# Wavelet Frames of Higher Riesz Transforms

Stefan Held

Technische Universität München, TUM,
Zentrum Mathematik M6

Summerschool
New Trends and Directions in Harmonic Analysis, Fractional Operator Theory and Image Analysis
Inzell, September 17, 2012

# Hilbert transform

**What is the amplitude and phase of a signal?**

Let $f \in L^2(\mathbb{R}^n)$,

$\mathscr{F}(f) = \widehat{f} := \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \bullet \rangle} dx, \; \forall f \in \mathscr{S}(\mathbb{R}^n)$

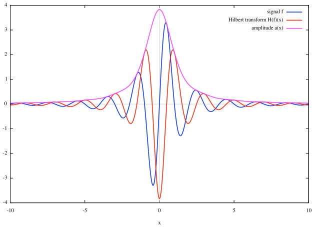
$n = 1$:

- $\mathscr{H} : L^2(\mathbb{R}, \mathbb{R}) \to L^2(\mathbb{R}, \mathbb{R}),\; f \mapsto \mathscr{F}^{-1}\left(\frac{i\xi}{|\xi|} \widehat{f}(\xi)\right)$
  Hilbert transform
- invertible $(i\mathscr{H})^2 = \mathrm{Id}$
- $f + i\mathscr{H}f$ analytical signal
- Phase decomposition of the analytic signal

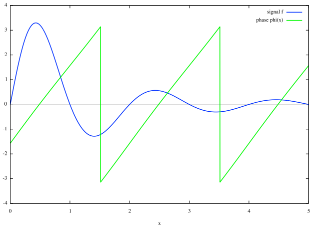
$$f + i\mathscr{H}f = a\big(\cos(\phi) + i \sin(\phi)\big)$$

  - $a := |f + i\mathscr{H}f| : \mathbb{R} \to \mathbb{R}_0^+$ amplitude
  - $\phi := \arg(f + i\mathscr{H}f) : \mathbb{R} \to [0, 2\pi[$ phase

Analytical signal




$f, \mathscr{H}f, a$



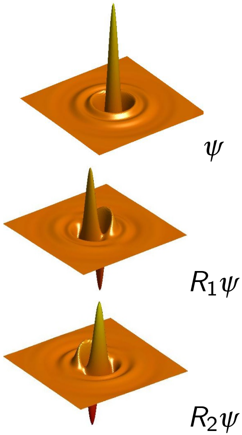

phase $\phi$ and signal $f$

# Riesz transform

What is the Hilbert transform of images?

$n \geq 2$:

- $R_\alpha : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $f \mapsto \mathscr{F}^{-1}(\frac{i\xi_\alpha}{|\xi|}\widehat{f}(\xi))$
  partial Riesz transform ($\alpha \in \{1,\ldots,n\}$)
- $R_\alpha R_\beta = R_\beta R_\alpha \neq \pm \mathrm{Id}$
- $R : L^2(\mathbb{R}^n,\mathbb{R}) \to L^2(\mathbb{R}^n,\mathbb{R}^n)$, $f \mapsto \sum_{\alpha=1}^n e_\alpha R_\alpha f$
  Riesz transform; $\{e_\alpha\}_{\alpha=0}^n$ canonical basis of $\mathbb{R}^{n+1}$
  Clifford algebra: $e_\alpha^2 = -e_0$, $e_\alpha e_\beta = -e_\beta e_\alpha, \forall \alpha \neq \beta \neq 0$
- invertible $\sum_{\alpha=1}^n R_\alpha^2 = -\mathrm{Id} \Rightarrow R^2 = \mathrm{Id}_{L^2(\mathbb{R}^n,\mathbb{R}_n)}$
- commutes with translation and dilation → implementation via wavelets
- S.H., et al. IEEE transactions on image processing 2010



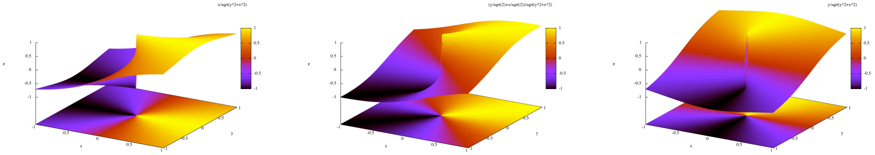

Function $\Psi$, and Riesz transforms

# Directionality and steerability

What is the difference to Hilbert transforms?

