Short-time Fourier transform for quaternionic signals

Joint work with Y. Fu and U. Kähler

P. Cerejeiras

Departamento de Matemática
Universidade de Aveiro
pceres@ua.pt

New Trends and Directions in Harmonic Analysis,
Fractional Operator Theory, and Image Analysis - Inzell 2012

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Complex and Hypercomplex Analysis

Frames

Definition

A discrete system $\{\psi_j, j \in J\}$ is a frame for a Hilbert space $\mathcal{H}$ if there exist $0 < A \leq B < \infty$ such that

$$A||f||^2 \leq \sum_{j\in J} |\langle f, \psi_j\rangle|^2 \leq B||f||^2$$

for all $f \in \mathcal{H}$.

Definition

We define the analysis operator as the linear operator $F : \mathcal{H} \to \ell_2(J)$, where $Ff = c$, with $c_j = \langle f, \psi_j\rangle$.

- Its synthesis operator F^* is given by $F^*c = \sum_{j\in J} c_j\psi_j$;
- The frame operator is given by $FF^* : \mathcal{H} \to \mathcal{H}$.

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Frames

Then one has

- a good characterization of f :

$$\langle f, \psi_j \rangle = 0, \ \forall j \in J \Rightarrow f = 0;$$

- a good reconstruction scheme for f :

$$f = \sum_{j \in J} \langle f, \psi_j \rangle \tilde{\psi}_j = \sum_{j \in J} \langle f, \tilde{\psi}_j \rangle \psi_j.$$

1. The frame is said to be tight if $A = B$;
2. If the frame is tight and $A = 1$ then the frame is orthogonal;
3. The frame is said to be exact if after redrawing an arbitrary ψ_{j_0} from the system it is no longer a frame.

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Windowed Fourier transform

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For a given window $g \in L_2(\mathbb{R})$ we define the windowed Fourier transform of $f \in L_2(\mathbb{R})$ as

$$\mathcal{F}^{win} f(t, \omega) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi\omega i x} g(x - t) dx.$$

Problem

For discretization of the WFT what choice of ω_0, t_0 such that

- the signal is characterized by its coefficients?
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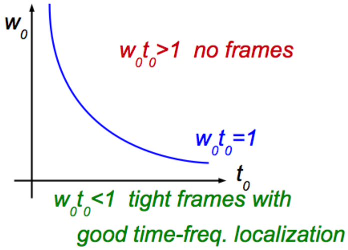
Discrete windowed Fourier transform



Lemma

For a given window g and its dual window $\tilde{g}$ in M^1 we have that the Gabor frame operator converges in $L_2(\mathbb{R})$.

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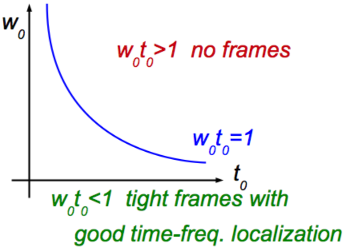

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The Balian-Low Theorem

The **Balian-Low theorem** expresses the fact that time-frequency concentration and non-redundancy are incompatible properties for Gabor systems

$$g_{m,n} = e^{-i2\pi m\cdot} g(\cdot - n), \;\; n, m \in \mathbb{Z},$$

that is to say, either

$$\int_{-\infty}^{+\infty} x^2 |g(x)|^2 dx = \infty$$

or

$$\int_{-\infty}^{+\infty} \xi^2 |\hat{g}(\xi)|^2 d\xi = \infty.$$

The Zak transform - 1

The Zak transform (J. Zak, 1967) is an important tool for studying the frame given by Gabor systems.

Definition

For $f \in L_2(\mathbb{R})$, we have

$$(Zf)(t,\omega) = \sum_{k\in\mathbb{Z}} e^{2\pi i k\omega} f(t-k), \quad (t,\omega) \in [0,1)^2.$$

- It defines a unitary operator from $L^2(\mathbb{R})$ to $L^2([0,1)^2)$. In abstract harmonic analysis the Zak transform is called the Weil-Brezin map;
- The harmonic waves $e^{2\pi i n\omega}$, $n \in \mathbb{Z}$ in the Zak transform have constant frequencies, which can be seen as the derivative of the linear phase $\phi(\omega) = 2\pi n\omega$;

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The Zak transform -2

- The nontrivial harmonic waves $e^{i\theta_a(2\pi\omega)}$ have positive time-varying frequencies and are expected to be better suitable and adaptable, along with different choices of a, to nonlinear and non-stationary time-frequency analysis.

