Frames and operators:
Basic properties and open problems

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Abstract The lectures will begin with an introduction to frames in general Hilbert spaces. This will be followed by a discussion of frames with a special structure, in particular, Gabor frames and wavelet frames in $L^2(\mathbb{R})$.
The lectures will highlight the connections to operator theory, and also present some open problems related to frames and operators.

Plan for the talks

- Part I: Frames in general Hilbert spaces
- Part II: Gabor frames and wavelet frames in $L^2(\mathbb{R})$
- Part III: Research topics related to frames and operator theory

What the talk really is about: Unification!

Part I:

Frames in general Hilbert spaces

Plan for the first part of the talk

- Bases and frames in general Hilbert spaces;
- Dual pairs of frames in general Hilbert spaces $\mathcal{H}$: expansions

$$f = \sum \langle f, g_k \rangle f_k, \; f \in \mathcal{H}.$$

Goal and scope

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space.

Want: Expansions

$$f = \sum c_k f_k$$

of signals $f \in \mathcal{H}$ in terms of convenient building blocks f_k.

Desirable properties could be:

- Easy to calculate the coefficients c_k
- Only few large coefficients c_k for the relevant signals f (as for wavelet ONB's!).

Various bases

Definition Consider a sequence $\{e_k\}_{k=1}^{\infty}$ of vectors in $\mathcal{H}$.

(i) The sequence $\{e_k\}_{k=1}^{\infty}$ is a basis for $\mathcal{H}$ if for each $f \in \mathcal{H}$ there exist unique scalar coefficients $\{c_k(f)\}_{k=1}^{\infty}$ such that

$$f = \sum_{k=1}^{\infty} c_k(f) e_k.$$

(ii) A basis $\{e_k\}_{k=1}^{\infty}$ is an unconditional basis if the series in (i) converges unconditionally for each $f \in \mathcal{H}$.

(iii) A basis $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis if $\{e_k\}_{k=1}^{\infty}$ is an orthonormal system, i.e., if

$$\langle e_k, e_j \rangle = \delta_{k,j}.$$

Characterizations of ONB's

Theorem For an orthonormal system $\{e_k\}_{k=1}^{\infty}$, the following are equivalent:

(i) $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis.

(ii) $f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k, \ \forall f \in \mathcal{H}$.

(iii) $\langle f, g \rangle = \sum_{k=1}^{\infty} \langle f, e_k \rangle \langle e_k, g \rangle, \ \forall f, g \in \mathcal{H}$.

(iv) $\sum_{k=1}^{\infty} |\langle f, e_k \rangle|^2 = ||f||^2, \ \forall f \in \mathcal{H}$.

(v) $\overline{\text{span}}\{e_k\}_{k=1}^{\infty} = \mathcal{H}$.

(vi) If $\langle f, e_k \rangle = 0, \ \forall k \in \mathbb{N}$, then $f = 0$.

Corollary If $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis, then each $f \in \mathcal{H}$ has an unconditionally convergent expansion

$$f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k.$$

Characterizations of ONB's

Theorem Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then the orthonormal bases for $\mathcal{H}$ are precisely the sets $\{Ue_k\}_{k=1}^{\infty}$, where $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator.

ONB's

Recall: If $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$, then each $f \in \mathcal{H}$ has an unconditionally convergent expansion

$$f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k. \tag{1}$$

ONB's are good - but the conditions quite restrictive!
How can we obtain expansions of the type (1), but under less restrictive conditions?

Riesz sequences

Definition: A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ is called a Riesz sequence if there exist constants $A, B > 0$ such that

$$A \sum |c_k|^2 \leq \left|\left|\sum c_k f_k\right|\right|^2 \leq B \sum |c_k|^2$$

for all finite sequences $\{c_k\}$.
A Riesz sequence $\{f_k\}_{k=1}^{\infty}$ for which $\overline{\text{span}}\{f_k\}_{k=1}^{\infty} = \mathcal{H}$ is called a Riesz basis.

Equivalent definition of Riesz bases: The Riesz bases are precisely the sequences which have the form $\{f_k\}_{k=1}^{\infty} = \{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator.

Key properties of Riesz bases $\{f_k\}_{k=1}^{\infty} = \{Ue_k\}_{k=1}^{\infty}$:

(i) $\frac{1}{||U^{-1}||^2}\,||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq ||U||^2\,||f||^2,\ \forall f \in \mathcal{H}.$

(ii) Letting $g_k := (U^{-1})^* e_k$,

$$f = \sum_{k=1}^{\infty} \langle f, g_k\rangle f_k,\ \forall f \in \mathcal{H}.$$

Proof. For $f \in \mathcal{H}$,

$$\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 = \sum_{k=1}^{\infty} |\langle f, Ue_k\rangle|^2 = \sum_{k=1}^{\infty} |\langle U^* f, e_k\rangle|^2 = ||U^* f||^2.$$

Now (i) follows from $||U^* f|| \leq ||U^*||\,||f|| = ||U||\,||f||$ and

$$||f|| = ||(U^*)^{-1} U^* f|| \leq ||(U^*)^{-1}||\,||U^* f|| = ||U^{-1}||\,||U^* f||.$$

Proof of (ii): for $f \in \mathcal{H}$,

$$f = UU^{-1}f = U\sum_{k=1}^{\infty}\langle U^{-1}f, e_k\rangle e_k = \sum_{k=1}^{\infty}\langle f, (U^{-1})^* e_k\rangle Ue_k = \sum_{k=1}^{\infty}\langle f, g_k\rangle f_k.$$

Riesz bases $\{f_k\}_{k=1}^{\infty} = \{Ue_k\}_{k=1}^{\infty}$

Definition: The sequence

$$\{g_k\}_{k=1}^{\infty} = \{(U^{-1})^* e_k\}_{k=1}^{\infty}.$$

is called the *dual* of $\{f_k\}_{k=1}^{\infty} = \{Ue_k\}_{k=1}^{\infty}$ is

Dual of $\{g_k\}_{k=1}^{\infty}$:

$$\left\{ \left(((U^{-1})^*)^{-1} \right)^* e_k \right\}_{k=1}^{\infty} = \{Ue_k\}_{k=1}^{\infty} = \{f_k\}_{k=1}^{\infty}.$$

So $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are duals of each other: they are called a *pair of dual Riesz bases.*

Consequence: For all $f \in \mathcal{H}$,

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k = \sum_{k=1}^{\infty} \langle f, f_k \rangle g_k.$$

Riesz bases

Two sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ in a Hilbert space are biorthogonal if

$$\langle f_j, g_k \rangle = \delta_{k,j}.$$

Proposition Any pair of dual Riesz bases $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are biorthogonal.

Proof. For any $j \in \mathbb{N}$,

$$f_j = \sum_{k=1}^{\infty} \langle f_j, g_k \rangle f_k.$$

Since $\{f_k\}_{k=1}^{\infty}$ is a basis, this implies that

$$\langle f_j, g_k \rangle = \begin{cases} 1 & \text{if } k = j; \\ 0 & \text{if } k \neq j, \end{cases}$$

as desired. □

The expansion property for a non-basis

Example The family

$$\{e_k\}_{k\in\mathbb{Z}} = \{e^{2\pi ikx}\}_{k\in\mathbb{Z}}$$

forms an ONB for $L^2(0,1)$.
Consider an open subinterval $I \subset]0,1[$ with $|I| < 1$.
Identify $L^2(I)$ with the subspace of $L^2(0,1)$ consisting of the functions which are zero on $]0,1[\setminus I$.
For a function $f \in L^2(I)$:

$$f = \sum_{k\in\mathbb{Z}} \langle f, e_k\rangle e_k \ \text{ in } \ L^2(0,1). \tag{2}$$

The expansion property for a non-basis

Since

$$\left\| f - \sum_{|k|\leq n} \langle f, e_k \rangle e_k \right\|^2_{L^2(I)} \to 0 \text{ as } n \to \infty,$$

we also have

$$f = \sum_{k\in\mathbb{Z}} \langle f, e_k \rangle e_k \ \text{ in } \ L^2(I). \tag{3}$$

The expansion property for a non-basis

That is, the (restrictions to I of the) functions $\{e_k\}_{k\in\mathbb{Z}}$ also have the expansion property in $L^2(I)$. However, they are not a basis for $L^2(I)$! To see this, define the function

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in I, \\ 1 & \text{if } x \notin I. \end{cases}$$

Then $\tilde{f} \in L^2(0,1)$ and we have the representation

$$\tilde{f} = \sum_{k\in\mathbb{Z}} \langle \tilde{f}, e_k\rangle e_k \;\text{ in }\; L^2(0,1). \tag{4}$$

By restricting to I, the expansion (4) is also valid in $L^2(I)$; since $f = \tilde{f}$ on I, this shows that

$$f = \sum_{k\in\mathbb{Z}} \langle \tilde{f}, e_k\rangle e_k \;\text{ in }\; L^2(I). \tag{5}$$

The expansion property for a non-basis

Thus, (3) and (5) are both expansions of f in $L^2(I)$, and they are non-identical; the argument is that since $f \neq \tilde{f}$ in $L^2(0,1)$, the expansions (2) and (4) show that

$$\{\langle f, e_k \rangle\}_{k\in\mathbb{Z}} \neq \{\langle \tilde{f}, e_k \rangle\}_{k\in\mathbb{Z}}.$$

Conclusion: The restriction of the functions $\{e_k\}_{k\in\mathbb{Z}}$ to I is not a basis for $L^2(I)$, but the expansion property is preserved.

It is natural to consider sequences which are not bases, but nevertheless have the expansion property. That is, cases where each $f \in \mathcal{H}$ has an expansion

$$f = \sum c_k e_k,$$

without $\{e_k\}$ being a basis.

Bessel sequences

Definition A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is called a Bessel sequence if there exists a constant $B > 0$ such that

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\, ||f||^2, \; \forall f \in \mathcal{H}.$$

Satisfied for all Riesz bases!

Bessel sequences

Definition A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is called a Bessel sequence if there exists a constant $B > 0$ such that

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Satisfied for all Riesz bases!

Theorem Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{H}$, and $B > 0$ be given. Then $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence with Bessel bound B if and only if

$$T : \{c_k\}_{k=1}^{\infty} \to \sum_{k=1}^{\infty} c_k f_k$$

defines a bounded operator from $\ell^2(\mathbb{N})$ into $\mathcal{H}$ and $||T|| \leq \sqrt{B}$.

Bessel sequences

Corollary If $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence in $\mathcal{H}$, then $\sum_{k=1}^{\infty} c_k f_k$ converges unconditionally for all $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$.

Pre-frame operator or *synthesis operator* associated to a Bessel sequence:

$$T : \ell^2(\mathbb{N}) \to \mathcal{H},\ T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k$$

The adjoint operator - the *analysis operator:*

$$T^* : \mathcal{H} \to \ell^2(\mathbb{N}),\ \ T^* f = \{\langle f, f_k \rangle\}_{k=1}^{\infty}.$$

The *frame operator:*

$$S : \mathcal{H} \to \mathcal{H},\ \ Sf = TT^* f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k.$$

The series defining S converges unconditionally for all $f \in \mathcal{H}$.

Frames

Definition: A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is a *frame* if there exist constants $A, B > 0$ such that

$$A\,||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\,||f||^2,\ \forall f \in \mathcal{H}.$$

A and B are called *frame bounds.* A frame is *tight* if we can take $A = B = 1$.
Note:

- The sequence $\{f_k\}_{k=1}^{\infty}$ is a *Bessel sequence* if at least the upper inequality holds.
- Any orthonormal basis is a Riesz basis;
- Any Riesz basis is a frame;
- Example of a frame which is not a basis:

$$\{e_1, e_1, e_2, e_3, \ldots\},$$

where $\{e_k\}_{k=1}^{\infty}$ is an ONB.

The frame operator $Sf = \sum \langle f, f_k \rangle f_k$

Lemma Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S and frame bounds A, B. Then the following holds:

- S is bounded, invertible, self-adjoint, and positive.
- $\{S^{-1}f_k\}_{k=1}^{\infty}$ is a frame with frame operator S^{-1} and frame bounds B^{-1}, A^{-1}.
- If A, B are the optimal frame bounds for $\{f_k\}_{k=1}^{\infty}$, then the bounds B^{-1}, A^{-1} are optimal for $\{S^{-1}f_k\}_{k=1}^{\infty}$.