- Riesz transforms are directional
- Direction of Riesz transforms is steerable:
- Representation of the rotation group:
  Let $\rho \in O(n)$ a rotation $S_\rho : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $f \to f(\rho\cdot)$

$$S_{\rho^{-1}} \circ R_\alpha \circ S_\rho = \sum_{\beta=1}^{n} \rho_{\alpha,\beta}(R_\beta).$$




$\widehat{R_1}$ $\quad$ $2^{-1/2}(\widehat{R_1} + \widehat{R_2})$ $\quad$ $\widehat{R_2}$

Figure : Steerability of the Fourier multiplier of the Riesz transform

# The monogenic signal

**What is the amplitude and phase of an image?**

- Let $x = e_0 x_0 + \sum_{\alpha=1}^{n} e_\alpha x_\alpha := e_0 x_0 + \vec{x} \in \mathbb{R}^{n+1}$
  $x = |x|\Big(\cos\big(\arg(\frac{x_0 + i|\vec{x}|}{|x|})\big) + \frac{\vec{x}}{|\vec{x}|}\sin\big(\arg(\frac{x_0 + i|\vec{x}|}{|x|})\big)\Big),$
  where $|\vec{x}| = (\sum_{\alpha=1}^{n} |x_\alpha|^2)^{1/2})$
- Monogenic signal: $f_m := e_0 f + Rf \in L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$
- Phase-amplitude decomposition

$$f + Rf = a\big(\cos(\phi) + \vec{d}\sin(\phi)\big)$$

  - $a := |f + Rf| : \mathbb{R}^n \to \mathbb{R}_0^+$ amplitude
  - $\phi := \arg(f + i|Rf|) : \mathbb{R}^n \to [0, 2\pi[$ phase
  - $\vec{d} = \frac{Rf}{|Rf|} : \mathbb{R}^n \to S^{n-1}$ phase direction

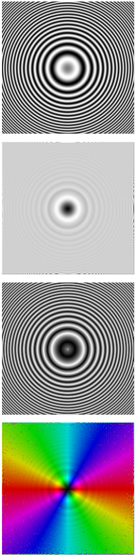
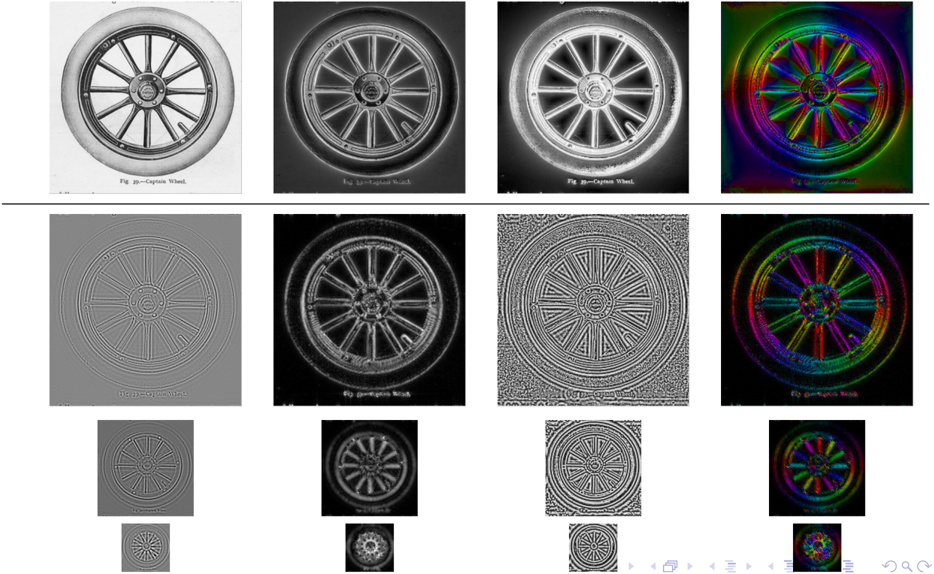