Analytic signals

Applying to a given signal $f(x)$ the Hilbert transform

$$Hf(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{f(y)}{y-x} dy, y \in \mathbb{R}.$$

we obtain the complex-valued function

$$F(x) = f(x) + iHf(x) = a(x)e^{i\theta(x)}$$

- $F(x)$ is the analytic signal, $a(x)$ the amplitude and $\theta(x)$ the phase;
- $\omega(x) = \theta'(x)$ is called instantaneous frequency;
- The pair (a, θ) is called canonical modulation pair of $f(x)$.

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Analytic signals - Mathematical background

Riemann-Hilbert problem:

$$\begin{aligned} \frac{\partial F}{\partial \bar{z}} &= 0 && \text{in } \mathrm{Im}(z) > 0 \\ \mathrm{Re}F(z) &= f(x) && \text{in } \mathrm{Im}(z) = 0 \\ \mathrm{Im}F(z_0) &= c && \text{in } \mathrm{Im}(z_0) > 0 \end{aligned}$$

- Cauchy integral transform:

$$F(z) := Cf(z) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\zeta - z} f(\zeta) d\zeta$$

- Plemelj-Sokhotzki formula:

$$\mathrm{tr}^{\pm} Cf(z_0) = \lim_{z \to z_0} Cf(z) = \frac{1}{2} \left[f(z_0) \pm \frac{1}{\pi i} \int_\Gamma \frac{1}{\zeta - z_0} f(\zeta) d\zeta \right]$$

Complex and Hypercomplex Analysis

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Why does it work for signal analysis?

- Hardy decomposition: $L_p(\Gamma) = H_p^+(\Gamma) \oplus H_p^-(\Gamma)$;
- Poisson kernel
$$P(z,\zeta) = \mathrm{Re}\frac{1}{\zeta - z_0}.$$
- Basic idea: Poisson kernel is also low-pass filter!
- Scale-space analysis by Gaussian kernel can be replaced by scale-space analysis using Poisson kernel;
- $f(x) = a(x)e^{i\theta(x)}$;
- Weyl-relation $[x, D] = i$ with $D = -i\partial_x$
- $< \omega >= \int \omega |\mathcal{F}f(\omega)|^2 d\omega = \int \theta'(x) a(x)^2 dx$

Extension to $\mathbb{C}^2$

- Riemann-Hilbert problem

$$
\begin{aligned}
\frac{\partial F}{\partial \overline{z}_1} &= 0 \qquad (z_1, z_2) \in \mathbb{C}^2, |z_1| < 1 \ \& \ |z_2| < 1, \\
\frac{\partial F}{\partial \overline{z}_2} &= 0 \qquad (z_1, z_2) \in \mathbb{C}^2, |z_1| < 1 \ \& \ |z_2| < 1, \\
\mathrm{Re} F(\xi_1, \xi_2) = f(\xi_1, \xi_2) & \qquad |\xi_1| = 1, |\xi_2| = 1.
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- Solution:

$$F(z_1, z_2) := Cf(z_1, z_2) = \frac{1}{4\pi^2} \int_{T^2} \frac{1}{(\xi_1 - z_1)(\xi_2 - z_2)} f(\xi_1, \xi_2) d\xi_1 d\xi_2.$$

Complex and Hypercomplex Analysis

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Hypercomplex signal by T. Bülow

- Two different imaginary units: $z_1 = x_1 + iy_1$ and $\mathbf{z}_2 = x_2 + jy_2$
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- Plemelj-Sokhotzki formula (just one side):

$$\begin{aligned} \mathrm{tr}Cf(x_1, x_2) &= \frac{1}{4}(I + iH_1)(I + jH_2)f(x_1, x_2) \\ &= \frac{1}{4}(f + iH_1 f + jH_2 f + kHf)(x_1, x_2). \end{aligned}$$