$\{S^{-1}f_k\}_{k=1}^{\infty}$ is called the *canonical dual frame* of $\{f_k\}_{k=1}^{\infty}$.

Frames versus Riesz bases

- Any Riesz basis is a frame.
- A frame $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis if and only if $\{f_k\}_{k=1}^{\infty}$ is a basis.
- A frame $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis if and only if

$$\sum_{k=1}^{\infty} c_k f_k = 0 \Rightarrow c_k = 0, \ \forall k.$$

- A frame which is not a Riesz basis, is said to be *overcomplete* or *redundant:* In that case there exists $\{c_k\}_{k=1}^{\infty} \setminus \{0\}$ such that

$$\sum_{k=1}^{\infty} c_k f_k = 0.$$

The frame decomposition

Theorem - the frame decomposition Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S. Then

$$f = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k \rangle f_k = \sum_{k=1}^{\infty} \langle f, f_k \rangle S^{-1} f_k, \quad \forall f \in \mathcal{H}.$$

Both series converge unconditionally for all $f \in \mathcal{H}$.

Proof. : Let $f \in \mathcal{H}$. Then

$$f = SS^{-1} f = \sum_{k=1}^{\infty} \langle S^{-1} f, f_k \rangle f_k = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k \rangle f_k.$$

Since $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence and

$$\{\langle f, S^{-1} f_k \rangle\}_{k=1}^{\infty} \in \ell^2(\mathbb{N}),$$

the series converges unconditionally. The second expansion follows from $f = S^{-1} S f$. □

Might be difficult to compute S^{-1} !

Tight frames: $\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 = A \, ||f||^2$

Corollary If $\{f_k\}_{k=1}^{\infty}$ is a tight frame with frame bound A, then the canonical dual frame is $\{A^{-1} f_k\}_{k=1}^{\infty}$, and

$$f = \frac{1}{A} \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \ \forall f \in \mathcal{H}. \tag{6}$$

Proof. For a tight frame,

$$\langle Sf, f \rangle = A||f||^2 = \langle Af, f \rangle;$$

since S is self-adjoint, this implies that

$$S = AI. \qquad \square$$

By scaling of the vectors $\{f_k\}_{k=1}^{\infty}$ in a tight frame, we can always obtain that $A = 1$; in that case (6) has exactly the same form as the representation via an orthonormal basis.

Tight frames can be used without any additional computational effort compared to the use of ONB's.

Tight frames $f = \frac{1}{A}\sum_{k=1}^{\infty}\langle f, f_k\rangle f_k, \ \forall f \in \mathcal{H}.$

Other advantages of tight frames:

- If $\{f_k\}_{k=1}^{\infty}$ consists of functions with compact support or fast decay, the same is the case for the functions in the canonical dual frame.
- If $\{f_k\}_{k=1}^{\infty}$ consists of functions with a special structure (Gabor structure or wavelet structure) the same is the case for the functions in the canonical dual frame.

The corresponding statements do not necessarily hold for a general frame $\{f_k\}_{k=1}^{\infty}$ and its canonical dual frame $\{S^{-1}f_k\}_{k=1}^{\infty}$!

$$f = \sum_{k=1}^{\infty}\langle f, S^{-1}f_k\rangle f_k, \ \ \forall f \in \mathcal{H}.$$

General frames versus Tight frames

To each frame $\{f_k\}_{k=1}^{\infty}$ one can associate a tight frame:

Corollary Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame operator S. Denote the positive square root of S^{-1} by $S^{-1/2}$. Then $\{S^{-1/2}f_k\}_{k=1}^{\infty}$ is a tight frame with frame bound equal to 1, and

$$f = \sum_{k=1}^{\infty} \langle f, S^{-1/2}f_k \rangle S^{-1/2}f_k, \ \forall f \in \mathcal{H}.$$

Problems:

- Not easy to find $\{S^{-1/2}f_k\}_{k=1}^{\infty}$;
- "Nice properties" of $\{f_k\}_{k=1}^{\infty}$ not necessarily inherited by $\{S^{-1/2}f_k\}_{k=1}^{\infty}$.

Characterizations of frames

Theorem (C. 1992): A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is a frame for $\mathcal{H}$ if and only if

$$T : \{c_k\}_{k=1}^{\infty} \to \sum_{k=1}^{\infty} c_k f_k$$

is a well-defined mapping of $\ell^2(\mathbb{N})$ onto $\mathcal{H}$.

Compare to the characterization of Bessel sequences in terms of T (T well-defined)!

Characterizations of frames

Theorem: Let $\{e_k\}_{k=1}^{\infty}$ be an arbitrary orthonormal basis for $\mathcal{H}$. The frames for $\mathcal{H}$ are precisely the families $\{Ue_k\}_{k=1}^{\infty}$, where $U : \mathcal{H} \to \mathcal{H}$ is a bounded and surjective operator.

Compare to the characterization of

- ONB's (U unitary);
- Riesz bases (U bounded and bijective).

A frame where no subsequence is a basis

Intuitively: A frame consists of a basis + some extra elements (redundance).
Good as intuitive feeling - but wrong in the technical sense:

Example: Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Let

$$\{f_k\}_{k=1}^{\infty} := \{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \cdots\};$$

that is, each vector $\frac{1}{\sqrt{\ell}}e_\ell, \ell \in \mathbb{N}$, is repeated ℓ times. Then $\{f_k\}_{k=1}^{\infty}$ is a tight frame for $\mathcal{H}$ with frame bound $A = 1$. No subfamily is a Riesz basis. □

More complicated: In each separable Hilbert space, there exists a frame for which no subfamily is a basis!

General dual frames

A frame which is not a Riesz basis is said to be *overcomplete*.

Theorem: *Assume that* $\{f_k\}_{k=1}^{\infty}$ *is an overcomplete frame. Then there exist frames*

$$\{g_k\}_{k=1}^{\infty} \neq \{S^{-1}f_k\}_{k=1}^{\infty}$$

for which

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \ \forall f \in \mathcal{H}.$$

$\{g_k\}_{k=1}^{\infty}$ is called a *dual frame* of $\{f_k\}_{k=1}^{\infty}$. The special choice

$$\{g_k\}_{k=1}^{\infty} = \{S^{-1}f_k\}_{k=1}^{\infty}$$

is called the *canonical dual frame*.

General dual frames

Note: Let $\{f_k\}_{k=1}^{\infty}$ be a Bessel sequence with pre-frame operator

$$T : \mathcal{H} \to \ell^2(\mathbb{N}),\ T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k$$

and $\{g_k\}_{k=1}^{\infty}$ be a Bessel sequence with pre-frame operator U. Then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are dual frames if and only if

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k,\ \forall f \in \mathcal{H},$$

i.e., if and only if

$$TU^* = I,$$

i.e., if and only if

$$UT^* = I.$$

Characterization of all dual frames

Result by Shidong Li, 1991:

Theorem: *Let* $\{f_k\}_{k=1}^{\infty}$ *be a frame with pre-frame operator* T*. The bounded operators* $U : \ell^2(\mathbb{N}) \to \mathcal{H}$ *for which*

$$UT^* = I,$$

i.e., the bounded left-inverses of T^**, are precisely the operators having the form*

$$U = S^{-1}T + W(I - T^*S^{-1}T),$$

where $W : \ell^2(\mathbb{N}) \to \mathcal{H}$ *is a bounded operator and* I *denotes the identity operator on* $\ell^2(\mathbb{N})$*.*

Characterization of all dual frames

Result by Shidong Li, 1991:

Theorem: *Let* $\{f_k\}_{k=1}^{\infty}$ *be a frame for* $\mathcal{H}$. *The dual frames of* $\{f_k\}_{k=1}^{\infty}$ *are precisely the families*

$$\{g_k\}_{k=1}^{\infty} = \left\{ S^{-1}f_k + h_k - \sum_{j=1}^{\infty} \langle S^{-1}f_k, f_j \rangle h_j \right\}_{k=1}^{\infty}, \tag{7}$$

where $\{h_k\}_{k=1}^{\infty}$ *is a Bessel sequence in* $\mathcal{H}$.

Allows us to *optimize* the duals:

- Which dual has the best approximation theoretic properties?
- Which dual has the smallest support?
- Which dual has the most convenient expression?
- Can we find a dual that is easy to calculate?

An example: Sigma-Delta quantization

Work by Lammers, Powell, and Yilmaz:

Consider a frame $\{f_k\}_{k=1}^N$ for $\mathbb{R}^d$. Letting $\{g_k\}_{k=1}^N$ denote a dual frame, each $f \in \mathbb{R}^d$ can be written

$$f = \sum_{k=1}^{N} \langle f, g_k \rangle f_k.$$

In practice: the coefficients $\langle f, g_k \rangle$ must be quantized, i.e., replaced by some coefficients d_k from a discrete set such that

$$d_k \approx \langle f, g_k \rangle,$$

which leads to

$$f \approx \sum_{k=1}^{N} d_k f_k.$$

Note: increased redundancy (large N) increases the chance of a good approximation.

An example: Sigma-Delta quantization

- For each $r \in \mathbb{N}$ there is a procedure (rth order sigma-delta quantization) to find appropriate coefficients d_k.
- $r$the order sigma-delta quantization with the canonical dual frame does not provide approximation order N^{-r}.
- Approximation order N^{-r} can be obtained using other dual frames.

Tight frames versus dual pairs

- For some years: focus on construction of tight frame.
- Do not forget the extra flexibility offered by convenient dual frame pairs!

Theorem: For each Bessel sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$, there exist a family of vectors $\{g_k\}_{k=1}^{\infty}$ such that

$$\{f_k\}_{k=1}^{\infty} \cup \{g_k\}_{k=1}^{\infty}$$

is a tight frame for $\mathcal{H}$.

Tight frames versus dual pairs

Example Let $\{e_j\}_{j=1}^{10}$ be an orthonormal basis for $\mathbb{C}^{10}$ and consider the frame

$$\{f_j\}_{j=1}^{10} := \{2e_1\} \cup \{e_j\}_{j=2}^{10}.$$

There exist 9 vectors $\{h_j\}_{j=1}^{9}$ such that

$$\{f_j\}_{j=1}^{10} \cup \{h_j\}_{j=1}^{9}$$

is a tight frame for $\mathbb{C}^{10}$ - and 9 is the minimal number to add.

A pair of dual frames can be obtained by adding just one element:

$$\{f_j\}_{j=1}^{10} \cup \{-3e_1\} \text{ and } \{f_j\}_{j=1}^{10} \cup \{e_1\}$$

form dual frames in $\mathbb{C}^{10}$.

Tight frames versus dual pairs

Theorem (Casazza and Fickus): Given a sequence of positive numbers $a_1 \geq a_2 \geq \cdots \geq a_M$, there exists a tight frame $\{f_j\}_{j=1}^M$ for $\mathbb{R}^N$ with $||f_j|| = a_j,\ j = 1, \ldots, M$, if and only if

$$a_1^2 \leq \frac{1}{N} \sum_{j=1}^{M} a_j^2. \tag{8}$$

Theorem (C., Powell, Xiao, 2010): Given any sequence $\{\alpha_j\}_{j=1}^M$ of real numbers, and assume that $M > N$. Then the following are equivalent:

(i) There exist a pair of dual frames $\{f_j\}_{j=1}^M$ and $\{\widetilde{f_j}\}_{j=1}^M$ for $\mathbb{R}^N$ such that $\alpha_j = \langle f_j, \widetilde{f_j} \rangle$ for all $j = 1, \ldots, M$.

(ii) $N = \sum_{j=1}^M \alpha_j$.

Part II:

Gabor frames and wavelet frames in $L^2(\mathbb{R})$

Operators on $L^2(\mathbb{R})$

Translation by $a \in \mathbb{R}$: $T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (T_a f)(x) = f(x-a).$

Modulation by $b \in \mathbb{R}$: $E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (E_b f)(x) = e^{2\pi i b x} f(x).$

Dilation by $a > 0$: $D_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (D_a f)(x) = \frac{1}{\sqrt{a}} f(\frac{x}{a}).$

Dyadic scaling: $D : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (Df)(x) = 2^{1/2} f(2x).$

All these operators are unitary on $L^2(\mathbb{R})$.

