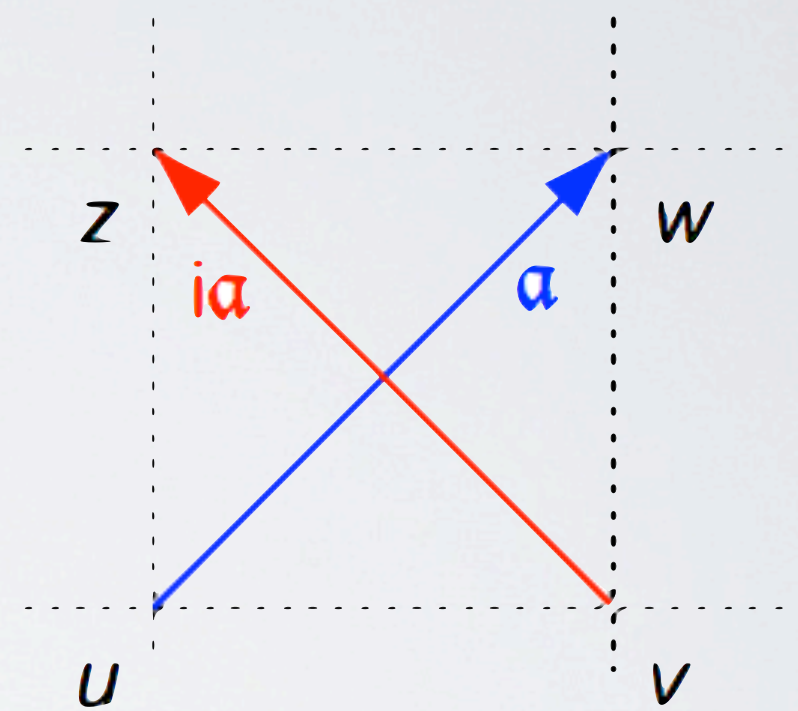
$$\Delta H = \sum_{v \text{ neighbour of } u} \omega_{uv}(H(v) - H(u))$$

- Discrete holomorphic functions: Isaacs 1941, Ferrand 1944, Duffin 1956

DISCRETE ANALYTIC FUNCTIONS

$$\partial_\alpha F = i\partial_{i\alpha} F$$





$$\frac{F(z)-F(u)}{z-u} = \frac{F(v)-F(u)}{v-u}$$

$$F(z)-F(u) = i(F(v)-F(u))$$

$$\frac{F(z)-F(v)}{z-v} = \frac{F(w)-F(u)}{w-u}$$

$$F(z)-F(v) = i(F(w)-F(u))$$

DISCRETE ANALYTIC FUNCTIONS ON SQUARE LATTICE

- F ± G ∈ DANAL
- Derivative F' is well-defined and ∈ DANAL
- Primitive $\int^z F$ is well-defined and ∈ DANAL
- $\int F = 0$ for closed contours
- Maximum principle
- $F = H + i\bar{H}$ ⇒ H discrete harmonic (on even sublattice)
- BUT: F, G ∈ DANAL does NOT imply F · G ∈ DANAL !

DISCRETE DIFFERENTIAL FORMS

- Construction goes back to Dimakis/Müller-Hoissen 1994
- Given a $\mathbb{Z}$ −graded associative algebra: $\Lambda^* \mathcal{A} = \sum_{r \geq 0} \Lambda^r \mathcal{A}$
- Given a linear map satisfying the graded Leibniz rule:

$$\mathbf{d} : \Lambda^r \mathcal{A} \to \Lambda^{r+1} \mathcal{A},$$
$$\mathbf{d}(\mathbf{d}\omega_r) = 0,$$
$$\mathbf{d}(\omega_r \omega_s) = (\mathbf{d}\omega_r)\omega_s + (-1)^r \omega_r \mathbf{d}\omega_s,$$

