Besov regularity for the solution of the Stokes system in polyhedral cones

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Outline

1. Motivation
2. Adaptive wavelet schemes
3. Polyhedral cones and weighted Sobolev spaces
4. Besov regularity for the Stokes system

Outline

1. Motivation

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- Numerical treatment of operator equations

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- Example: Poisson equation on a Lipschitz domain $\Omega \subset \mathbb{R}^d$:

$$\begin{aligned} \Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

$\Delta : H_0^1(\Omega) \to H^{-1}(\Omega).$

Stokes system

Consider on a Lipschitz domain $\Omega \subset \mathbb{R}^3$:

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

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Weak formulation: For given $f \in H^{-1}(\Omega)^3$, determine $u \in H_0^1(\Omega)^3$ and $p \in L_{2,0}(\Omega) := \{q \in L_2(\Omega) : \int_\Omega q(x)\mathrm{dx} = 0\}$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= f(v) \text{ for all } v \in H_0^1(\Omega)^3 \\ b(u,q) &= 0 \text{ for all } q \in L_{2,0}(\Omega), \end{aligned}$$

where

$$a(u,v) := \int_\Omega \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \mathrm{dx},$$

$$b(v,q) := -\int_\Omega q(x) \operatorname{div}(v)(x) \mathrm{dx}.$$

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  - Based on uniform space refinements
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- **Convergence rate depends on regularity of the solution (which regularity?)**

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2. Adaptive wavelet schemes

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- $\Omega \subset \mathbb{R}^d$: bounded domain
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- $V_{j+1} = V_j \oplus W_{j+1}, \ V_0 = W_0 \longrightarrow$

$$L_2(\Omega) = \bigoplus_{j\geq 0} W_j, \quad W_j = \overline{\text{span}\{\psi_{j,k} : \ k \in \mathcal{I}_j\}}$$

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- Get a orthonormalbasis for $L_2(\Omega)$

Solving with adaptive schemes I

- Determine approximations $u_{\Lambda_0}, u_{\Lambda_1}, \ldots,$

$$u_{\Lambda_j} = \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda,$$

such that for a given $\varepsilon > 0$ after a finite number of steps we get a solution $u_{\Lambda_{k_\varepsilon}}$ with

$$||u - u_{\Lambda_{k_\varepsilon}}||_{L_2(\Omega)} < \varepsilon.$$

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- The sets Λ_i, $i \geq 0$ are determined adaptively by the algorithm.

Solving with adaptive schemes II

- "ideal" algorithm: best m-term approximation.
-

$$\mathcal{M}_m := \left\{ f = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda : |\Lambda| = m \right\}.$$

$$\sigma_m(u)_{L_2(\Omega)} := \inf_{g \in \mathcal{M}_m} ||u - g||_{L_2(\Omega)} \sim ||u - g_m||_{L_2(\Omega)},$$

$$g_m = \sum_{\lambda \in \Lambda_m} c_\lambda \psi_\lambda, \;\; \Lambda_m \widehat{=} m \text{ biggest wavelet coefficients}$$

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- $\sigma_m(u)_{L_2(\Omega)} = \mathcal{O}(m^{-\alpha/d}) \Longleftarrow$
  $$u \in B^\alpha_\tau(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{1}{2} + \frac{\alpha}{d}.$$

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- Implemented schemes from Cohen, Dahmen and DeVore.

Comparison with linear approximation

- Convergence using adaptive schemes: $\sigma_n(u)_{L_2(\Omega)} = \mathcal{O}(n^{-\alpha/d}) \Longleftarrow$

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$$E_j(u) := \inf_{g \in V_j} ||u - g||_{L_2(\Omega)} \lesssim 2^{-\beta j} |u|_{H^\beta(\Omega)}$$

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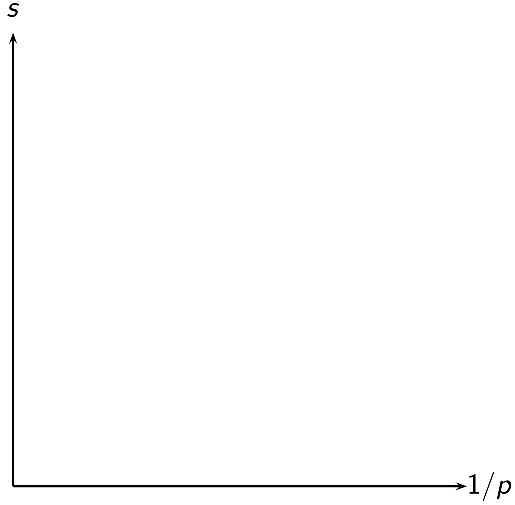
- Natural question:

Besov regularity α > Sobolev regularity β?