image
amplitude
phase
direction

Real part $f$ | Amplitude | Phase | Phase direction



# What is analytic about the analytical signal?

Let

- $f, f_\alpha \in L^2(\mathbb{R}^n)$ for $\alpha = 1, \ldots, n$.
- $u(x_0, x) := e_0 P_{x_0} * f(x_0, x) + \sum_{\alpha=1}^n e_\alpha P_{x_0} * f_\alpha(x_0, x)$
  $P_{x_0}(x) := \int_{\mathbb{R}^n} e^{2\pi i \langle t, x \rangle} e^{-2\pi |t| x_0} dt$ Poisson kernel
- $\partial := e_0 \frac{d}{d_{x_0}} + \sum_{\alpha=1}^n e_\alpha \frac{d}{dx_\alpha}$

Then

$f_1 = \mathscr{H} f$ $\qquad$ $f_\alpha = R_\alpha f, \ \forall \alpha = 1, \ldots, n$

if and only if $\partial u = 0$ which is equivalent to the Cauchy Riemann equations

$$\frac{d}{dx_0} u_0 - \frac{d}{dx_1} u_1 = 0$$
$$\frac{d}{dx_0} u_1 + \frac{d}{dx_1} u_0 = 0$$

$u$ is an analytical function

$$\frac{d}{dx_0} u_0 - \sum_{\alpha=1}^n \frac{d}{dx_\alpha} u_\alpha = 0$$
$$\frac{d}{dx_0} u_\alpha + \frac{d}{dx_\alpha} u_0 = 0, \ \forall \alpha = 1, \ldots, n$$
$$\frac{d}{dx_\alpha} u_\beta - \frac{d}{dx_\beta} u_\alpha = 0, \ \forall \alpha \neq \beta = 1, \ldots, n$$

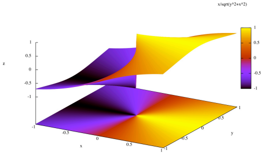
$u$ is a monogenic function

# What are higher Riesz transforms?

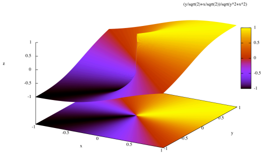
- Higher Riesz transforms: $R^\alpha := R_1^{\alpha_1} \cdots R_n^{\alpha_n}$, $\alpha \in \mathbb{N}_0^n$
- Fourier multiplier $\left(\frac{ix}{|x|}\right)^\alpha = \frac{(ix)^\alpha}{|x|^{|\alpha|}}$ homogenous polynomials

$\Rightarrow$ The Fourier multipliers of a higher Riesz transform should be a set of polynomials $\{H_l\}_l$ such that

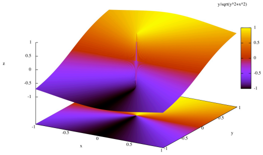
- $\sum_l H_l^2(x) = \pm 1 \; \forall x \in S^{n-1}$ $\Rightarrow$ self inverting
- homogenous: $\exists k \in \mathbb{N} : H(ax) = a^k H(x), \; \forall a \in \mathbb{R}, x \in \mathbb{R}$
  $\Rightarrow$ dilation invariance, boundedness
- Representation of the rotation group
  $\forall \rho \in \mathrm{SO}(n) \exists \mathfrak{D} : H_l(\rho x) = \sum_r \mathfrak{D}_{l,r} H_r(x)$ $\Rightarrow$ steerability




$\widehat{R_1}$




$2^{-1/2}(\widehat{R_1} + \widehat{R_2})$

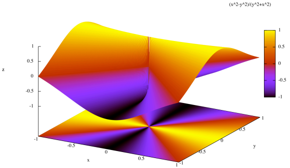



$\widehat{R_2}$

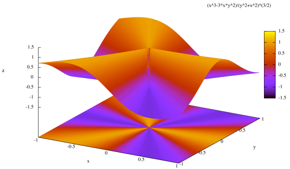
# Polynomial spaces for higher Riesz transforms

Which polynomial spaces shall we use?