- Basis generated by using discrete delta functions: $b_l(m) = \delta_{lm}$

USING THE ADDITIVE STRUCTURE OF THE LATTICE - TRANSLATION OPERATORS

- Assume lattice equipped with group action ***l+m=p*** (Cayley graphs)

- Introduce translation operator:

$$T_l f = \sum_m f_{m+l} b_m \quad = \quad \sum_m f_m b_{l-m}$$

- New representation of 1-forms:

$$\Theta^l = \sum_m b_{m,m+l}, \quad \text{if } \textstyle\sum_m \Theta^m b_l \neq 0$$

DISCRETE DIFFERENTIAL FORMS AND EXTERIOR DERIVATIVE

- In terms of new representation *n*-forms are given by:

$$\omega_r = \sum_{j=0}^{r} \sum_{m^j} F_{m^0,m^1,\ldots,m^r} \Theta^{m^1-m^0} \ldots \Theta^{m^r-m^{r-1}}$$

- Action of **d** on *n*-forms:

- $$\mathbf{d}f = \sum_{l,m} f_l(b_{m,l} - b_{l,m}) = \sum_{l} (T_l f - f)\Theta^l$$

LINK WITH GRAPH STRUCTURE

- Idea of lattice reduction by Dimakis/Müller-Hoissen 1994
- Consider a cell complex indexed by set of vectors $\{v_j\}$
- Lattice reduction:

$$b_{m,p} \begin{cases} \neq 0 & \text{if } p = m + v_j,\ j \in [n] \cup [n]' \\ = 0 & \text{otherwise} \end{cases}$$

- 2*n*-nonzero differentials:

$$\Theta^{v_j} = \sum_{m \in \mathcal{L}} b_{m,m+v_j}, \quad j \in [n] \cup [n]'$$

- Translations $m + v_j$ and $m + v_{j'}$ are symmetric!

COMING BACK TO SQUARE LATTICE - FORWARD/BACKWARD DIFFERENCES

- Introducing directions by split: $G = \overleftarrow{G} + \overrightarrow{G}$
- Introducing local coordinates: $dx_j = [G, x_j]$
- Foward/backward difference action:

$$(\partial_h^{+j} f)(x) = \frac{1}{h}(\partial^{v_j} f)(x), \quad (\partial_h^{-j} f)(x) = -\tfrac{1}{h}(\partial^{v_{j'}} f)(x), \quad \forall f \in \mathcal{A}$$

- Switch to operators: $\nabla_h^j = \frac{1}{2}(\partial_h^{-j} + \partial_h^{+j}), \quad \tilde{\nabla}_h^j = \frac{1}{2i}(\partial_h^{-j} - \partial_h^{+j})$
- Discrete Dirac operator: $\partial_{X|} := -i(\partial_z + \partial_z^\dagger) \quad = \quad \sum_{j=1}^{n} e_j \tilde{\nabla}_h^j - e_{j+n} \nabla_h^j$

DISCRETE DIRAC OPERATORS

- N. Faustino, U. Kähler, F. Sommen, Discrete Dirac operators in Clifford analysis, Adv. Appl. Cliff. Alg. 17 (3) (2007), 451-467
- E. Forgy, U. Schreiber, Discrete differential geometry on causal graphs, preprint (2004), arXiv:math-ph/0407005v1
- I. Kanamori, N. Kawamoto, Dirac-Kähler Fermion from Clifford Product with Noncommutative Differential Form on a Lattice, Int. J. Mod. Phys. A19 (2004), 695-736
- K. Gürlebeck, A. Hommel, On Finite Difference Potentials and their Applications in a Discrete Function Theory, Math. Meth. Appl. Sci. 25 (2002), 1563-1576
- J. Vaz, Clifford-like calculus over lattices, Adv. Appl. Cliff. Alg. 7 (1) (1997), 37-70.