DeVore-Triebel diagram



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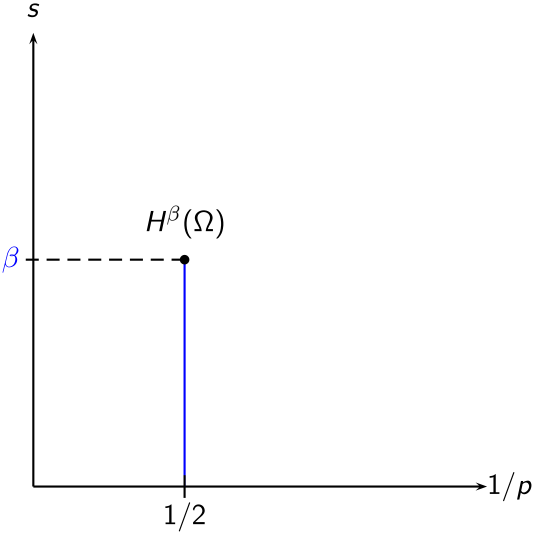

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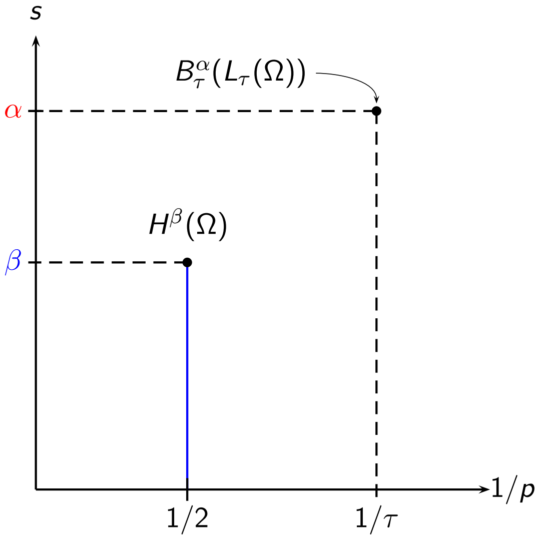

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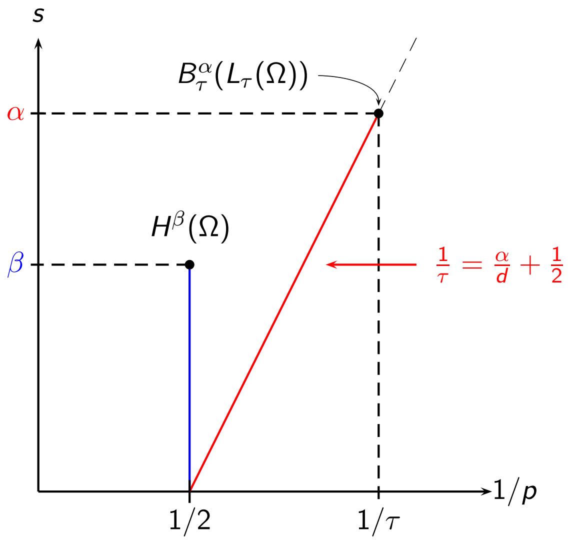
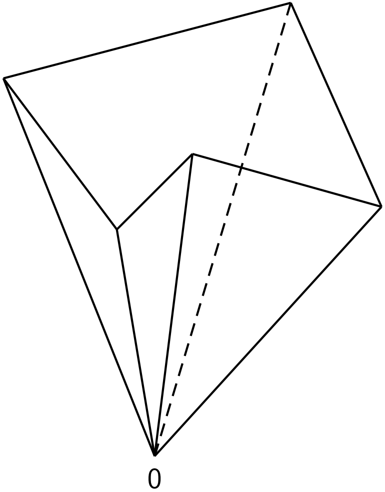


Outline

3. Polyhedral cones and weighted Sobolev spaces

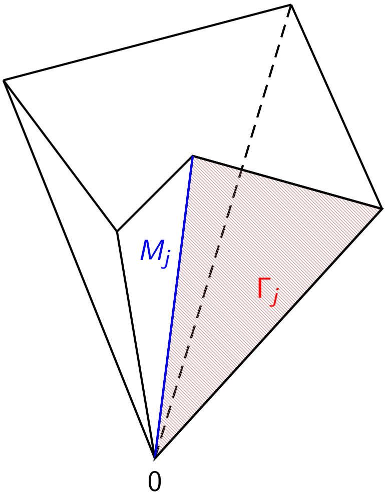
Polyhedral cones



$$\mathcal{K} := \{x \in \mathbb{R}^3 : x = \rho \cdot \omega, 0 < \rho < \infty, \omega \in \Omega\},$$