Important commutator relations:

$$T_a E_b = e^{-2\pi i b a} E_b T_a,\ \ T_b D_a = D_a T_{b/a},\ \ D_a E_b = E_{b/a} D_a$$

The Fourier transform

For $f \in L^1(\mathbb{R})$, the *Fourier transform* is defined by

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\gamma}\, dx, \ \gamma \in \mathbb{R}.$$

The Fourier transform can be extended to a unitary operator on $L^2(\mathbb{R})$.

Plancherel's equation:

$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle, \ \forall f, g \in L^2(\mathbb{R}), \text{ and } ||\hat{f}|| = ||f||.$$

Important commutator relations:

$$\begin{aligned} \mathcal{F}T_a &= E_{-a}\mathcal{F}, \ \mathcal{F}E_a = T_a\mathcal{F}, \\ \mathcal{F}D_a &= D_{1/a}\mathcal{F}, \ \mathcal{F}D = D^{-1}\mathcal{F}. \end{aligned}$$

B-Splines

The B-splines $B_N, \ N \in \mathbb{N}$, are given by

$$B_1 = \chi_{[0,1]}, \ \ B_{N+1} = B_N * B_1.$$

Theorem: Given $N \in \mathbb{N}$, the B-spline B_N has the following properties:

(i) *supp* $B_N = [0, N]$ and $B_N > 0$ on $]0, N[$.

(ii) $\int_{-\infty}^{\infty} B_N(x)dx = 1$.

(iii) $\sum_{k\in\mathbb{Z}} B_N(x-k) = 1$

(iv) For any $N \in \mathbb{N}$,

$$\widehat{B_N}(\gamma) = \left(\frac{1 - e^{-2\pi i\gamma}}{2\pi i\gamma}\right)^N.$$

B-Splines

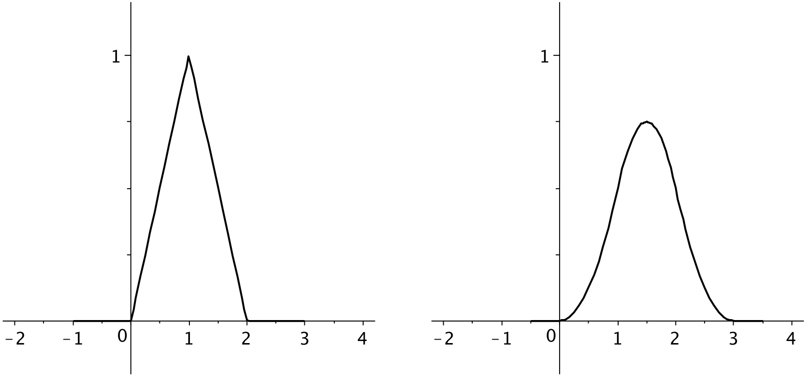

Figure: The B-splines B_2 and B_3.

Centered B-Splines $\widetilde{B_N}$

The centered B-splines $\widetilde{B_N},\ N \in \mathbb{N}$, are given by

$$\widetilde{B_1} = \chi_{[-1/2,1/2]},\ \ \widetilde{B_{N+1}} = \widetilde{B_N} * \widetilde{B_1}.$$

Theorem: Given $N \in \mathbb{N}$, the centered B-spline $\widetilde{B_N}$ has the following properties:

(i) *supp* $\widetilde{B_N} = [-N/2, N/2]$ and $\widetilde{B_N} > 0$ on $]-N/2, N/2[$.

(ii) $\int_{-\infty}^{\infty} \widetilde{B_N}(x)dx = 1$.

(iii) $\sum_{k\in\mathbb{Z}} \widetilde{B_N}(x-k) = 1$

(iv) For any $N \in \mathbb{N}$,

$$\widehat{\widetilde{B_N}}(\gamma) = \left(\frac{\sin \pi\gamma}{\pi\gamma}\right)^N.$$

Classical wavelet theory

- Given a function $\psi \in L^2(\mathbb{R})$ and $j, k \in \mathbb{Z}$, let

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), \; x \in \mathbb{R}.$$

- In terms of the operators $T_k f(x) = f(x-k)$ and $Df(x) = 2^{1/2} f(2x)$,

$$\psi_{j,k} = D^j T_k \psi, \; j, k \in \mathbb{Z}.$$

- If $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, the function ψ is called a *wavelet.* In this case every $f \in L^2(\mathbb{R})$ has the representation

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}.$$

Multiresolution analysis - a tool to construct a wavelet

Definition: *A multiresolution analysis for* $L^2(\mathbb{R})$ *consists of a sequence of closed subspaces* $\{V_j\}_{j\in\mathbb{Z}}$ *of* $L^2(\mathbb{R})$ *and a function* $\phi \in V_0$*, such that the following conditions hold:*

(i) $\cdots V_{-1} \subset V_0 \subset V_1 \cdots$.

(ii) $\overline{\cup_j V_j} = L^2(\mathbb{R})$ *and* $\cap_j V_j = \{0\}$.

(iii) $f \in V_j \Leftrightarrow [x \to f(2x)] \in V_{j+1}$.

(iv) $f \in V_0 \Rightarrow T_k f \in V_0, \ \forall k \in \mathbb{Z}$.

(v) $\{T_k \phi\}_{k\in\mathbb{Z}}$ *is an orthonormal basis for* V_0.

Construction of wavelet ONB

- The function ϕ in a multiresolution analysis satisfies a scaling equation,

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma), \;\; a.e.\ \gamma \in \mathbb{R},$$

  for some 1-periodic function $H_0 \in L^2(0,1)$.
- Let

$$H_1(\gamma) = \overline{H_0(\gamma + \frac{1}{2})}e^{-2\pi i\gamma}.$$

- Then the function ψ defined via

$$\hat{\psi}(2\gamma) = H_1(\gamma)\hat{\phi}(\gamma)$$

  generates a wavelet orthonormal basis $\{D^j T_k \psi\}_{j,k\in\mathbb{Z}}$.
- Explicitly: if $H_1(\gamma) = \sum_{k\in\mathbb{Z}} d_k e^{2\pi ik\gamma}$, then $\psi(x) = 2\sum_{k\in\mathbb{Z}} d_k \phi(2x+k)$.

Construction of wavelet ONB

Theorem: Let $\phi \in L^2(\mathbb{R})$, and let

$$V_j := \overline{\text{span}}\{D^j T_k \phi\}_{k\in\mathbb{Z}}.$$

Assume that the following conditions hold:

(i) $\inf_{\gamma\in]-\epsilon,\epsilon[} |\hat{\phi}(\gamma)| > 0$ for some $\epsilon > 0$;

(ii) The scaling equation

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma),$$

is satisfied for a bounded 1-periodic function H_0;

(iii) $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal system.

Then ϕ generates a multiresolution analysis, and there exists a wavelet ψ of the form

$$\psi(x) = 2\sum_{k\in\mathbb{Z}} d_k \phi(2x+k).$$

Desirable properties for wavelet bases

- that ψ has a computationally convenient form, for example that ψ is a piecewise polynomial (a spline);
  Not satisfied for the conventional wavelets like the Daubechies' wavelets and the Battle-Lemarié wavelets
- regularity of ψ;
- symmetry (or anti-symmetry) of ψ, i.e., that

$$\psi(x) = \psi(-x) \text{ or } \psi(x) = -\psi(-x), \ x \in \mathbb{R};$$

- compact support of ψ, or at least fast decay;
- that ψ has *vanishing moments*, i.e., that for a certain $N \in \mathbb{N}$,

$$\int_{-\infty}^{\infty} x^{\ell} \psi(x)\, dx = 0 \text{ for } \ell = 0, 1, \ldots, N-1.$$

Vanishing moments

- Vanishing moments are essential in the context of *compression.* Assuming that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, every $f \in L^2(\mathbb{R})$ has the representation

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}. \tag{9}$$

All information about f is stored in the coefficients $\{\langle f, \psi_{j,k}\rangle\}_{j,k\in\mathbb{Z}}$, and (9) tells us how to reconstruct f based on the coefficients. In practice one can not store an infinite sequence of non-zero numbers, so one has to select a finite number of the coefficients to keep. Done by

Thresholding for wavelet bases, $f = \sum \langle f, \psi_{j,k} \rangle \psi_{j,k}$.

Thresholding: Chooses a certain $\epsilon > 0$ and keep only the coefficients $\langle f, \psi_{j,k} \rangle$ for which

$$|\langle f, \psi_{j,k} \rangle| \geq \epsilon.$$

If ψ has a large number of vanishing moments, then only relatively few coefficients $\langle f, \psi_{j,k} \rangle$ will be large:
Theorem: Assume that the function $\psi \in L^2(\mathbb{R})$ is compactly supported and has $N-1$ vanishing moments. Then, for any N times differentiable function $f \in L^2(\mathbb{R})$ for which $f^{(N)}$ is bounded, there exists a constant $C > 0$ such that

$$|\langle f, \psi_{j,k} \rangle| \leq C\, 2^{-jN} 2^{-j/2}, \ \forall j, k \in \mathbb{Z}.$$

Desirable properties for wavelet bases

Further relevant properties:

- Compact support (or at least fast decay) of ψ is essential for the use of computer-based methods, where a function with unbounded support always has to be truncated. For the same reason we often want the support to be small.
- The condition of ψ being symmetric is helpful in image processing, where a non-symmetric wavelet will generate non-symmetric errors, which are more disturbing to the human eye than symmetric errors.

Non-existence of symmetric wavelets

One can not combine the classical multiresolution analysis with the desire of having a symmetric wavelet ψ:

Proposition: Assume that $\phi \in L^2(\mathbb{R})$ is real-valued and compactly supported, and let

$$V_j = \overline{\text{span}}\{D^j T_k \phi\}_{k \in \mathbb{Z}},\ j \in \mathbb{Z}.$$

Assume that $(\phi, \{V_j\})$ constitute a multiresolution analysis. Then, if the associated wavelet ψ is real-valued and compactly supported and has either a symmetry axis or an antisymmetry axis, then ψ is necessarily the Haar wavelet.

Thus, under the above assumptions we are back at the function we want to avoid!

Can we overcome the shortcomings (no explicit formula for ψ, no symmetry) by considering wavelet frames instead of bases - while keeping the convenient properties of multiresolution analysis?

Spline wavelets B_N

- The B-splines $B_N,\ N \in \mathbb{N}$, are given by

$$B_1 = \chi_{[0,1]}, \;\; B_{N+1} = B_N * B_1.$$

- One can consider any order splines B_N and define associated multiresolution analyses, which leads to wavelets of the type

$$\psi(x) = \sum_{k \in \mathbb{Z}} c_k B_N(2x + k).$$

- These wavelets are called *Battle–Lemarié wavelets.*
- **Only shortcoming:** except for the case $N = 1$, all coefficients c_k are non-zero, which implies that the wavelet ψ has support equal to $\mathbb{R}$.

Spline wavelets - can we do better for $N > 1$?

Can show:

- There does not exists an ONB $\{D^j T_k \psi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ generated by a finite linear combination

$$\psi(x) = \sum c_k B_N(2x+k).$$

- There does not exists a tight frame $\{D^j T_k \psi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ generated by a finite linear combination

$$\psi(x) = \sum c_k B_N(2x+k).$$

- There does not exists a pairs of dual wavelet frames $\{D^j T_k \psi\}_{j,k\in\mathbb{Z}}$ and $\{D^j T_k \tilde{\psi}\}_{j,k\in\mathbb{Z}}$ for which ψ and $\tilde{\psi}$ are finite linear combinations of functions $DT_k B_N,\ j,k \in \mathbb{Z}$.

Solution: consider systems of the wavelet-type, but generated by more than one function.