- $k$-homogeneous polynomials
  ($k$-homogenous: $p(\epsilon x) = \epsilon^k p(x)$, $\forall \epsilon > 0$)
  $\mathfrak{P}_k(\mathbb{R}^n) := \left\{ \sum_{|\alpha|=k} c_\alpha x^\alpha \, , \, c_\alpha \in \mathbb{R}, \, x \in \mathbb{R}^n \right\}$
  $\mathfrak{P}_k = \mathfrak{P}_k(S^{n-1}) = \mathfrak{P}_k(\mathbb{R}^n)\big|_{S^{n-1}}$
  - rotation invariant
  - $\ldots \mathfrak{P}_{k-2} \subset \mathfrak{P}_k \subset \mathfrak{P}_{k+2} \cdots$
  - unflexible, too big
- $\mathfrak{H}_k(S^{n-1})$ spherical harmonics :
  $\mathfrak{P}_k = \mathfrak{H}_k \oplus \mathfrak{P}_{k-2}$
  - $p \in \mathfrak{H}_k \to \triangle p(\frac{x}{|x|})|x|^k = 0$, $\forall x \in \mathbb{R}^n$
  - minimal rotation invariant




An element of $\mathfrak{H}_2(S^1)$




An element of $\mathfrak{H}_3(S^1)$

# Properties of spherical harmonics

**Do spherical harmonics meet our requirements?**

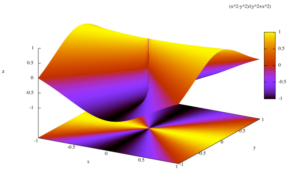
- $\bigoplus_{k\in\mathbb{N}_0} \mathscr{H}_k = L^2(S^{n-1})$
- $\mathscr{H}_k \perp \mathscr{H}_l,\ \forall k \neq l \in \mathbb{N}_0$
- $\dim(\mathscr{H}_k) = d_{n,k} := \binom{n+k-1}{k} - \binom{n+k-3}{k-2}$
- Let $S^k := \{S_l^k\}_{l=1}^{d_{n,k}} \subset \mathscr{H}_k$ ONB
- Addition theorem

$$\sum_{l=1}^{d_{n,k}} |d_{n,k}^{-1/2} S_l^k|^2 = 1$$

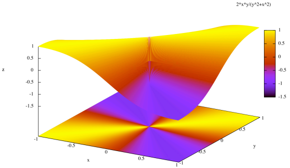
- unitary irreduzible representation of SO$(n)$:
  Let $\rho \in \mathrm{SO}(n) \Rightarrow \exists \mathfrak{D}_\rho^k \in \mathrm{SO}(d_{n,k})$ :

$$(S_1^k(\rho x), \dots, S_{d_{n,k}}^k(\rho x)) = \mathfrak{D}_\rho^k (S_1^k(x), \dots, S_{d_{n,k}}^k(x))$$

Spherical harmonics in $\mathbb{R}^2$




$y_1^2 = x_1^2 - x_2^2$




$y_2^2 = 2x_1x_2$

# Examples for spherical harmonics in $\mathbb{R}^2$

$k = 0:$ $\quad y_1^0 = 1, \quad y_2^0 = 0$

$k = 1:$ $\quad y_1^1 = \frac{x_1}{|x|}, \quad y_2^1 = \frac{x_2}{|x|}$

$k = 2:$ $\quad y_1^2 = \frac{x_1^2 - x_2^2}{|x|^2}, \quad y_2^2 = \frac{2x_1x_2}{|x|^2}$

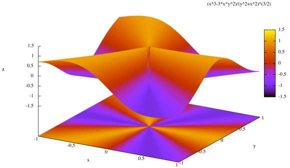
$k = 3:$ $\quad y_1^3 = \frac{x_1^3 - 3x_2^2x_1}{|x|^3}, \quad y_2^3 = \frac{3x_1^2x_2 - x_2^3}{|x|^3}$