$$\mathcal{K}_0 := \{x \in \mathcal{K} : ||x||_2 \leq r_0\}.$$

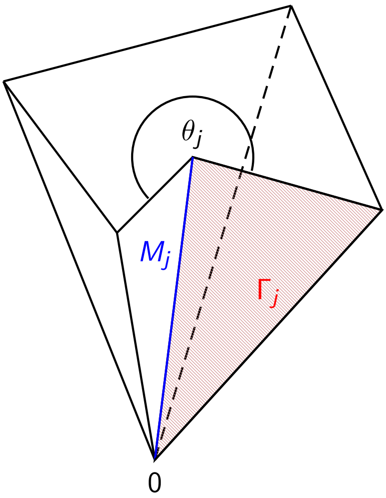
Polyhedral cones



Boundary of $\mathcal{K}$ consists of

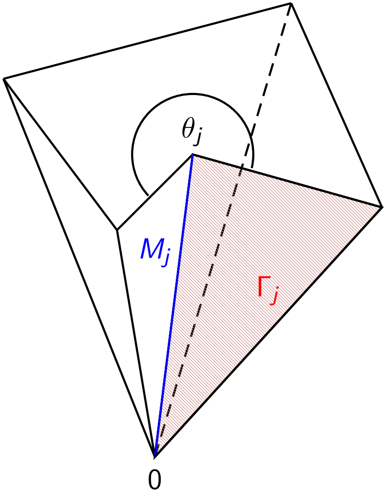
vertex $x = 0$, edges M_j, faces Γ_j.

Polyhedral cones



θ_j : angel at the edge M_j.

Polyhedral cones



$$\rho(x) := |x|$$

$$r_j(x) := \mathrm{dist}(x, M_j)$$

Weighted Sobolev Spaces

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- Weighted Sobolev norm

$$||u||_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})} :=$$

$$\left(\int_{\mathcal{K}} \sum_{|\alpha| \le l} |D^\alpha u(x)|^2 \mathrm{d}x \right)^{1/2}$$

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- $W^{l,2}_{\beta,\delta}(\mathcal{K}) := \overline{\mathcal{C}_0^\infty(\overline{\mathcal{K}} \backslash \{0\})}^{||\cdot||_{W^{l,2}_{\beta,\delta}}}$

Outline

The Stokes system

We consider the Stokes system

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in } \mathcal{K}, \\ \operatorname{div} u &= g \quad \text{in } \mathcal{K}, \\ u &= 0 \quad \text{on } \Gamma_j,\ j = 1, ..., d. \end{aligned}$$

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\end{aligned}
$$

$$
\nabla p = \left(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3} \right), \quad \Delta u = \left(\sum_{i=1}^{3} \frac{\partial^2 u_j}{\partial^2 x_i} \right)_{j=1,2,3}.
$$

$$
\operatorname{div} u = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}
$$

Sobolev regularity of the solution

Consider

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in } \mathcal{K}_0, \\ \operatorname{div} u &= g \quad \text{in } \mathcal{K}_0, \\ u &= 0 \quad \text{on } \Gamma_j,\ j = 1, \dots, d. \end{aligned}$$

Proposition (M. Dauge, 89)

Assume $(f, g) \in L_2(\mathcal{K}_0)^3 \times H^{\alpha_0}(\mathcal{K}_0)$ *for* $\alpha_0 < 1/2$. *Let* g *fulfill*

$$\int_{\mathcal{K}_0} g(x)\,\mathrm{d}\,x = 0.$$

Then there exists a unique solution

$$(u, p) \in H^{\alpha_0+1}(\mathcal{K}_0)^3 \times H^{\alpha_0}(\mathcal{K}_0).$$

Weighted Sobolev estimates of the solution

Proposition (V. Maz'ya, J. Rossmann, 01)

Suppose $(f,g) \in W^{l-2,2}_{\beta,\delta}(\mathcal{K})^3 \times W^{l-1,2}_{\beta,\delta}(\mathcal{K})$ *where* $l \geq 2$ *is an integer. Then there exists a countable set* $E \subset \mathbb{C}$ *such that the following holds. If* $\beta \in \mathbb{R}$ *and the vector* $\delta \in (\mathbb{R}\backslash\mathbb{Z})^d$ *are chosen such that*

$$Re\, \lambda \neq l - \beta - \frac{3}{2} \quad \textit{for all } \lambda \in E$$

and

$$\max\left(0, l-1-\frac{\pi}{\theta_k}\right) < \delta_k < l-1, \quad k = 1, \dots, d,$$

then there exists a uniquely determined solution

$$(u,p) \in W^{l,2}_{\beta,\delta}(\mathcal{K}) \times W^{l-1,2}_{\beta,\delta}(\mathcal{K})$$

Besov regularity of the solution

Theorem (E., 12)

Assume that the conditions in the above propositions are fulfilled. For $\beta < l-1$ we obtain

$$u \in B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))^3, \quad s_1 < \min\left(l, \frac{3}{2}\cdot(\alpha_0+1), 3\cdot(l-|\vec{\delta}|)\right),$$

$$\frac{1}{\tau_1} = \frac{s_1}{3} + \frac{1}{2},$$

$$p \in B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0)), s_2 < \min\left(l-1, \frac{3}{2}\cdot\alpha_0, 3\cdot(l-(|\vec{\delta}|+1))\right),$$

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- Investigate the Besov regularity of the solution of the Stokes system.
- Proof is performed by using characterization of Besov spaces by wavelet expansions.
- For valid parameters the Besov regularity is 3/2 times higher than its Sobolev regularity.