Setup for construction of tight wavelet frames by Ron and Shen:
Let $\psi_0 \in L^2(\mathbb{R})$ and assume that

(i) There exists a function $H_0 \in L^\infty(\mathbb{T})$ such that

$$\widehat{\psi}_0(2\gamma) = H_0(\gamma)\widehat{\psi}_0(\gamma).$$

(ii) $\lim_{\gamma \to 0} \widehat{\psi}_0(\gamma) = 1.$

Further, let $H_1, \dots, H_n \in L^\infty(\mathbb{T})$, and define $\psi_1, \dots, \psi_n \in L^2(\mathbb{R})$ by

$$\widehat{\psi_\ell}(2\gamma) = H_\ell(\gamma)\widehat{\psi}_0(\gamma), \ \ell = 1, \dots, n.$$

The unitary extension principle

- We want to find conditions on the functions $H_1, \dots, H_n$ such that $\psi_1, \dots, \psi_n$ generate a multiwavelet frame for $L^2(\mathbb{R})$.
- Let H denote the $(n+1) \times 2$ matrix-valued function defined by

$$H(\gamma) = \begin{pmatrix} H_0(\gamma) & T_{1/2}H_0(\gamma) \\ H_1(\gamma) & T_{1/2}H_1(\gamma) \\ \cdot & \cdot \\ \cdot & \cdot \\ H_n(\gamma) & T_{1/2}H_n(\gamma) \end{pmatrix}, \ \gamma \in \mathbb{R}.$$

The unitary extension principle

Theorem (Ron and Shen, 1997): *Let $\{\psi_\ell, H_\ell\}_{\ell=0}^n$ be as in the general setup, and assume that $H(\gamma)^*H(\gamma) = I$ for a.e. $\gamma \in \mathbb{T}$. Then the multiwavelet system $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$ constitutes a tight frame for $L^2(\mathbb{R})$ with frame bound equal to* 1.

The matrix $H(\gamma)^*H(\gamma)$ has four entries, but it is enough to verify two sets of equations:

Corollary: Let $\{\psi_\ell, H_\ell\}_{\ell=0}^n$ be as in the general setup and assume that

$$\sum_{\ell=0}^{n} |H_\ell(\gamma)|^2 = 1,$$

and

$$\sum_{\ell=0}^{n} \overline{H_\ell(\gamma)} T_{1/2} H_\ell(\gamma) = 0,$$

for a.e. $\gamma \in \mathbb{T}$. Then $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$ constitutes a tight frame for $L^2(\mathbb{R})$ with frame bound equal to 1.

The UEP can be applied to any order B-spline!

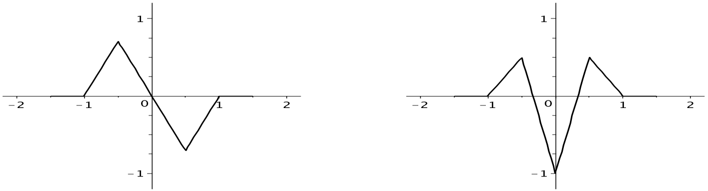

Figure: The two wavelet frame generators ψ_1 and ψ_2 associated with $\psi_0 = \widetilde{B_2}$.

The unitary extension principle and B-splines

Exmple: For any $m = 1, 2, \ldots,$ we consider the centered B-spline

$$\psi_0 := \widetilde{B_{2m}}$$

of order $2m$. Then

$$\widehat{\psi}_0(\gamma) = \left(\frac{\sin(\pi\gamma)}{\pi\gamma} \right)^{2m}.$$

It is clear that $\lim_{\gamma \to 0} \widehat{\psi}_0(\gamma) = 1$, and by direct calculation,

$$\widehat{\psi}_0(2\gamma) = \left(\frac{\sin(2\pi\gamma)}{2\pi\gamma} \right)^{2m} = \left(\frac{2\sin(\pi\gamma)\cos(\pi\gamma)}{2\pi\gamma} \right)^{2m} = \cos^{2m}(\pi\gamma)\widehat{\psi}_0(\gamma).$$

Thus ψ_0 satisfies a refinement equation with two-scale symbol

$$H_0(\gamma) = \cos^{2m}(\pi\gamma).$$

The unitary extension principle and B-splines

Now, consider the binomial coefficient

$$\begin{pmatrix} 2m \\ \ell \end{pmatrix} := \frac{(2m)!}{(2m-\ell)!\ell!},$$

and define the functions $H_1, \ldots, H_{2m} \in L^\infty(\mathbb{T})$ by

$$H_\ell(\gamma) = \sqrt{\begin{pmatrix} 2m \\ \ell \end{pmatrix}} \sin^\ell(\pi\gamma) \cos^{2m-\ell}(\pi\gamma).$$

Using that $\cos(\pi(\gamma - 1/2)) = \sin(\pi\gamma)$ and $\sin(\pi(\gamma - 1/2)) = -\cos(\pi\gamma)$,

$$T_{1/2}H_\ell(\gamma) = \sqrt{\begin{pmatrix} 2m \\ \ell \end{pmatrix}} (-1)^\ell \cos^\ell(\pi\gamma) \sin^{2m-\ell}(\pi\gamma).$$

The unitary extension principle and B-splines

Thus, the matrix H is given by

$$H(\gamma) = \begin{pmatrix} H_0(\gamma) & T_{1/2}H_0(\gamma) \\ H_1(\gamma) & T_{1/2}H_1(\gamma) \\ \cdot & \cdot \\ \cdot & \cdot \\ H_{2m}(\gamma) & T_{1/2}H_{2m}(\gamma) \end{pmatrix} =$$

$$\begin{pmatrix} \cos^{2m}(\pi\gamma) & \sin^{2m}(\pi\gamma) \\ \sqrt{\begin{pmatrix} 2m \\ 1 \end{pmatrix}} \sin(\pi\gamma)\cos^{2m-1}(\pi\gamma) & -\sqrt{\begin{pmatrix} 2m \\ 1 \end{pmatrix}} \cos(\pi\gamma)\sin^{2m-1}(\pi\gamma) \\ \cdot & \cdot \\ \cdot & \cdot \\ \sqrt{\begin{pmatrix} 2m \\ 2m \end{pmatrix}} \sin^{2m}(\pi\gamma) & \sqrt{\begin{pmatrix} 2m \\ 2m \end{pmatrix}} \cos^{2m}(\pi\gamma) \end{pmatrix}.$$

The unitary extension principle and B-splines

Using the binomial formula

$$(x+y)^{2m} = \sum_{\ell=0}^{2m} \binom{2m}{\ell} x^{\ell} y^{2m-\ell},$$

$$\begin{aligned}\sum_{\ell=0}^{2m} |H_\ell(\gamma)|^2 &= \sum_{\ell=0}^{2m} \binom{2m}{\ell} \sin^{2\ell}(\pi\gamma)\cos^{2(2m-\ell)}(\pi\gamma) \\ &= \left(\sin^2(\pi\gamma)+\cos^2(\pi\gamma)\right)^{2m} = 1,\ \gamma \in \mathbb{T}.\end{aligned}$$

Using the binomial formula with $x=-1, y=1$,

$$\sum_{\ell=0}^{2m} \overline{H_\ell(\gamma)} T_{1/2} H_\ell(\gamma) = \sin^{2m}(\pi\gamma)\cos^{2m}(\pi\gamma) \sum_{\ell=0}^{2m} (-1)^{\ell} \binom{2m}{\ell}$$

$$= \sin^{2m}(\pi\gamma)\cos^{2m}(\pi\gamma)(1-1)^{2m} = 0.$$

Thus the $2m$ functions $\psi_1, \ldots, \psi_{2m}$ defined by

$$\begin{aligned}\widehat{\psi_\ell}(\gamma) &= H_\ell(\gamma/2)\widehat{\psi}_0(\gamma/2) \\ &= \sqrt{\begin{pmatrix} 2m \\ \ell \end{pmatrix}} \frac{\sin^{2m+\ell}(\pi\gamma/2)\cos^{2m-\ell}(\pi\gamma/2)}{(\pi\gamma/2)^{2m}}\end{aligned}$$

generate a tight frame $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,2m}$ for $L^2(\mathbb{R})$. $\square$

Shortcomings of the UEP

- The computational effort increases with the order of the B-spline $\widetilde{B_{2m}}$: For higher orders, we need more generators, and more non-zero coefficients appear in ψ_ℓ.
- There is a limitation on the possible number of vanishing moments ψ_ℓ can have: in the B-spline case, at least one of the functions ψ_ℓ can only have one vanishing moment. This leads to sub-optimal approximation properties.

More recent extension principles

- Mixed extension principle: construction of dual wavelet frames
- Oblique extension principle: equivalent to the UEP, but provides more natural constructions of frames with high approximation orders and optimal number of vanishing moments
- Mixed oblique extension principle: dual frame variant of the OEP, but computationally much simpler (avoids spectral factorization)

General theory for wavelet frames

Definition: Let $\psi \in L^2(\mathbb{R})$. A frame for $L^2(\mathbb{R})$ of the form $\{D^j T_k \psi\}_{j,k \in \mathbb{Z}}$ is called a dyadic wavelet frame.

The associated frame operator:

$$S : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ Sf = \sum_{j,k \in \mathbb{Z}} \langle f, D^j T_k \psi \rangle D^j T_k \psi.$$

The frame decomposition:

$$f = \sum_{j,k \in \mathbb{Z}} \langle f, S^{-1} D^j T_k \psi \rangle D^j T_k \psi,\ f \in L^2(\mathbb{R}).$$

Inconvenient - one needs to calculate

$$\langle f, S^{-1} D^j T_k \psi \rangle,\ \forall j, k \in \mathbb{Z}.$$

Wavelet frame coefficients $\langle f, D^j T_k \psi \rangle$

Improvement: can show that

$$S^{-1} D^j T_k \psi = D^j S^{-1} T_k \psi.$$

Unfortunately, in general

$$D^j S^{-1} T_k \psi \neq D^j T_k S^{-1} \psi.$$

We can not expect the canonical dual frame of a wavelet frame to have wavelet structure.
Bownik and Weber: example of a wavelet system $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ for which

- The canonical dual does not have the wavelet structure;
- There exist infinitely many functions $\tilde{\psi}$ for which $\{\tilde{\psi}_{j,k}\}_{j,k \in \mathbb{Z}}$ is a dual frame.

General theory for wavelet frames

Example: Let $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ be a wavelet ONB for $L^2(\mathbb{R})$. Given $\epsilon \in]0,1[$, let

$$\theta = \psi + \epsilon D\psi.$$

Then

- $\{\theta_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Riesz basis
- The canonical dual frame of $\{\theta_{j,k}\}_{j,k\in\mathbb{Z}}$ does *not* have the wavelet structure.
- For a Riesz basis, the dual is unique: so no dual with wavelet structure exists!
- If ψ has compact support, then θ also has compact support, and all the functions $\{\theta_{j,k}\}_{j,k\in\mathbb{Z}}$ have compact support.
- For the canonical dual frame $\{S^{-1}\theta_{j,k}\}_{j,k\in\mathbb{Z}}$, the functions have compact support when $k \neq 0$. However, the functions $S^{-1}\theta_{j,0}$ do not have compact support.

Gabor systems

Gabor systems: have the form

$$\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$$

for some $g \in L^2(\mathbb{R}),\ a,b>0$. Short notation:

$$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$$

Example:

- $\{e^{2\pi imx}\chi_{[0,1]}(x)\}_{m\in\mathbb{Z}}$ is an ONB for $L^2(0,1)$
- For $n\in\mathbb{Z}$, $\{e^{2\pi im(x-n)}\chi_{[0,1]}(x-n)\}_{m\in\mathbb{Z}} = \{e^{2\pi imx}\chi_{[0,1]}(x-n)\}_{m\in\mathbb{Z}}$ is an ONB for $L^2(n,n+1)$
- $\{e^{2\pi imx}\chi_{[0,1]}(x-n)\}_{m,n\in\mathbb{Z}}$ is an ONB for $L^2(\mathbb{R})$

Problem: The function $\chi_{[0,1]}$ is discontinuous and has very slow decay in the Fourier domain:

$$\widehat{\chi}_{[0,1]}(\gamma) = \int_0^1 e^{-2\pi ix\gamma}\,dx = e^{-\pi i\gamma}\frac{\sin\pi\gamma}{\pi\gamma}.$$

Thus, the function is not suitable for time-frequency analysis.

Question: Can we obtain more suitable Gabor bases by replacing $\chi_{[0,1]}$ by a smoother function g?

Gabor systems

A related short coming - the *Balian–Low Theorem:*

Theorem: Assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Riesz basis. Then

$$\left(\int_{\mathbb{R}} |xg(x)|^2 dx\right)\left(\int_{\mathbb{R}} |\gamma\hat{g}(\gamma)|^2 d\gamma\right) = \infty.$$

A function g generating a Gabor Riesz basis can not be well localized in both time and frequency.

For example: impossible that the estimates

$$\begin{aligned} |g(x)| &\leq \frac{C}{(1+x^2)^{1/2}}, \\ |\hat{g}(\gamma)| &\leq \frac{C}{(1+\gamma^2)^{1/2}} \end{aligned}$$

hold simultaneously.

This motives the construction of Gabor frames!