$k \in \mathbb{N}:$ Tchebichef polynomials

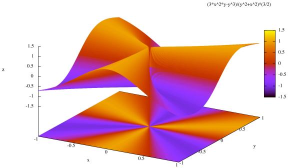
$$y_1^k = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^j}{|x|^k} \binom{k}{2j} x_1^{2j} x_2^{k-2j},$$

$$y_2^k = \sum_{j=0}^{\lceil \frac{k}{2} \rceil} \frac{(-1)^{j+1}}{|x|^k} \binom{k}{2j-1} x_2^{2j-1} x_1^{k-2j+1}$$

Spherical harmonics in $\mathbb{R}^2$



$y_1^3 = x_1^3 - 3x_2^2x_1$



$y_2^3 = 3x_1^2x_2 - x_2^3$

# Examples for spherical harmonics in $\mathbb{R}^3$

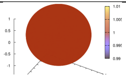
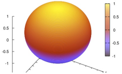
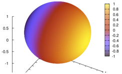
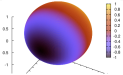
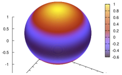
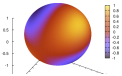
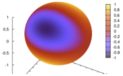
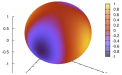
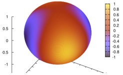
| | | | |
|---|---|---|---|
| $k=0$ | m=0 | $y_0^0 = 1$  | |
| $k=1$ | m=0 | $y_1^0 = \frac{x_3}{\lvert x\rvert}$  | |
| | m=1 | $y_{1c}^1 = \frac{x_1}{\lvert x\rvert}$  | $y_{1s}^1 = \frac{x_2}{\lvert x\rvert}$  |
| $k=2$ | m=0 | $y_2^0 = \frac{2x_3^2 - x_1^2 - x_2^2}{\lvert x\rvert^2}$  | |
| | m=1 | $y_{2,c}^1 = \frac{x_2 x_3}{\lvert x\rvert^2}$  | $y_{2,s}^1 = \frac{x_1 x_3}{\lvert x\rvert^2}$  |
| | m=2 | $y_{2,c}^2 = \frac{x_1 x_2}{\lvert x\rvert^2}$  | $y_{2,s}^2 = \frac{x_1^2 - x_2^2}{\lvert x\rvert^2}$  |

Figure : Basis elements of spherical harmonics for $\mathbb{R}^3$ (not normalized).

# Higher Riesz transforms

## What is a higher Riesz transform?

- $\mathfrak{I} : \mathbb{R}^n \to S^{n-1}, x \to \frac{x}{|x|}$
- Let $\mathfrak{p} \in \mathfrak{P}_k$ partial higher Riesz transform of degree $k \in \mathbb{N}$:

$$R_{\mathfrak{p}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),\ f \to \mathscr{F}^{-1}(i^k(\mathfrak{p} \circ \mathfrak{I})\mathscr{F}(f))$$

- Let $S^k := \{S_l^k\}_{l=1}^{d_{n,k}} \subset \mathscr{H}_k$ ONB
  $R_k := d_{n,k}^{-1/2} \sum_{l=1}^{d_{n,k}} e_l R_{S_l^k}$ higher Riesz transform of degree $k$
  $\{e_\alpha\}_{\alpha=0}^{d_{n,k}}$ canonical basis of $\mathbb{R}^{d_{n,k}+1}$ Clifford algebra: $e_\alpha^2 = -e_0,\ e_\alpha e_\beta = -e_\beta e_\alpha, \forall \alpha \neq \beta$
- Invertible: $\sum_{l=1}^{d_{n,k}} (R_{S_l^k})^2 = 1 \Rightarrow R_k^2 = (-1)^{k+1} \mathrm{Id}$
- Irreducible representation of the rotation group: Let $\rho \in \mathrm{SO}(n)$

$$\rho \circ (R_{S_1^k}, \dots, R_{S_{d_{n,k}}^k}) \circ \rho = \mathfrak{D}_\rho^k (R_{S_1^k}, \dots, R_{S_{d_{n,k}^k}}) \Rightarrow \text{steerable.}$$

# Combined higher Riesz transforms

Are there more steerable higher Riesz transforms?