THREE ESSENTIAL APPROACHES TO THE INFINITE LATTICE

- One difference operator (does not give Star-Laplacian): **Pontryagin duality**
- Central difference operator (does only give Star-Laplacian on even sublattice): **Sheffer map/Weyl algebra**
- Forward/backward difference operator (does give Star-Laplacian): **Sommen-Weyl relations**

FIRST CASE - „PURE APPROACH“ USING ONE DIFFERENCE OPERATOR

- Using only forward difference operator
- Corresponds to representation theory of unitary group on a torus (periodic functions, phase space $\mathbb{T}^n \times \mathbb{Z}^n$):

$$\partial^{+j} \mathcal{F}_h f = \mathcal{F}_h((e^{i2\pi x_j} - 1)f)$$

- Weyl relation:

$$[\partial^{+j}, X_j^+] = 1 \text{ with } X_j^+ = x_j T_j^+$$

- Taylor series given by Newton interpolation

THIRD CASE - DISCRETE DIRAC OPERATOR BASED ON FORWARD AND BACKWARD DIFFERENCES

- Basis elements (Witt basis):

$$e_j^- e_k^- + e_k^- e_j^- = 0$$
$$e_j^+ e_k^+ + e_k^+ e_j^+ = 0$$
$$e_j^+ e_k^- + e_k^- e_j^+ = \delta_{jk}$$

- Dirac operators:

$$D^+ = \sum_j (e^{+j}\partial^{+j} + e^{-j}\partial^{-j}) \quad D^- = \sum_j (e^{+j}\partial^{-j} + e^{-j}\partial^{+j})$$

WEYL RELATIONS FOR DISCRETE DIRAC OPERATOR ON INFINITE LATTICE

- Classic Weyl relations

$$\partial^{+j} X_j^+ - X_j^+ \partial^{+j} = 1$$

$$\partial^{-j} X_j^- - X_j^- \partial^{-j} = 1$$

- Forward/backward differences are treated independently

$$X_j^+ f = x_j T_j^+ f, X_j^- f = x_j T_j^- f$$

- No commutativity

$$[X_j^+, X_j^-] = 2x_j$$

- Sommen-Weyl relations

$$\partial^{+j} X_j^+ - X_j^- \partial^{-j} = 1$$

$$\partial^{-j} X_j^- - X_j^+ \partial^{+j} = 1$$

- No direct independence between forward/backward differences

- X^2 is not a scalar, but a „skew“ scalar, i.e. still creates a one-dimensional space

REMARKS ON BOTH APPROACHES

- Classic Weyl relations: Easy expression of position operators, but complicated algebraic structure (no direct Weyl-Heisenberg structure)
- N. Faustino: Finite Difference operators with potentials
- Sommen-Weyl relations: No explicit expression of position operators, but given by recursive action on polynomials, easy realization of sl_2 /osp(1|2)

BASIC RELATIONS IN SECOND CASE

$$E = \sum e_j^+ e_j^- X_j^- \Delta_j^- + e_j^- e_j^+ X_j^+ \Delta_j^+$$

$$DX + XD = 2E + m$$

$$EX = XE + X$$

$$DE = ED + D$$

$$\Gamma = \frac{XD - DX + m}{2}$$

CALCULATION OF $X_j^{\pm}$

- Weyl relations are used as defining relations for $X_j^{\pm}$
- Basic polynomials are obtained by letting $X_j^{\pm}$ act on ground state $\Rightarrow$ Rodriguez formula
- Given in terms of coefficients of even Euler polynomials
- For details see: de Schepper, de Ridder, K., Sommen, *Discrete function theory based on skew Weyl relations*, Proc. AMS 2010

DISCRETE DIRAC OPERATORS ON FINITE LATTICE -SOME REMARKS ON THE FOURIER TRANSFORM

- Continuous Fourier transform maps delta functionals into exponentials: $\mathcal{F}(\delta(x-y)) = e^{i2\pi y\xi}$

- Finite Fourier transform:

$$\mathcal{F}_h u(l) = \frac{1}{d^{n/2}} \sum_{m:m_i\in\{0,\ldots,d-1\}} e^{i2\pi\frac{kl}{d^n}} u(m)$$