Gabor frames and Riesz bases

- If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame, then $ab \leq 1$;
- If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame, then

$$\{f_k\}_{k=1}^{\infty} \text{ is a Riesz basis } \Leftrightarrow \; ab = 1.$$

For the sake of time-frequency analysis: we want the Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be generated by a continuous function g with compact support.

Gabor systems $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$

Lemma: *If g is be a continuous function with compact support, then*

- $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *can not be an ONB.*
- $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *can not be a Riesz basis.*
- $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *can be a frame if* $0 < ab < 1$;

Thus, it is necessary to consider frames if we want Gabor systems with good properties.

Gabor frames $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, necessary conditions

The following necessary condition for $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$ depends on the interplay between the function g and the translation parameter a, and is expressed in terms of the function

$$G(x) := \sum_{n\in\mathbb{Z}} |g(x-na)|^2, \; x \in \mathbb{R}.$$

Proposition: Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given, and assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame with bounds A, B. Then

$$bA \leq \sum_{n\in\mathbb{Z}} |g(x-na)|^2 \leq bB, \; a.e. \; x \in \mathbb{R}.$$

Gabor systems $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$

Lemma: Suppose that f is a bounded measurable function with compact support and that the function

$$G(x) = \sum_{n\in\mathbb{Z}} |g(x-na)|^2, \; x \in \mathbb{R}$$

is bounded. Then

$$\sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 = \frac{1}{b}\int_{-\infty}^{\infty} |f(x)|^2 \sum_{n\in\mathbb{Z}} |g(x-na)|^2 \, dx$$
$$+ \quad \frac{1}{b}\sum_{k\neq 0}\int_{-\infty}^{\infty} \overline{f(x)} f(x-k/b) \sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-k/b)} \, dx.$$

Important special case: The formula simplifies if g has (small) compact support.

Gabor frames $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, sufficient conditions

Theorem: Let $g \in L^2(\mathbb{R})$, $a, b > 0$ and suppose that

$$B := \frac{1}{b} \sup_{x\in[0,a]} \sum_{k\in\mathbb{Z}} \left| \sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-k/b)} \right| < \infty.$$

Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence with bound B. If also

$$A := \frac{1}{b} \inf_{x\in[0,a]} \left[\sum_{n\in\mathbb{Z}} |g(x-na)|^2 - \sum_{k\neq 0} \left| \sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-k/b)} \right| \right] > 0,$$

then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B.

Gabor systems with compactly supported g

Corollary: Let $g \in L^2(\mathbb{R})$ be bounded and compactly supported. Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for any $a, b > 0$.

Recall:

$$G(x) = \sum_{n\in\mathbb{Z}} |g(x-na)|^2, \; x \in \mathbb{R}.$$

Corollary: Let $a, b > 0$ be given. Suppose that $g \in L^2(\mathbb{R})$ has support in an interval of length $\frac{1}{b}$ and that the function G is bounded above and below. Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B. The frame operator S and its inverse S^{-1} are given by

$$Sf = \frac{G}{b}f, \;\; S^{-1}f = \frac{b}{G}f, \;\; f \in L^2(\mathbb{R}).$$

Gabor frames and B-splines

Corollary: For $n \in \mathbb{N}$, the B-splines B_n and $\widetilde{B_n}$ generate Gabor frames for all $(a, b) \in]0, n[\times]0, 1/n[$.

Question: Characterization of $(a, b) \in \mathbb{R}^2$ for which B_n generates a Gabor frame?

The exact answer is unknown, and is bound to be complicated:

1) $\{E_{mb}T_{na}B_2\}_{m,n\in\mathbb{Z}}$ can not be a frame for any $b > 0$ whenever $a \geq 2$.
2) [Gröchenig, Janssen, Kaiblinger, Pfander, 2002]: For $b = 2, 3, \ldots,$ $\{E_{mb}T_{na}B_2\}_{m,n\in\mathbb{Z}}$ can not be a frame for any $a > 0$.

Gabor frames $\{E_mT_{na}\chi_{[0,c[}\}_{m,n\in\mathbb{Z}}$

Example:Surprisingly complicated to find the exact range of $c > 0$ and parameters $a, b > 0$ for which $\{E_mT_{na}\chi_{[0,c[}\}_{m,n\in\mathbb{Z}}$ is a frame!
Via a scaling, assume that $b = 1$. Janssen (2002) solved 8 cases:

(i) $\{E_mT_{na}\chi_{[0,c[}\}_{m,n\in\mathbb{Z}}$ is not a frame if $c < a$ or $a > 1$.

(ii) $\{E_mT_{na}\chi_{[0,c[}\}_{m,n\in\mathbb{Z}}$ is a frame if $1 \geq c \geq a$.

Assuming now that $a < 1, c > 1$, we further have

(iv) $\{E_mT_{na}\chi_{[0,c[}\}_{m,n\in\mathbb{Z}}$ is a frame if $a \notin \mathbb{Q}$ and $c \in]1, 2[$.

(v) $\{E_mT_{na}\chi_{[0,c[}\}_{m,n\in\mathbb{Z}}$ is not a frame if $a = p/q \in \mathbb{Q}$, $gcd(p, q) = 1$, and $2 - \frac{1}{q} < c < 2$.

(vi) $\{E_mT_{na}\chi_{[0,c[}\}_{m,n\in\mathbb{Z}}$ is not a frame if $a > \frac{3}{4}$ and $c = L - 1 + L(1 - a)$ with $L \in \mathbb{N}, L \geq 3$.

Complete solution 2012 by Q. Sun (14 cases, 88 pages!)

The duals of a Gabor frame

For a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ with associated frame operator S, the frame decomposition shows that

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, S^{-1}E_{mb}T_{na}g\rangle E_{mb}T_{na}g, \ \forall f \in L^2(\mathbb{R}).$$

We need to be able to calculate the canonical dual frame $\{S^{-1}E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ – difficult!

A simplification can be obtained via

Lemma: Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given, and assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence with frame operator S. Then the following holds:

(i) $SE_{mb}T_{na} = E_{mb}T_{na}S$ for all $m, n \in \mathbb{Z}$.

(ii) If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame, then also

$$S^{-1}E_{mb}T_{na} = E_{mb}T_{na}S^{-1}, \ \forall m, n \in \mathbb{Z}.$$

The duals of a Gabor frame

Theorem: Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given, and assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame. Then the following holds:

(i) The canonical dual frame also has the Gabor structure and is given by $\{E_{mb}T_{na}S^{-1}g\}_{m,n\in\mathbb{Z}}$.

(ii) The canonical tight frame associated with $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is given by $\{E_{mb}T_{na}S^{-1/2}g\}_{m,n\in\mathbb{Z}}$.

The duals of a Gabor frame

Proposition: Let $g \in L^2(\mathbb{R})$, and assume that g as well as $\hat{g}$ decay exponentially. Let $a, b > 0$ be given and assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame. Then $\{E_{mb}T_{na}S^{-1/2}g\}_{m,n\in\mathbb{Z}}$ is a tight frame, for which $S^{-1/2}g$ as well as $\mathcal{F}(S^{-1/2}g)$ decay exponentially.

Theoretically perfect!

Can be applied to the Gaussian $g(x) = e^{-x^2/2}$ – but the resulting generators are not given explicitly.

The duals of a Gabor frame

Example

- The Gaussian

$$g(x) = e^{-x^2}$$

  generates a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for all $a, b \in]0,1[$. The Fourier transform

$$\hat{g}(x) = \sqrt{\pi}e^{-\pi^2 x^2}$$

  has exponential decay. Also, the dual generator $S^{-1}g$ has exponential decay in time and frequency.

- The function

$$h(x) = S^{-1/2}e^{-x^2}$$

  generates a tight Gabor frame $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ for all $a, b \in]0,1[$, and h as well as $\hat{h}$ decay exponentially.

Problem: $S^{-1/2}e^{-x^2}$ and $S^{-1}e^{-x^2}$ are not given explicitly.

The duals of a Gabor frame

Solution: Don't construct a nice frame and *expect* the canonical dual to be nice.

Construct simultaneously dual pairs $\{E_{mb}T_{na}g\}$**,**$\{E_{mb}T_{na}h\}$ **such that** g **and** h **have the required properties, and**

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g, \forall f \in L^2(\mathbb{R}).$$

Pairs of dual Gabor frames

Two Bessel sequences $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ form dual frames if

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g, \ \forall f \in L^2(\mathbb{R}).$$

Ron & Shen, A.J.E.M. Janssen (1998):

Theorem: *Two Bessel sequences* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ *form dual frames if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g(x-n/b-ka)}h(x-ka) = b\delta_{n,0}, \ a.e.\ x \in [0,a].$$

History

- Bölsckei, Janssen, 1998-2000: For b rational, characterization of Gabor frames with compactly supported window having a compactly supported dual window;
- Feichtinger, Gröchenig (1997): window in $\mathcal{S}_0$ implies that the canonical dual window is in $\mathcal{S}_0$;
- Krishtal, Okoudjou, 2007: window in $W(L^\infty, \ell^1)$ implies that the canonical dual window is in $W(L^\infty, \ell^1)$.

Duality principle

The duality principle (Janssen, Daubechies, Landau, and Landau, and Ron and Shen: concerns the relationship between frame properties for a function g with respect to the lattice $\{(na, mb)\}_{m,n\in\mathbb{Z}}$ and with respect to the so-called *dual lattice* $\{(n/b, m/a)\}_{m,n\in\mathbb{Z}}$:

Theorem: *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the following are equivalent:*

(i) $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a frame for $L^2(\mathbb{R})$ with bounds A, B;*

(ii) $\{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ *is a Riesz sequence with bounds A, B.*

Intuition: If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then $ab \leq 1$, i.e., the sampling points $\{(na, mb)\}_{m,n\in\mathbb{Z}}$ are "sufficiently dense." Therefore the points $\{(n/b, m/a)\}_{m,n\in\mathbb{Z}}$ are "sparse," and therefore $\{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ are linearly independent and only span a subspace of $L^2(\mathbb{R})$.

Wexler-Raz' Theorem

Wexler-Raz' Theorem: If the Gabor systems $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ are Bessel sequences, then the following are equivalent:

(i) The Gabor systems $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ are dual frames;

(ii) The Gabor systems $\{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ and $\{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}h\}_{m,n\in\mathbb{Z}}$ are biorthogonal, i.e.,

$$\langle\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g,\frac{1}{\sqrt{ab}}E_{m'/a}T_{n'/b}h\rangle=\delta_{m,m'}\delta_{n,n'}.$$

Explicit construction of dual pairs of Gabor frames

In order for a frame $\{f_k\}_{k=1}^{\infty}$ to be useful, we need a dual frame $\{g_k\}_{k=1}^{\infty}$, i.e., a frame such that

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \ \forall f \in \mathcal{H}.$$

How can we construct convenient dual frames?

Ansatz/suggestion: Given a window function $g \in L^2(\mathbb{R})$generating a frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, look for a dual window of the form

$$h(x) = \sum_{k=-K}^{K} c_k g(x+k).$$

The structure of h makes it easy to derive properties of h based on properties of g (regularity, size of support, membership un various vector spaces,....)

Explicit construction of dual pairs of Gabor frames

Theorem:(C., 2006) *Let* $N \in \mathbb{N}$. *Let* $g \in L^2(\mathbb{R})$ *be a real-valued bounded function for which*

- *supp* $g \subseteq [0, N]$,
- $\sum_{n \in \mathbb{Z}} g(x-n) = 1$.

Let $b \in]0, \frac{1}{2N-1}]$. *Then the function* g *and the function* h *defined by*

$$h(x) = bg(x) + 2b \sum_{n=1}^{N-1} g(x+n)$$

generate dual frames $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$ *and* $\{E_{mb}T_n h\}_{m,n \in \mathbb{Z}}$ *for* $L^2(\mathbb{R})$.

Candidates for g - the B-splines

For the B-spline case:

- The functions B_N and the dual window

$$h(x) = bB_N(x) + 2b\sum_{n=1}^{N-1} B_N(x+n)$$

  are splines;
- B_N and h have compact support, i.e., perfect time–localization;
- By choosing N sufficiently large, polynomial decay of $\widehat{B_N}$ and h of any desired order can be obtained.