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Quaternionic algebra

Quaternion algebra $\mathbb{H}$

The quaternion algebra is an extension of complex numbers to a 4D algebra. Every element of $\mathbb{H}$ is a linear combination of a real scalar and three orthogonal imaginary units (denoted by $\mathbf{i}, \mathbf{j}, \mathbf{k}$) with real coefficients

$$\mathbb{H} = \{q \ : \ q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3, q_0, \ \ q_1, q_2, q_3 \in \mathbb{R}\},$$

where the elements $\mathbf{i}, \mathbf{j}, \mathbf{k}$ obey the Hamilton's multiplication rules

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

The scalar part is denoted as $Sc(q) = q_0$ and has the cyclic multiplication symmetry

$$Sc(qrs) = Sc(rsq), \forall q, r, s \in \mathbb{H}.$$

$\mathbb{H}$-valued function space

$\mathbb{H}$-**conjugation** of a given $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 = Sc(q) + Vec(q)$ is

$$\overline{q} = Sc(q) - Vec(q).$$

Consider the quaternion-valued function space $L_2(\mathbb{R}^2; \mathbb{H})$ equipped with the quaternionic-valued inner product

$$(f, g) := \int_{\mathbb{R}^2} f(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x}.$$

Additionally, consider also the complex-valued inner product

$$\langle f, g \rangle := Sc(f, g) = \int_{\mathbb{R}^2} Sc[f(\mathbf{x})\overline{g(\mathbf{x})}]d\mathbf{x}.$$

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Quaternionic windowed Fourier transform

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Given a 2D quaternion-valued signal $f \in L_2(\mathbb{R}^2, \mathbb{H})$, we defined its quaternionic Fourier transform as

$$\mathcal{F}_{\mathbb{H}}[f](\mathbf{w}) = \int_{\mathbb{R}^2} e^{-2\pi \mathbf{i} x_1 \omega_1} f(\mathbf{x}) e^{-2\pi \mathbf{j} x_2 \omega_2} d\mathbf{x}, \quad \mathbf{w} = (\omega_1, \omega_2), \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2.$$

Definition

Given a windowed function $g \in L_2(\mathbb{R}^2, \mathbb{R})$, we set the windowed quaternionic Fourier transform (WQFT) of a 2D quaternion-valued signal $f \in L_2(\mathbb{R}^2, \mathbb{H})$ as

$$\mathcal{Q}_g[f](\mathbf{w}, \mathbf{b}) = \int_{\mathbb{R}^2} e^{-2\pi \mathbf{i} x_1 \omega_1} f(\mathbf{x}) g(\mathbf{x} - \mathbf{b}) e^{-2\pi \mathbf{j} x_2 \omega_2} d\mathbf{x}.$$

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Multiplication operator

Aim: to express the kernel of the WQFT!

Solution: to introduce the *multiplication* operator.

For $f, g \in L_2(\mathbb{R}^2, \mathbb{H})$, we define their pointwise product as

$$\mathcal{C}[f, g](\mathbf{x}) = f(\mathbf{x})g(\mathbf{x}).$$

Based on this, we define the left and right multiplicative operators, $\mathcal{C}[\lambda, \cdot]$, resp. $\mathcal{C}[\cdot, \lambda]$, as

$$\mathcal{C}[\lambda, \cdot]f(\mathbf{x}) := f(\mathbf{x})\lambda, \quad \textit{resp. } f(\mathbf{x})\mathcal{C}[\cdot, \lambda] := \lambda f(\mathbf{x}).$$

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$$\overline{\mathcal{C}[\lambda,\cdot]} := \mathcal{C}[\cdot,\overline{\lambda}] \quad \text{and} \quad \overline{\mathcal{C}[\cdot,\lambda]} := \mathcal{C}[\overline{\lambda},\cdot].$$