Let $K \subset \mathbb{N}_0$

- Let $P = \bigoplus_{k \in K} \mathscr{H}_k = \{p = \sum_{k\in K} \sum_{l=1}^{d_{n,k}} p_l^k S_l^k,\ p_l^k \in \mathbb{R},\ S_l^k \in S_k\}$
- $S_P := \{S^k\}_{k \in K}$ ONB of $P$
- Steerable: reducible representation of the rotation group
  Let $\rho \in \mathrm{SO}(n)$ $\mathfrak{D}_\rho^P := \mathrm{diag}(\mathfrak{D}_\rho^k)_{k\in K}$

$$p(\rho x) = \mathfrak{D}_\rho^P p(x) = \sum_k \sum_m (\mathfrak{D}_\rho^k)_{l,m} p_m^k S_m^k(x)$$

- Higher Riesz transform $R_K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n, \mathbb{R}^{\sum d_{n,k}})$,
  $f \mapsto R_K f = (\sum_{k\in K} d_{n,k})^{-1/2} (R_p f)_{p \in S_P}$
- Invertible: $\sum_{k\in K} \sum_{l=1}^{d_{n,k}} (-1)^{k-1} R^2_{S_l^k} = \mathrm{Id}$
- Special case: $P = \mathfrak{P}_k$ Riesz transforms of higher order
  [Michael Unser, Dimitri Van De Ville 2009, 2010]

# Customized higher Riesz transforms

Do higher Riesz transforms corresponding to a differential operator exist?

Let $\mathfrak{p} : S^{n-1} \to \mathbb{R}$ a polynomial e.g. $\mathfrak{p} \in \mathfrak{P}_k$

- There exists a minimal space $\mathfrak{P}(S^{n-1}) = \bigoplus \mathfrak{H}_k$ that contains $\mathfrak{p}|_{S^{n-1}}$
- Complement $\{\mathfrak{p}\}$ to get an ONB $S_{\mathfrak{P}}$ of $\mathfrak{P}$
- $R_{\mathfrak{P}}$ higher Riesz transform with geometrical properties of $\mathfrak{p}$
- Interpretation: Directional information of a derivative
  - Let $\mathfrak{p} \in \mathfrak{P}_k$, $f \in W^{k,2}$ $(W^{k,2}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : D^\alpha f \in L^2(\mathbb{R}^n), \forall |\alpha| \leq k\})$.
  - $\mathfrak{p}(D)f = R_{\mathfrak{p}}(-\triangle)^{k/2} f$ $((-\triangle)^{k/2}$ fractional Laplacian)
  - i.e. $\mathfrak{p}(D)f = \mathscr{F}^{-1}(\mathfrak{p}(2\pi i \bullet)\mathscr{F}(f)) = R_{\mathfrak{p}}\mathfrak{p}(2\pi)\mathscr{F}^{-1}(|\bullet|^k \mathscr{F}(f)) = R_{\mathfrak{p}}\widetilde{f}$ such that $\widetilde{f} \in L^2(\mathbb{R}^n)$
  - If $f$ is radial, so is $\widetilde{f}$.

# Higher monogenic signals

A phase decomposition using higher Riesz transforms?