Note:

$$\widehat{B_N}(\gamma) = \left(\frac{\sin \pi\gamma}{\pi\gamma}\right)^N e^{-\pi i N\gamma}.$$

Example

For the B-spline

$$B_2(x) = \begin{cases} x & x \in [0,1[, \\ 2-x & x \in [1,2[, \\ 0 & x \notin [0,2[, \end{cases}$$

we can use the result for $b \in]0, 1/3]$. For $b = 1/3$ we obtain the dual generator

$$\begin{aligned} h(x) &= \frac{1}{3}B_2(x) + \frac{2}{3}B_2(x+1) \\ &= \begin{cases} \frac{2}{3}(x+1) & x \in [-1,0[, \\ \frac{1}{3}(2-x) & x \in [0,2[, \\ 0 & x \notin [-1,2[. \end{cases} \end{aligned}$$

Example

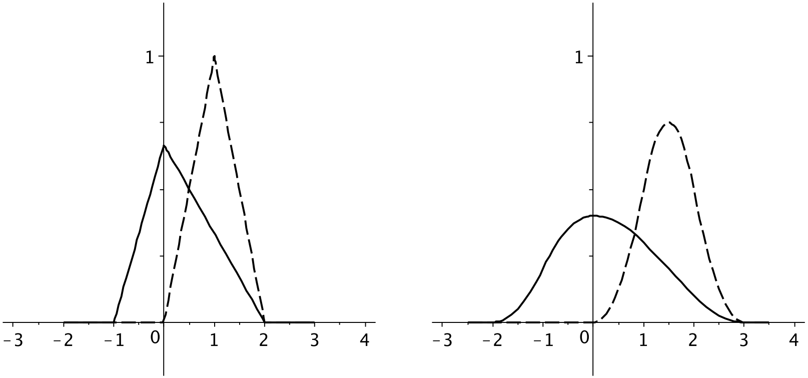

Figure: The B-spline N_2 and the dual generator h for $b = 1/3$; and the B-spline N_3 and the dual generator h with $b = 1/5$.

Other choices? **Yes!**

Theorem: (C., Kim, 2007) *Let* $N \in \mathbb{N}$. *Let* $g \in L^2(\mathbb{R})$ *be a real-valued bounded function for which*

- *supp* $g \subseteq [0, N]$,
- $\sum_{n\in\mathbb{Z}} g(x-n) = 1$.

Let $b \in]0, \frac{1}{2N-1}]$. *Define* $h \in L^2(\mathbb{R})$ *by*

$$h(x) = \sum_{n=-N+1}^{N-1} a_n g(x+n),$$

where

$$a_0 = b, \;\; a_n + a_{-n} = 2b, \; n = 1, 2, \cdots, N-1.$$

Then g *and* h *generate dual frames* $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ *for* $L^2(\mathbb{R})$.

Example: B-splines revisited

1) Take

$$a_0 = b, a_n = 0 \text{ for } n = -N+1, \dots, -1, a_n = 2b, n = 1, \dots N-1.$$

This is the previous Theorem. This choice gives the shortest support.
2) Take

$$a_{-N+1} = a_{-N+2} = \cdots = a_{N-1} = b:$$

if g is symmetric, this leads to a symmetric dual generator

$$h(x) = b \sum_{n=-N+1}^{N-1} g(x+n).$$

Note: $h(x) = b$ on supp g.

B-splines revisited

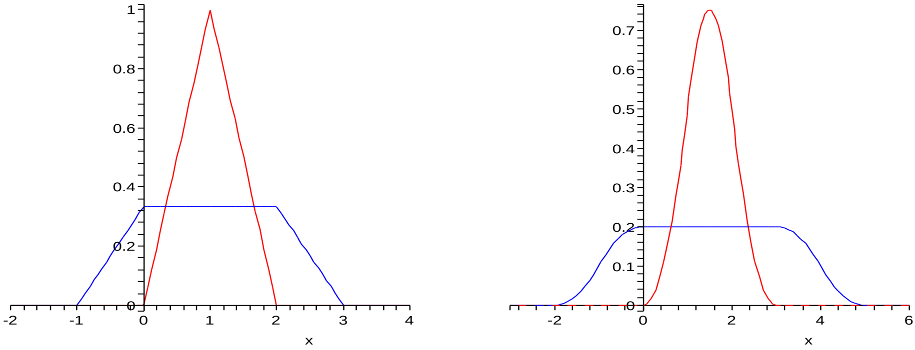

Figure: The generators B_2 and B_3 and their dual generators via 2).

Another class of examples:

Exponential B-splines: Given a sequence of scalars $\beta_1, \beta_2, \ldots, \beta_N \in \mathbb{R}$, let

$$\mathcal{E}_N := e^{\beta_1(\cdot)}\chi_{[0,1]}(\cdot) * e^{\beta_2(\cdot)}\chi_{[0,1]}(\cdot) * \cdots * e^{\beta_N(\cdot)}\chi_{[0,1]}(\cdot).$$

- The function $\mathcal{E}_N$ is supported on $[0, N]$.
- If $\beta_k = 0$ for at least one $k = 1, \ldots, N$, then $\mathcal{E}_N$ satisfies the partition of unity condition (up to a constant).
- If $\beta_j \neq \beta_k$ for $j \neq k$, an explicit expression for $\mathcal{E}_N$ is known (C., Peter Massopust, 2010).

Explicit constructions

Assume that $\beta_k = (k-1)\beta, k = 1, \ldots, N$. Then

$$\mathcal{E}_N(x) =$$

$$\begin{cases} \frac{1}{\beta^{N-1}} \sum_{k=0}^{N-1} \frac{1}{\prod_{\substack{j=1 \\ j\neq k+1}}^{N} (k+1-j)} e^{\beta k x}, & x \in [0,1]; \\ \frac{(-1)^{\ell-1}}{\beta^{N-1}} \sum_{k=0}^{N-1} \left(\frac{\sum_{\substack{0\le j_1<\cdots<j_{\ell-1}\le N-1 \\ j_1,\ldots,j_{\ell-1}\neq k-1}} [e^{\beta j_1} + \cdots + e^{\beta j_{\ell-1}}]}{\prod_{\substack{j=1 \\ j\neq k+1}}^{N} (k+1-j)} \right) e^{\beta k(x-\ell+1)}, & \begin{matrix} x \in [\ell-1,\ell] \\ \ell = 2,\ldots,N \end{matrix}. \end{cases}$$

Explicit constructions

$$\sum_{k\in\mathbb{Z}} \mathcal{E}_N(x-k) = \frac{\prod_{m=1}^{N-1}\left(e^{\beta m}-1\right)}{\beta^{N-1}(N-1)!}.$$

Via the Theorem: construction of dual Gabor frames with generators

$$\mathcal{E}_N, \qquad h_N(x) = \sum_{k=-N+1}^{N-1} c_k \mathcal{E}_N(x+k).$$

From Gabor frames to wavelet frames - duality conditions:

Theorem: *Two Bessel sequences* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ *form dual frames if and only if*

(i) $\sum_{k\in\mathbb{Z}} \overline{g(x-ka)}h(x-ka) = b,\ a.e.\ x \in [0,a].$

(ii) $\sum_{k\in\mathbb{Z}} \overline{g(x-ka-n/b)}h(x-ka) = 0,\ a.e.\ x \in [0,a],\ n \in \mathbb{Z}\setminus\{0\}.$

Theorem: *Given* $a>1$, $b>0$, *two Bessel sequences* $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ *and* $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$, *where* $\psi,\widetilde{\psi}\in L^2(\mathbb{R})$, *form dual wavelet frames for* $L^2(\mathbb{R})$ *if and only if the following two conditions are satisfied:*

(i) $\sum_{j\in\mathbb{Z}} \overline{\widehat{\psi}(a^j\gamma)}\widehat{\widetilde{\psi}}(a^j\gamma) = b$ *for a.e.* $\gamma\in\mathbb{R}$.

(ii) *For any number* $\alpha\neq 0$ *of the form* $\alpha = m/a^j,\ m,j\in\mathbb{Z}$,

$$\sum_{\{(j,m)\in\mathbb{Z}^2\ |\ \alpha=m/a^j\}} \overline{\widehat{\psi}(a^j\gamma)}\widehat{\widetilde{\psi}}(a^j\gamma+m/b) = 0,\ a.e.\ \gamma\in\mathbb{R}.$$

From Gabor frames to wavelet frames

C., Say Song Goh, 2012: metod for construction of dual pairs of wavelet frames based on dual pairs of Gabor frames.

Let $\theta > 1$ be given. Associated with a function $g \in L^2(\mathbb{R})$ with the property that $g(\log_\theta |\cdot|) \in L^2(\mathbb{R})$ we define a function $\psi \in L^2(\mathbb{R})$ by

$$\widehat{\psi}(\gamma) = g(\log_\theta(|\gamma|)).$$

Then

$$\widehat{\psi}(a^j\gamma) = g(j\log_\theta(a) + \log_\theta(|\gamma|)).$$

When applied to the dual Gabor frames $\{E_{mb}T_n\mathcal{E}_N\}_{m,n\in\mathbb{Z}},\ \{E_{mb}T_n h_N\}_{m,n\in\mathbb{Z}}$:

Construction of dual pairs of wavelet frames with generators ψ and $\widetilde{\psi}$, for which $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$ are compactly supported splines with geometrically distributed knot sequences.

Example

The exponential B-spline $\mathcal{E}_2$ with $\beta_1 = 0, \beta_2 = 1$:

$$\mathcal{E}_2(x) = \begin{cases} 0, & x \notin [0,2], \\ e^x - 1, & x \in [0,1], \\ e - e^{-1}e^x, & x \in [1,2]. \end{cases}$$

Then

$$\sum_{k \in \mathbb{Z}} \mathcal{E}_2(x-k) = e - 1,\ x \in \mathbb{R},$$

so we consider the function

$$g(x) := (e-1)^{-1}\mathcal{E}_2(x).$$

Example

Thus, the function $g(x) := (e-1)^{-1}\mathcal{E}_2(x)$ has support in $[0,2]$ and satisfies that

$$\sum_{n\in\mathbb{Z}} g(x-n) = 1.$$

Taking

$$h(x) := b\sum_{n=-1}^{1} g(x+n)$$

leads to a pair of dual Gabor frames $\{E_{\frac{m}{15}}T_n g\}_{m,n\in\mathbb{Z}},\ \{E_{\frac{m}{15}}T_n h\}_{m,n\in\mathbb{Z}}$.

Example

- Let ψ be defined via

$$\widehat{\psi}(\gamma) = g(\log_e(|\gamma|)) = g(\ln(|\gamma|)) = \begin{cases} 0, & |\gamma| \notin [1, e^2], \\ \frac{|\gamma|-1}{e-1}, & |\gamma| \in [1, e], \\ \frac{e-e^{-1}|\gamma|}{e-1}, & |\gamma| \in [e, e^2]. \end{cases}$$

- $\widehat{\psi}$ is a geometric spline with knots at the points $\pm 1, \pm e, \pm e^2$.
- Let $b = 15^{-1}$. Then the function $\widetilde{\psi}$ defined by

$$\widehat{\widetilde{\psi}}(\gamma) = h(\ln(|\gamma|)) = \frac{1}{15} \sum_{n=-1}^{1} \widehat{\psi}(|e^n \gamma|)$$

is a dual generator.
- $\widehat{\widetilde{\psi}}$ is a geometric spline with knots at $\pm e^{-1}, \pm 1, \pm e^2, \pm e^3$.

Example

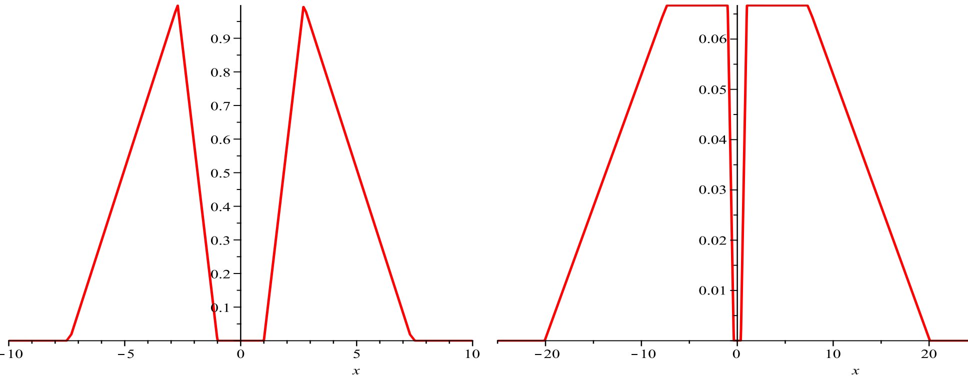

Figure: Plots of the geometric splines $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$.