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For a windowed function $g \in L_2(\mathbb{R}^2, \mathbb{R})$, then

$$g_{\mathbf{w},\mathbf{b}}(\mathbf{x}) = e^{2\pi \mathbf{j} x_2 \omega_2} g(\mathbf{x}-\mathbf{b}) \mathcal{C}[e^{2\pi \mathbf{i} x_1 \omega_1}, \cdot]$$

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Consequences

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- $f(\mathbf{x})\overline{g_{\mathbf{w},\mathbf{b}}} = g_{\mathbf{w},\mathbf{b}}\overline{f(\mathbf{x})}$; so that

$$(f, g_{\mathbf{w},\mathbf{b}}) := \int_{\mathbb{R}^2} f(\mathbf{x})\overline{g_{\mathbf{w},\mathbf{b}}} d^2\mathbf{x}$$

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$$\mathcal{Q}_g[f](\mathbf{w}, \mathbf{b}) = (f, g_{\mathbf{w},\mathbf{b}}).$$

Theorem [Reconstruction formula]

For $g \in L^2(\mathbb{R}^2; \mathbb{R})$ non-zero real-valued window function, then every $f \in L^2(\mathbb{R}^2; \mathbb{H})$ can be reconstructed via

$$f(\mathbf{x}) = \frac{1}{\|g\|_{L^2}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{2\pi \mathbf{i} x_1 \omega_1} \mathcal{Q}_g[f](\mathbf{w}, \mathbf{b}) g(\mathbf{x} - \mathbf{b}) e^{2\pi \mathbf{j} x_2 \omega_2} d^2\mathbf{w} d^2\mathbf{b}.$$

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Then

- $f(\mathbf{x})\overline{g_{\mathbf{w},\mathbf{b}}} = g_{\mathbf{w},\mathbf{b}}\overline{f(\mathbf{x})}$; so that

$$(f, g_{\mathbf{w},\mathbf{b}}) := \int_{\mathbb{R}^2} f(\mathbf{x})\overline{g_{\mathbf{w},\mathbf{b}}} d^2\mathbf{x}$$

- the WQFT can be write as

$$\mathcal{Q}_g[f](\mathbf{w}, \mathbf{b}) = (f, g_{\mathbf{w},\mathbf{b}}).$$

Theorem [Reconstruction formula]

For $g \in L^2(\mathbb{R}^2; \mathbb{R})$ non-zero real-valued window function, then every $f \in L^2(\mathbb{R}^2; \mathbb{H})$ can be reconstructed via

$$f(\mathbf{x}) = \frac{1}{\|g\|_{L^2}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{2\pi \mathbf{i} x_1 \omega_1} \mathcal{Q}_g[f](\mathbf{w}, \mathbf{b}) g(\mathbf{x} - \mathbf{b}) e^{2\pi \mathbf{j} x_2 \omega_2} d^2\mathbf{w} d^2\mathbf{b}.$$

Expansion

Moreover, if the system $\{1\overline{g_{-m,n}} : \quad m,n \in \mathbb{Z}^2\}$ is a Gabor orthonormal basis, a function $f \in L^2(\mathbb{R}^2;\mathbb{H})$ admits the expansion

$$f(\mathbf{x}) = \sum_{m,n\in\mathbb{Z}^2} c_{m,n}(1\overline{g_{-m,n}})(\mathbf{x})$$

with $c_{m,n} = \langle f, 1\overline{g_{-m,n}}\rangle$.

This expansion can be regarded as the discrete form of the previous continuous reconstruction formula.

Quaternionic Zak transform

Definition

Given a function $f \in L_2(\mathbb{R}^2, \mathbb{H})$, we set its quaternionic Zak transform as

$$Z_{\mathbb{H}}[f](\mathbf{x}, \mathbf{w}) = \sum_{\mathbf{k} \in \mathbb{Z}^2} e^{2\pi \mathbf{j} k_2 \omega_2} f(\mathbf{x} - \mathbf{k}) e^{2\pi \mathbf{i} k_1 \omega_1}, \quad \mathbf{x}, \mathbf{w} \in [0,1)^2.$$