Let $P = \bigoplus_{k \in K} \mathscr{H}_k$ for some $K \subset \mathbb{N}$

- Higher monogenic signal $f_{m,k} := (f, R_k f)$
- Phase decomposition $f_m = a\big(\cos(\phi) + \vec{d}\sin(\phi)\big)$
  - $a := |R_k f|_{\mathbb{R}^{\dim(P)}}$ amplitude
  - $d := \frac{R_k f}{|R_k f|_{\mathbb{R}^{\dim(P)}}} \in \mathbb{R}^{\dim(P)}$ phase direction
  - $\phi = \arg(f + iR_k f)$ phase
- Interpretation of the phase direction: $d$ coefficient vector in the basis of $P$ such that $R_d f$ is maximal

## What is monogenic about the higher monogenic signal?

Let

- $k \in 2\mathbb{N}-1,\ f, f_\alpha \in L^2(\mathbb{R}^n),\ \alpha = 1, \dots, d_{n,k}.$
- $u(x_0, x) = e_0 P_{x_0} * f(x_0, x) + \sum_{l=1}^{d_{n,k}} e_l P_{x_0} * f_\alpha(x_0, x),$
  $\left(P_{x_0}(x) := \int_{\mathbb{R}^n} e^{2\pi i \langle t, x\rangle} e^{-2\pi |t| x_0}\, dt \text{ Poisson kernel}\right)$
- $\partial_k = \frac{\partial^k}{\partial x_0^k} + \sum_{l=1}^{d_{n,k}} e_l S_l^k(D), \qquad \underline{\partial}_k = \frac{\partial^k}{\partial x_0^k} - \sum_{l=1}^{d_{n,k}} e_l S_l^k(D)$

Then

$$f_\alpha = R_{S_\alpha^k} f$$

if and only if

$$\partial_k u = 0.$$

Which is equivalent to a set of generalized Cauchy Riemann equations.
Furthermore $\partial_k \underline{\partial}_k = \frac{\partial^{2k}}{\partial x_0^{2k}} + \left(\sum_{\alpha=1}^n \frac{\partial^2}{\partial x_\alpha^2}\right)^k$.

# Implementation via wavelet frames

How are the higher Riesz transforms implemented?

The (higher) Riesz transform

- maps (tight) frames to (tight) multiframes (Proof uses Clifford frames – frames with Clifford algebra valued coefficients. See S.H., et al. IEEE transactions on image processing 2010)
- maps wavelets to wavelets
- of a radial function is steerable

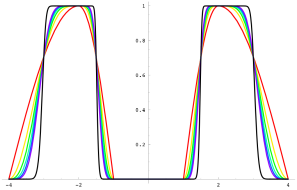
⇒ Implementation via a tight steerable wavelet frame

We need a tight wavelet frame with radial mother wavelet

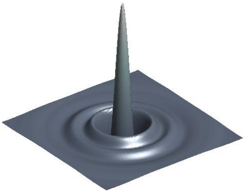
# Isotrope Waveletframes

Do suitable radial wavelet frames exist?

- $$\widehat{\Psi}(\xi) := \begin{cases} \cos(2\pi q(\|\xi\|)), & \forall \|\xi\| \in (\frac{1}{8}, \frac{1}{4}], \\ \sin(2\pi q(\frac{1}{2}\|\xi\|)), & \forall \|\xi\| \in (\frac{1}{4}, \frac{1}{2}], \\ 0, & \text{otherwise}, \end{cases}$$
  where $q \in C([\frac{1}{8}, \frac{1}{4}]) : 0 \leq q(t) \leq \frac{1}{4}, \ \forall t \in [\frac{1}{8}, \frac{1}{4}]$, $q(\frac{1}{8}) = \frac{1}{4}, \ q(\frac{1}{4}) = 0$
- $\{2^j T_a \Psi : j \in \mathbb{Z}, a \in \mathbb{Z}^n\}$ is a tight wavelet frame of $L^2(\mathbb{R}^n)$ with frame bound 1.
- S.H., et al. IEEE transactions on image processing 2010

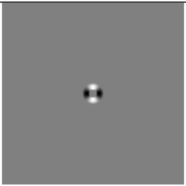
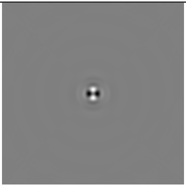
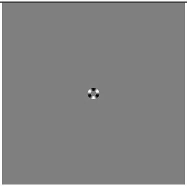
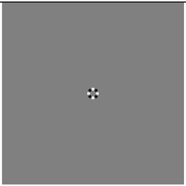
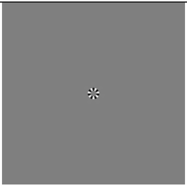