From wavelet frames to Gabor frames

The *Meyer wavelet:* the first example of a function $\psi \in L^2(\mathbb{R})$ such that $\widehat{\psi} \in C^\infty(\mathbb{R})$ has compact support.
The Meyer wavelet is really a class of wavelets.

From wavelet frames to Gabor frames

The *Meyer wavelet*is the function $\psi \in L^2(\mathbb{R})$ defined via

$$\widehat{\psi}(\gamma) = \begin{cases} e^{i\pi\gamma}\sin(\frac{\pi}{2}(\nu(3|\gamma|-1))), & \text{if } 1/3 \le |\gamma| \le 2/3, \\ e^{i\pi\gamma}\cos(\frac{\pi}{2}(\nu(3|\gamma|/2-1))), & \text{if } 2/3 \le |\gamma| \le 4/3, \\ 0, & \text{if } |\gamma| \notin [1/3, 4/3], \end{cases}$$

where $\nu : \mathbb{R} \to \mathbb{R}$ is any continuous function for which

$$\nu(x) = \begin{cases} 0, & \text{if } x \le 0, \\ 1, & \text{if } x \ge 1, \end{cases}$$

and

$$\nu(x) + \nu(1-x) = 1,\ x \in \mathbb{R}.$$

Known: $\{D_{2^j}T_k\psi\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, and

$$\text{supp}\,\widehat{\psi} = [-4/3, -1/3] \cup [1/3, 4/3].$$

From wavelet frames to Gabor frames

Let

$$\nu_0(x) = \begin{cases} \exp\left[-\left\{\exp\left[x/(1-x)\right]-1\right\}^{-1}\right], & \text{if } 0 < x < 1, \\ 0, & \text{if } x \leq 0, \\ 1, & \text{if } x \geq 1, \end{cases}$$

$$\nu(x) := \frac{1}{2}(\nu_0(x) - \nu_0(1-x) + 1), \; x \in \mathbb{R}.$$

$$\tau(x) := \begin{cases} \frac{1}{\sqrt{2}} \sin(\frac{\pi}{2}(\nu(3 \cdot 2^x - 1))), & \text{if } -\frac{\ln 3}{\ln 2} \leq x \leq 1 - \frac{\ln 3}{\ln 2}, \\ \frac{1}{\sqrt{2}} \cos(\frac{\pi}{2}(\nu(\frac{3}{2} \cdot 2^x - 1))), & \text{if } 1 - \frac{\ln 3}{\ln 2} \leq x \leq 2 - \frac{\ln 3}{\ln 2}, \\ 0, & \text{if } x \notin [-\frac{\ln 3}{\ln 2}, 2 - \frac{\ln 3}{\ln 2}], \end{cases}$$

Then τ is real-valued, compactly supported, belongs to $C^\infty(\mathbb{R})$, and $\{E_{m/2}T_n\tau\}_{m,n\in\mathbb{Z}}$ is a tight frame with bound $A = 1$.

From wavelet frames to Gabor frames

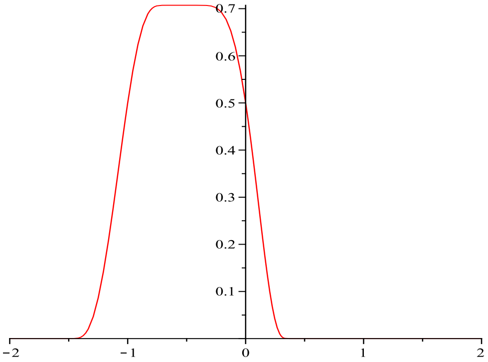

Figure: The function τ, which is C^∞ and has compact support. The Gabor system $\{E_{m/2}T_n\tau\}_{m,n\in\mathbb{Z}}$ is a tight frame with bound $A = 1$.

Part III:

Research problems related to frames and operator theory

Open problems

- An extension problem for wavelet frames
- The duality principle in general Hilbert spaces

An extension problem for wavelet frames

Extension of Bessel sequences to tight frames

Theorem: For each Bessel sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$, there exist a family of vectors $\{g_k\}_{k=1}^{\infty}$ such that

$$\{f_k\}_{k=1}^{\infty} \cup \{g_k\}_{k=1}^{\infty}$$

is a tight frame for $\mathcal{H}$.

Extension of Bessel sequences to dual frames

Thorem: Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be Bessel sequences in a Hilbert space $\mathcal{H}$. Then there exist Bessel sequences $\{p_j\}_{i\in J}$ and $\{q_j\}_{i\in J}$ in $\mathcal{H}$ such that $\{f_i\}_{i\in I} \cup \{p_j\}_{i\in J}$ and $\{g_i\}_{i\in I} \cup \{q_j\}_{i\in J}$ form a pair of dual frames for $\mathcal{H}$.

Proof. Let T and U denote the preframe operators for $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$, respectively, i.e.,

$$T, U : \ell^2(I) \to \mathcal{H},\ T\{c_i\}_{i\in I} = \sum_{i\in I} c_i f_i,\ \ U\{c_i\}_{i\in I} = \sum_{i\in I} c_i g_i.$$

Let $\{a_j\}_{i\in J}, \{b_j\}_{i\in J}$ denote any pair of dual frames for $\mathcal{H}$. Then

$$\begin{aligned} f = UT^*f + (I - UT^*)f &= \sum_{i\in I} \langle f, f_i\rangle g_i + \sum_{j\in I} \langle (I - UT^*)f, a_j\rangle b_j \\ &= \sum_{i\in I} \langle f, f_i\rangle g_i + \sum_{j\in I} \langle f, (I - UT^*)^* a_j\rangle b_j \end{aligned}$$

The sequences $\{f_i\}_{i\in I}, \{g_i\}_{i\in I}$, and $\{b_j\}_{i\in J}$ are Bessel sequences by definition, and one can verify that $\{(I - UT^*)^* a_j\}_{j\in J}$ is a Bessel sequence as well. □

Extension of Gabor Bessel sequences to tight frames

Theorem (D. Li and W.Sun, 2009): Let $\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}}$ be Bessel sequences in $L^2(\mathbb{R})$, and assume that $ab \le 1$. Then the following hold:

- There exists a Gabor systems $\{E_{mb}T_{na}g_2\}_{m,n\in\mathbb{Z}}$ such that

$$\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}g_2\}_{m,n\in\mathbb{Z}}$$

  is a tight frame for $L^2(\mathbb{R})$.
- If g_1 has compact support and $|\text{supp} g_1| \le b^{-1}$, then g_2 can be chosen to have compact support.

Extension of Gabor Bessel sequences to dual frame pairs

Theorem (C., Kim, Kim, 2011): Let $\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h_1\}_{m,n\in\mathbb{Z}}$ be Bessel sequences in $L^2(\mathbb{R})$, and assume that $ab \leq 1$. Then the following hold:

- There exist Gabor systems $\{E_{mb}T_{na}g_2\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h_2\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ such that

  $$\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}g_2\}_{m,n\in\mathbb{Z}} \text{ and } \{E_{mb}T_{na}h_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}h_2\}_{m}$$

  form a pair of dual frames for $L^2(\mathbb{R})$.
- If g_1 and h_1 have compact support, the functions g_2 and h_2 can be chosen to have compact support.

Note: closely related to work by Han (2009), where it is assumed that $\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h_1\}_{m,n\in\mathbb{Z}}$ are dual frames for a subspace.

Extension of Gabor Bessel sequences to dual frame pairs

Key step in the proof: Take $\{a_j\}_{i\in J}, \{b_j\}_{i\in J}$ to have Gabor structure, i.e., $a_j = E_{mb}T_{na}g_1$. Then

$$(I - UT^*)^*a_j = (I - TU^*)E_{mb}T_{na}g_1 = E_{mb}T_{na}(I - TU^*)g_1,$$

which has Gabor structure!

The wavelet case

Let $\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ be Bessel sequences in $L^2(\mathbb{R})$. Take $\{a_j\}_{i\in J}, \{b_j\}_{i\in J}$ to have wavelet structure. Then

$$(I - UT^*)^* a_j = (I - TU^*) D^j T_k \varphi.$$

Unfortunately, the operator TU^* in general does not commute with $D^j T_k \varphi$! Thus, we can not copy the technique from the Gabor analysis.

Open question: Can a pair of wavelet Bessel systems $\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ be extended to a pair of dual frames by adding just one pair of wavelet systems?

Known: Any pair of wavelet Bessel systems $\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ can be extended to a pair of dual frames by adding *two* pairs of wavelet systems.

The wavelet case

Theorem (C., Kim, Kim, 2011): Let $\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ be Bessel sequences in $L^2(\mathbb{R})$. Assume that the Fourier transform of $\widetilde{\psi_1}$ satisfies

$$\text{supp}\ \widehat{\widetilde{\psi_1}} \subseteq [-1,1].$$

Then there exist wavelet systems $\{D^j T_k \psi_2\}_{j,k\in\mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi_2}\}_{j,k\in\mathbb{Z}}$ such that

$$\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}} \cup \{D^j T_k \psi_2\}_{j,k\in\mathbb{Z}} \text{ and } \{D^j T_k \widetilde{\psi_1}\}_{j,k\in\mathbb{Z}} \cup \{D^j T_k \widetilde{\psi_2}\}_{j,k\in\mathbb{Z}}$$

form dual frames for $L^2(\mathbb{R})$.

Corollary: (C., Kim, Kim, 2011): In the above setup, assume that $\widehat{\psi_1}$ is compactly supported and that

$$\text{supp}\ \widehat{\widetilde{\psi_1}} \subseteq [-1,1] \setminus [-\epsilon,\epsilon]$$

for some $\epsilon > 0$. Then the functions ψ_2 and $\widetilde{\psi_2}$ can be chosen to have compactly supported Fourier transforms as well.

The wavelet case - Open problems:

- Let $\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ be Bessel sequences in $L^2(\mathbb{R})$. Assume that supp $\widehat{\widetilde{\psi_1}}$ is NOT contained in $[-1, 1]$. Does there exist wavelet systems $\{D^j T_k \psi_2\}_{j,k\in\mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi_2}\}_{j,k\in\mathbb{Z}}$ such that

  $$\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}} \cup \{D^j T_k \psi_2\}_{j,k\in\mathbb{Z}} \text{ and } \{D^j T_k \widetilde{\psi_1}\}_{j,k\in\mathbb{Z}} \cup \{D^j T_k \widetilde{\psi_2}\}_{j,k\in\mathbb{Z}}$$

  form dual frames for $L^2(\mathbb{R})$?
- Or can we find just one example of a pair of Bessel sequences $\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}}$ and $\{D^j T_k \widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ that can not be extended to a pair of dual wavelet frames, each with 2 generators?
- Any positive or negative conclusion will be new!

Note: A pair of wavelet Bessel sequences can always be extended to dual wavelet frame pairs by adding two pairs of wavelet systems.

The wavelet case

- Conjecture, Han, 2009: Let $\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}}$ be a wavelet frame with upper frame bound B. Then there exists $D > B$ such that for each $K \geq D$, there exists $\widetilde{\psi_1} \in L^2(\mathbb{R})$ such that

$$\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}} \cup \{D^j T_k \widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$$

  is a tight frame for $L^2(\mathbb{R})$ with bound K.
- Based on an example, where

$$\text{supp } \widehat{\psi_1} \subseteq [-1, 1].$$

The duality principle in general Hilbert spaces

Gabor systems

Gabor systems: have the form

$$\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$$

for some $g \in L^2(\mathbb{R}),\ a,b>0$. Short notation:

$$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imbx}g(x-na)\}$$

Duality principle, Wexler-Raz' Theorem

The duality principle:

Theorem: *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the following are equivalent:*

(i) $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$ *with bounds* A, B;

(ii) $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ *is a Riesz sequence with bounds* A, B.

Wexler-Raz' Theorem: If the Gabor systems $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ are Bessel sequences, then the following are equivalent:

(i) The Gabor systems $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ are dual frames;

(ii) The Gabor systems $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ and $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}h\}_{m,n\in\mathbb{Z}}$ are biorthogonal, i.e.,

$$\langle \frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g, \frac{1}{\sqrt{ab}} E_{m'/a}T_{n'/b}h \rangle = \delta_{m,m'}\delta_{n,n'}.$$

Duality principle

Can the duality principle in Gabor analysis be recast as a special case of a general theory, valid for general frames?