Properties

1. $Z_{\mathbb{H}}[f]$ is a well defined function in $L_2([0,1)^2 \times [0,1)^2, \mathbb{H})$;
2. $Z_{\mathbb{H}}[f](\mathbf{x} + \mathbf{n}, \mathbf{w}) = e^{2\pi \mathbf{j} n_2 \omega_2} Z_{\mathbb{H}}[f](\mathbf{x}, \mathbf{w}) e^{2\pi \mathbf{i} n_1 \omega_1}$, and $Z_{\mathbb{H}}[f](\mathbf{x}, \mathbf{w} + \mathbf{n}) = Z_{\mathbb{H}}[f](\mathbf{x}, \mathbf{w})$, where $\mathbf{n} \in \mathbb{Z}^2$;
3. $Z_{\mathbb{H}}$ is a unitary operator from $L_2(\mathbb{R}^2, \mathbb{H})$ to $L_2([0,1)^2 \times [0,1)^2, \mathbb{H})$, i.e.,

$$\langle Zf, Zg \rangle = \langle f, g \rangle;$$

Complex and Hypercomplex Analysis

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Quaternionic Zak transform

Properties (cont.)

4.
$$f(\mathbf{x}) = \int_{[0,1)^2} Z_{\mathbb{H}}[f](\mathbf{x},\mathbf{w})d\mathbf{w}, \quad \mathbf{x} \in \mathbb{R}^2;$$

5.
$$\mathcal{F}_{\mathbb{H}}[\bar{f}](-\mathbf{w}) = \int_{[0,1)^2} e^{2\pi \mathbf{i} n_1\omega_1} \overline{Z_{\mathbb{H}}[f](\mathbf{x},\mathbf{w})} e^{2\pi \mathbf{j} n_2\omega_2} d\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^2;$$

Space $\mathcal{Z}$ of all $\phi : \mathbb{R}^2 \to \mathbb{H}$ such that

$$\phi(\mathbf{x}+\mathbf{n},\mathbf{w}) = e^{2\pi \mathbf{j} n_2\omega_2}\phi(\mathbf{x},\mathbf{w})e^{2\pi \mathbf{i} n_1\omega_1}, \phi(\mathbf{x},\mathbf{w}+\mathbf{n}) = \phi(\mathbf{x},\mathbf{w}), \quad \forall \mathbf{n} \in \mathbb{Z}^2,$$

$$||\phi||^2 = \int_{[0,1)^2}\int_{[0,1)^2} |\phi(\mathbf{x},\mathbf{w})|^2 d\mathbf{x}d\mathbf{w} < \infty.$$

Quaternionic Zak transform

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Complex and Hypercomplex Analysis

Consequences

The quaternion Zak transform $Z_{\mathbb{H}}$ is an unitary map between $L_2(\mathbb{R}^2; \mathbb{H})$ and $\mathcal{Z}$.

We know the inverse map as well: for every $\phi \in \mathcal{Z}$,

$$Z_{\mathbb{H}}^{-1}\phi(\mathbf{x}) = \int_{[0,1)^2} \phi(\mathbf{x}, \mathbf{w}) d\mathbf{w}.$$

Lemma

Given $f \in L_2(\mathbb{R}^2, \mathbb{H})$, and a Gabor system $\{1\overline{g_{-\mathbf{m},\mathbf{n}}}, \mathbf{m}, \mathbf{n} \in \mathbb{Z}^2\}$ we have

$$\sum_{\mathbf{m},\mathbf{n}\in\mathbb{Z}^2} |\langle f, g_{\mathbf{m},\mathbf{n}}\rangle|^2 = ||Z_{\mathbb{H}}[f]\overline{Z_{\mathbb{H}}[1\overline{g_{-\mathbf{m},\mathbf{n}}}]}||^2.$$

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Link with Weyl-Heisenberg algebra

- Annihilation/creation operators:

$$\mathbf{a}_1 = \frac{1}{2\pi}\partial_{x_1} + x_1, \mathbf{a}_1^\dagger = -\frac{1}{2\pi}\partial_{x_1} + x_1, \quad \mathbf{a}_2 = \frac{1}{2\pi}\partial_{x_2} + x_2, \mathbf{a}_2^\dagger = -\frac{1}{2\pi}\partial_{x_2} + x_2$$