- Finite Fourier transforms map discrete delta functions into exponentials $\mathcal{F}_h \delta_k(l) = \frac{1}{d^{n/2}} e^{i2\pi\frac{kl}{d^n}}$

- Eigenvalues of finite Fourier transform: 1,-1,i,-i

FINITE „POSITION STATES“ AND „MOMENTUM STATES“ FOLLOWING IDEAS OF A. VOURDAS

- Denote integers modulo *d by* $\mathbb{Z}(d)$

- Consider an orthonormal basis of „position states“:

$$\{x_m\} : m \in (\mathbb{Z}(d))^n$$

- „Momentum states“: $\{p_m : p_m = \mathcal{F}_h x_m\}$

- „Position and momentum operators“:

$$X x_m = m x_m, P p_m = m p_m, \mathcal{F}_h X \mathcal{F}_h^\dagger = P, \mathcal{F}_h P \mathcal{F}_h^\dagger = -X$$

- *X* and *P* are finite-dimensional operators!

DISCRETE DIRAC-TYPE OPERATOR

- Extension to Clifford-Weyl algebra: $\mathrm{Alg}\{X_m, P_n, e_j\}$
- Dirac-type operator: $D = \sum_{j=0}^{n} e_j P_j$
- Dirac-type operator is finite dimensional
- D^d not independent of lower powers (Theorem of Cayley-Hamilton)
- Commutator: $[D, X] \neq I$ (but not really important here)

DISPLACEMENT OPERATORS

- Phase space: toroidal lattice
- Displacement operators:

$$\mathcal{X} = \exp(i\frac{2\pi}{d}X) \qquad \mathcal{D} = \exp(i\frac{2\pi}{d}D)$$

- Basic properties:

$$\mathcal{X}x_m = \left[e^{i\frac{2\pi}{d}|m|}\chi_+(m) + e^{-i\frac{2\pi}{d}|m|}\chi_-(m)\right]x_m$$

$$\mathcal{D}p_m = \left[e^{i\frac{2\pi}{d}|m|}\chi_+(m) + e^{-i\frac{2\pi}{d}|m|}\chi_-(m)\right]p_m$$

$$\mathcal{X}^d = \mathcal{D}^d = I$$

- Commutation property: $\mathcal{D}^\alpha\mathcal{X}^\beta = (e^{i\frac{2\pi n}{d}\alpha\beta})\mathcal{X}^\beta\mathcal{D}^\alpha$

CONNECTION WITH HEISENBERG-WEYL GROUP

- General displacement operators: $D(\alpha, \beta) = \mathcal{X}^\alpha \mathcal{D}^\beta e^{-\frac{n}{2}\alpha\beta}$
- Heisenberg-Weyl group is discrete => NO Lie algebra
- Instead of calculus by D, X use $\mathcal{D}, \mathcal{X}$

$$D(\alpha, \beta) X [D(\alpha, \beta)]^\dagger = X - \beta I$$
$$D(\alpha, \beta) D [D(\alpha, \beta)]^\dagger = D - \alpha I$$

- Symplectic transformations: $\mathcal{D}' = S\mathcal{D}S^\dagger = D(\lambda, \kappa)$

$$\mathcal{X}' = S\mathcal{X}S^\dagger = D(\nu, \mu)$$

MODIFICATIONS FOR GENERAL DIFFERENCE OPERATORS

- Start with definition of P, P_j
- Position operator defined by $X_j = -\mathcal{F}_h P_j \mathcal{F}_h$
- Commutating relation: $X_j^\alpha P_j^\beta = -\mathcal{F}_h P_j^\beta X_j^\alpha \mathcal{F}_h$
- Position operator to discrete Dirac operator

$$X = \mathcal{F}_h[\sum_j e_j^-(e^{-i2\pi x_j} - 1) + e_j^+(1 - e^{i2\pi x_j})]\mathcal{F}_h^\dagger$$

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THE END