Abstract duality in a Hilbert space $\mathcal{H}$

R-dual of a sequence $\{f_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$, introduced by Casazza, Kutyniok, and Lammers:

Definition: *Let* $\{e_i\}_{i\in I}$ *and* $\{h_i\}_{i\in I}$ *denote orthonormal bases for* $\mathcal{H}$, *and let* $\{f_i\}_{i\in I}$ *be any sequence in* $\mathcal{H}$ *for which* $\sum_{i\in I} |\langle f_i, e_j\rangle|^2 < \infty$ *for all* $j \in I$. *The R-dual of* $\{f_i\}_{i\in I}$ *with respect to the orthonormal bases* $\{e_i\}_{i\in I}$ *and* $\{h_i\}_{i\in I}$ *is the sequence* $\{\omega_j\}_{j\in I}$ *given by*

$$\omega_j = \sum_{i\in I} \langle f_i, e_j\rangle h_i, \ i \in I. \tag{10}$$

Abstract duality - results by Casazza, Kutyniok, Lammers:

Theorem: *Define the R-dual* $\{\omega_j\}_{j\in I}$ *of a sequence* $\{f_i\}_{i\in I}$ *as above. Then:*

(i) *For all* $i \in I$,

$$f_i = \sum_{j\in I} \langle \omega_j, h_i \rangle e_j, \tag{11}$$

i.e., $\{f_i\}_{i\in I}$ *is the R-dual sequence of* $\{\omega_j\}_{j\in I}$ *w.r.t. the orthonormal bases* $\{h_i\}_{i\in I}$ *and* $\{e_i\}_{i\in I}$.

(ii) $\{f_i\}_{i\in I}$ *is a Bessel sequence if and only* $\{\omega_i\}_{i\in I}$ *is a Bessel sequence.*

(iii) $\{f_i\}_{i\in I}$ *satisfies the lower frame condition with bound A if and only if* $\{\omega_j\}_{j\in I}$ *satisfies the lower Riesz sequence condition with bound A.*

(iv) $\{f_i\}_{i\in I}$ *is a frame for* $\mathcal{H}$ *with bounds* A, B *if and only if* $\{\omega_j\}_{j\in I}$ *is a Riesz sequence in* $\mathcal{H}$ *with bounds* A, B.

(v) *Two Bessel sequences* $\{f_i\}_{i\in I}$ *and* $\{g_i\}_{i\in I}$ *in* $\mathcal{H}$ *are dual frames if and only if the associated R-dual sequences* $\{\omega_j\}_{j\in I}$ *and* $\{\gamma_j\}_{j\in I}$ *satisfy that*

$$\langle \omega_j, \gamma_k \rangle = \delta_{j,k}, \ j,k \in I. \tag{12}$$

The duality principle in Gabor analysis

The *duality principle:*

Theorem: *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the Gabor system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B if and only if $\{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ is a Riesz sequence with bounds A, B.*

- Can this result be derived as a consequence of the abstract duality concept?
- That is, can $\{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ be realized as the R-dual of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ w.r.t. certain choices of orthonormal bases $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$ and $\{h_{m,n}\}_{m,n\in\mathbb{Z}}$?

Abstract duality

Result by Casazza, Kutyniok, Lammers:

- If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame and $ab = 1$, then $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ can be realized as the R-dual of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ w.r.t. certain choices of orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$.
- If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a tight frame, then $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ can be realized as the R-dual of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ w.r.t. certain choices of orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$.

Abstract duality - a general approach

Question A: What are the conditions on two sequences $\{f_i\}_{i\in I}, \{\omega_j\}_{j\in I}$ such that $\{\omega_j\}_{j\in I}$ is the R-dual of $\{f_i\}_{i\in I}$ with respect to *some* choice of the orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$?

- We will always assume that $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$.
- We arrive at the following equivalent formulation of Question A:

Question B: Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ and $\{\omega_j\}_{j\in I}$ a Riesz sequence in $\mathcal{H}$. Under what conditions can we find orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ for $\mathcal{H}$ such that

$$f_i = \sum_{j\in I} \langle \omega_j, h_i \rangle e_j, \tag{13}$$

holds?

Idea: Fix an ONB $\{e_i\}_{i\in I}$ and search for $\{h_i\}_{i\in I}$ such that (13) holds!

Abstract duality

Theorem: (C., Kim, Kim) *Let* $\{\omega_j\}_{j\in I}$ *be a Riesz basis for the subspace* W *of* $\mathcal{H}$*, with dual Riesz basis* $\{\widetilde{\omega_k}\}_{k\in I}$*. Let* $\{e_i\}_{i\in I}$ *be an orthonormal basis for* $\mathcal{H}$*. Given any frame* $\{f_i\}_{i\in I}$ *for* $\mathcal{H}$*, the following hold:*

(i) *There exists a sequence* $\{h_i\}_{i\in I}$ *in* $\mathcal{H}$ *such that*

$$f_i = \sum_{j\in I} \langle \omega_j, h_i \rangle e_j, \ \forall i \in I. \tag{14}$$

(ii) *The sequences* $\{h_i\}_{i\in I}$ *satisfying* (14) *are characterized as*

$$h_i = m_i + n_i, \tag{15}$$

where $m_i \in W^\perp$ *and*

$$n_i := \sum_{k=1}^{\infty} \langle e_k, f_i \rangle \widetilde{\omega_k}, \ i \in I.$$

Question C: When is it possible to find an orthonormal basis $\{h_i\}_{i\in I}$ for $\mathcal{H}$ of the form (15)?

Abstract duality

Theorem: (C., Kim, Kim) *Let* $\{\omega_j\}_{j\in I}$ *be a Riesz sequence spanning a proper subspace* W *of* $\mathcal{H}$ *(bounds* C, D*) and* $\{e_i\}_{i\in I}$ *an orthonormal basis for* $\mathcal{H}$*. Given any frame* $\{f_i\}_{i\in I}$ *for* $\mathcal{H}$ *(bounds* A, B*) the following are equivalent:*

(i) $\{\omega_j\}_{j\in I}$ *is an R-dual of* $\{f_i\}_{i\in I}$ *w.r.t.* $\{e_i\}_{i\in I}$ *and some orthonormal basis* $\{h_i\}_{i\in I}$.

(ii) *There exists an orthonormal basis* $\{h_i\}_{i\in I}$ *for* $\mathcal{H}$ *satisfying*

$$f_i = \sum_{j\in I} \langle \omega_j, h_i \rangle e_j, \ \forall i \in I.$$

(iii) *The sequence* $\{n_i\}_{i\in I}$ *is a tight frame for* W *with frame bound* $E = 1$,

$$n_i := \sum_{k=1}^{\infty} \langle e_k, f_i \rangle \widetilde{\omega_k}, \ i \in I.$$

Can show: $\{n_i\}_{i\in I}$ is always a frame for W with bounds $A/D, B/C$!

The duality principle in Gabor analysis

Key question: Given a frame $\{f_i\}_{i\in I}$ and a Riesz sequence $\{\omega_j\}_{j\in I}$, both with bounds A, B. Let $\{\widetilde{\omega_k}\}_{k\in I}$ denote the dual Riesz basis. Finally, let

$$W := \overline{\text{span}}\{\omega_j\}_{j\in I}.$$

- Can we find an ONB $\{e_i\}_{i\in I}$ such that $\{n_i\}_{i\in I}$, given by

$$n_i := \sum_{k=1}^{\infty} \langle e_k, f_i\rangle \widetilde{\omega_k}, \ i \in I,$$

  is a tight frame for W with bound 1?
- Or can we find a case where NO choice of $\{e_i\}_{i\in I}$ makes $\{n_i\}_{i\in I}$ a tight frame for W with bound 1?
  Yes!

Abstract duality

Example (Diana Stoeva): Let $\{z_i\}$ be an orthonormal basis. Consider the overcomplete frame

$$\{f_i\} = \{z_1, z_1, z_2, z_3, z_3, z_4, z_5, z_5, z_6, \dots\}$$

for $\mathcal{H}$ and the Riesz sequence

$$\{g_i\} = \{\sqrt{2}z_1, z_3, z_5, z_7, z_9, \dots\},$$

which have the same optimal bounds $A = 1$, $B = 2$.

Abstract duality

Assume that there exist orthonormal bases $\{e_i\}$ and $\{h_i\}$ so that $\{g_i\}$ is an R-dual of $\{f_i\}$ wrt $\{e_i\}$ and $\{h_i\}$, i.e.

$$g_j = \sum_{i=1}^{\infty} \langle f_i, e_j \rangle h_i, j \in \mathbb{N}. \tag{16}$$

Then

$$z_3 = g_2 = \sum_{i=1}^{\infty} \langle f_i, e_2 \rangle h_i,$$

which implies that

$$\langle z_3, h_1 \rangle = \langle z_1, e_2 \rangle, \ \langle z_3, h_2 \rangle = \langle z_1, e_2 \rangle, \ \langle z_3, h_3 \rangle = \langle z_2, e_2 \rangle,$$
$$\langle z_3, h_4 \rangle = \langle z_3, e_2 \rangle, \ \langle z_3, h_5 \rangle = \langle z_3, e_2 \rangle, \ \langle z_3, h_6 \rangle = \langle z_4, e_2 \rangle, \ \ldots,$$

Abstract duality

and thus,

$$\begin{aligned} 1 &= \|z_3\|^2 = \sum_{i=1}^{\infty} |\langle z_3, h_i \rangle|^2 = \sum_{i=1}^{\infty} |\langle z_i, e_2 \rangle|^2 + \sum_{i=1}^{\infty} |\langle z_{2i-1}, e_2 \rangle|^2 \\ &= 1 + \sum_{i=1}^{\infty} |\langle z_{2i-1}, e_2 \rangle|^2. \end{aligned}$$

Therefore, $e_2 \perp z_{2i-1}, \forall i \in \mathbb{N}$. In the same way, $e_3 \perp z_{2i-1}, \forall i \in \mathbb{N}$, $e_4 \perp z_{2i-1}, \forall i \in \mathbb{N}$, etc. In particular,

$$z_1 \perp e_i, \ \forall i \geq 2, \ \text{ and } \ z_3 \perp e_i, \ \forall i \geq 2,$$

which is a contradiction. □

Abstract duality

Not all Riesz sequences with given bounds are R-duals of any frame with the same bounds.

Conclusion:

- Either the theory of R-duals is not an extension of the duality principle for Gabor frames;
- Or the theory for R-duals is an extension of the duality principle, but for some complicated reasons that is not just related to the value of the frame bounds.

Abstract duality

Special case - a toy problem: Assume that $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$ with bounds A, B, and that $\{\omega_j\}_{j\in I}$ is a Riesz basis for $\mathcal{H}$, also with bounds A, B. Then, for any ONB $\{e_i\}_{i\in I}$ the equation

$$f_i = \sum_{j\in I} \langle \omega_j, h_i \rangle e_j, \ \forall i \in I$$

has the unique solution

$$h_i = n_i = \sum_{k=1}^{\infty} \langle e_k, f_i \rangle \widetilde{\omega_k}, \ i \in I.$$

Question: Can we find an ONB $\{e_i\}_{i\in I}$ such that $\{n_i\}_{i\in I}$ is an ONB for $\mathcal{H}$?

Abstract duality

Corollary: (C., Kim, Kim) *Assume that* $\{\omega_j\}_{j\in I}$ *is a tight orthogonal Riesz sequence spanning a proper subspace of* $\mathcal{H}$ *with (Riesz) bound A and that* $\{f_i\}_{i\in I}$ *is a tight frame for* $\mathcal{H}$ *with frame bound A. Then the following hold:*

(i) *Given any orthonormal basis* $\{e_i\}_{i\in I}$ *for* $\mathcal{H}$, *there exists an orthonormal basis* $\{h_i\}_{i\in I}$ *for* $\mathcal{H}$ *such that*

$$f_i = \sum_{j\in I} \langle \omega_j, h_i\rangle e_j.$$

(ii) $\{\omega_i\}_{i\in I}$ *is an R-dual of* $\{f_i\}_{i\in I}$.

As a special case we obtain the following known result:

Corollary: (Casazza, Kutyniok, Lammers) *If* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a tight frame then* $\{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ *can be realized as the R-dual of* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$.

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