- Link with Zak transform:

$$Z_{\mathbb{H}}[\mathbf{a}_2 f](\mathbf{x}, \mathbf{w}) = A_2 Z_{\mathbb{H}}[f](\mathbf{x}, \mathbf{w}), Z_{\mathbb{H}}[f\mathbf{a}_1](\mathbf{x}, \mathbf{w}) = Z_{\mathbb{H}} A_2[f](\mathbf{x}, \mathbf{w})$$

where $A_1 = \frac{1}{2\pi i}(\partial_{\omega_1} + i\partial_{x_1}) + x_1$, $A_2 = \frac{1}{2\pi j}(\partial_{\omega_2} + j\partial_{x_2}) + x_2$;

- Quaternionic time-frequency shift:

$$Z_{\mathbb{H}}[M_\theta T_p f](\mathbf{x}, \mathbf{w}) = e^{2\pi j(p_2\omega_2 + \theta_2 x_2)} Z_{\mathbb{H}}[f](\mathbf{x}, \mathbf{w}) e^{2\pi i(p_1\omega_1 + \theta_1 x_1)}$$

Complex and Hypercomplex Analysis

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Case of Gaussian window

- Gaussian window $M_\theta T_p \phi(x), \phi(x) = 2^{1/2} e^{-\pi|x|^2}$:

  $$Z_{\mathbb{H}}[M_\theta T_p \phi](\mathbf{x}, \mathbf{w}) = e^{2\pi j(p_2\omega_2 + \theta_2 x_2)} e^{-\pi|\mathbf{x}|^2} \Theta(\mathbf{z}_2)\Theta(z_1) e^{2\pi i(p_1\omega_1 + \theta_1 x_1)}$$

  with $z_1 = \omega_1 + ix_1, \mathbf{z}_2 = \omega_2 + jx_2$ and $\Theta(z) = 2^{1/2}\theta_3(z; i)$ - Jacobi elliptic function;

- Link with annihilation operator:

  $$\begin{aligned} (M_\theta T_p \phi)\mathbf{a}_1 &= (M_\theta T_p \phi)(\omega_1 + ip_1) \\ \mathbf{a}_2(M_\theta T_p \phi) &= (\omega_2 + jp_2)(M_\theta T_p \phi) \end{aligned}$$

Frame condition

Theorem

A Gabor system $\{1\overline{g_{-\mathbf{m},\mathbf{n}}}, \mathbf{m}, \mathbf{n} \in \mathbb{Z}^2\}$ is

1. a frame for $L_2(\mathbb{R}^2, \mathbb{H})$ if there exist bounds $0 < A \leq B < \infty$ such that
$$A \leq |Z_{\mathbb{H}}[g](\mathbf{x}, \mathbf{w})|^2 \leq B, \quad a.e.\ \mathbf{x}, \mathbf{w} \in \mathbb{R}^2.$$
2. an orthonormal basis for $L_2(\mathbb{R}^2, \mathbb{H})$ iff additionally
$$|Z_{\mathbb{H}}[g](\mathbf{x}, \mathbf{w})|^2 = 1,$$
for a.e. $\mathbf{x}, \mathbf{w} \in \mathbb{R}^2$.

The Balian-Low Theorem

Balian-Low Theorem

Let $g \in L_2(\mathbb{R}^2, \mathbb{R})$ be such that the associated Gabor system

$$\{1\overline{g_{-\mathbf{m},\mathbf{n}}}, \mathbf{m}, \mathbf{n} \in \mathbb{Z}^2\}$$

is a frame for $L_2(\mathbb{R}^2, \mathbb{H})$. Then

$$\Delta x_k \Delta \omega_k = \infty, \quad k = 1, 2,$$

where

$$\Delta x_k = \frac{||x_k f||^2}{||f||^2}, \quad \Delta \omega_k = \frac{||\omega_k \mathcal{F}_{\mathbb{H}}[f]||^2}{||\mathcal{F}_{\mathbb{H}}[f]||^2}, \quad k = 1, 2.$$

Complex and Hypercomplex Analysis

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