Fourier transforms of $\psi$ for different $q$



The wavelet $\psi$

# Examples of Higher Riesz transforms of the given wavelets

| k | Fourier domain | image domain | k | Fourier domain | image domain |
|---|---|---|---|---|---|
| 2 |  |  | 3 |  |  |
| 4 |  |  | 7 |  |  |

# Conclusion

- Higher Riesz transforms yield steerable wavelets
- Multiscale decomposition: amplitude, phase and geometrical information
- Higher Riesz transforms taylored from minimal rotation invariant spaces
- Simple implementation via tight multiwavelet frames using fast Fourier transform
- Based on a isotropic, tight wavelet frame construction in arbitrary dimension

Thank you for your attention

S. H., Martin Storath, Brigitte Forster, and Peter Massopust.
Steerable wavelet frames based on the riesz transform.
*IEEE Transactions on Image Processing*, 19(3):653–667, 2010.

Martin Storath and Stefan Held.
Monogenic Wavelet Toolbox.

Michael Unser and D. Van De Ville.
Wavelet steerability and the higher order riesz transform.
*IEEE Transactions on Image Processing*, 2010.

What is monogenic about the higher monogenic signal?

Let $f, f_\alpha \in L^2(\mathbb{R}^n)$, $\alpha = 1, \dots, d_{n,k}$.
Then $f_\alpha = R_{S_\alpha^k} f$
if and only if $u_0(x_0, x) := P_{x_0} * f(x_0, x)$, $u_\alpha(x_0, x) := P_{x_0} * f_\alpha(x_0, x)$,
where $P_{x_0}(x) := \int_{\mathbb{R}^n} e^{2\pi i \langle t, x \rangle} e^{-2\pi |t| x_0} dt$ Poisson kernel
satisfy the generalized Cauchy Riemann equations:

$$S_\beta^k(D) u_\alpha(x) = S_\alpha^k(D) u_\beta(x), \ \forall \alpha, \beta = 1, \dots, d_{n,k};$$

$$\frac{\partial^k u_\alpha}{\partial x_0^k}(x) + (-1)^{k+1} S_\alpha^k(D) u_0(x) = 0, \ \forall \alpha = 1, \dots, d_{n,k};$$

$$\sum_{\alpha=1}^{n} S_\alpha^k(D) u_\alpha(x) = \frac{\partial^k u_0}{\partial x_0^k}(x), \text{ f.a.a. } x_0 \in \mathbb{R}^+, x \in \mathbb{R}^n.$$

## Are there differential operators behind higher monogenicity?

- Let $k \in 2\mathbb{N}-1$, $u = \sum_{\alpha=0}^{d_{n,k}} e_\alpha u_\alpha \in L^p(\mathbb{R}_+^{n+1}, \mathbb{R}_+^{d_{n,k}+1})$, $1 < p < \infty$
  Then $u$ satisfies the generalized Cauchy Riemann equations if and only if $u = \sum_{\alpha=0}^{d_n^k} u_\alpha$ satisfies
  $$\partial_k u = 0,$$
  where $\partial_k = \frac{\partial^k}{\partial x_0^k} + \sum_{l=1}^{d_{n,k}} e_l S_l^k(D)$.
- $\left(\sum_{l=1}^{d_{n,k}} e_l S_l^k(D)\right)^2 = \sum_{l=1}^{d_{n,k}} \left(S_l^k(D)\right)^2 = -\triangle^k$
- Let $\underline{\partial}_k = \frac{\partial^k}{\partial x_0^k} - \sum_{l=1}^{d_{n,k}} e_l S_l^k(D)$.
  $$\text{Then } \partial_k \underline{\partial}_k = \frac{\partial^{2k}}{\partial x_0^{2k}} + \left(\sum_{\alpha=1}^{n} \frac{\partial^2}{\partial x_\alpha^2}\right)^k.$$