References

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- M. Dauge. Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with corners. I: Linearized equations. *SIAM J. Math. Anal*, 20:7497, 1989.
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Thanks a lot for your attention!

Sketch of the proof I

Idea: Estimate the coefficients of the wavelet decomposition of the solution.

Proposition

Let $s \in \mathbb{R}$ *and* $0 < p, q < \infty$. *Suppose* $r > \max\left(s, n \max\left(0, \frac{1}{p} - 1\right) - s\right)$. *Then*

$$f \in B^s_q(L_p(\mathbb{R}^n))$$

$$\Longleftrightarrow$$

$$\left(\sum_{i=1}^{2^n-1} \sum_{j=0}^{\infty} 2^{j(s+n(1/2-1/p))q} \left(\sum_{k \in \mathbb{Z}^n} | \langle f, \psi_{i,j,k} \rangle |^p \right)^{q/p} \right)^{1/q} < \infty$$

Sketch of the proof II

Idea: Estimate the coefficients of the wavelet decomposition of the solution.

- Wavelets are assumed to be compact supported.
- Then there exists a cube Q centered at the origin, s.t.

$$Q_{j,k} := 2^{-j}k + 2^{-j}Q, \; j \in \mathbb{N}_0, \; k \in \mathbb{Z}^3$$

contains the support of the wavelets $\psi_{i,j,k}$.

Sketch of the proof II

Idea: Estimate the coefficients of the wavelet decomposition of the solution.

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$$Q_{j,k} := 2^{-j}k + 2^{-j}Q, \;\; j \in \mathbb{N}_0, \;\; k \in \mathbb{Z}^3$$

  contains the support of the wavelets $\psi_{i,j,k}$.
- Consider the set of indices

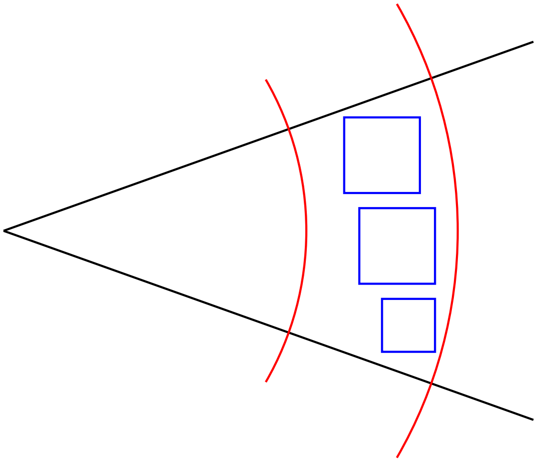
$$\Lambda_\iota := \{(i,j,k) : Q_{j,k} \subset \mathcal{K}_0, 2^{-3\iota} \leq 2^{-j} \leq 2^{-3\iota+2}\}.$$

Sketch of the proof III

Estimate the coefficients $|\langle v, \psi_{i,j,k}\rangle|$ in three steps:

- Consider for $\kappa \in \mathbb{N}$ the set

$$\Lambda_{\iota,\kappa} := \{(i,j,k) \in \Lambda_\iota : \kappa 2^{-\iota} \leq \mathrm{dist}(Q_{j,k}, 0) < (\kappa+1)2^{-\iota}\}.$$



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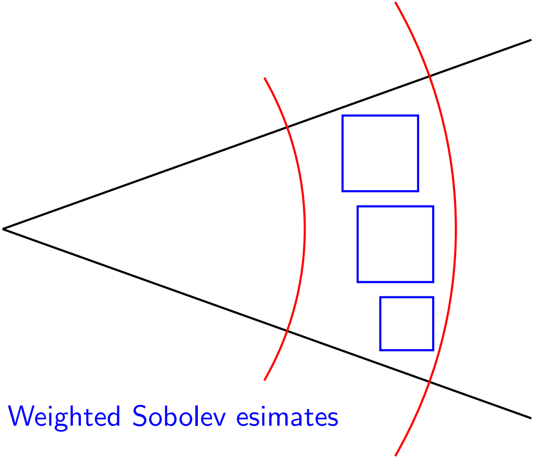

$\rightarrow$ Weighted Sobolev esimates

Sketch of the proof III

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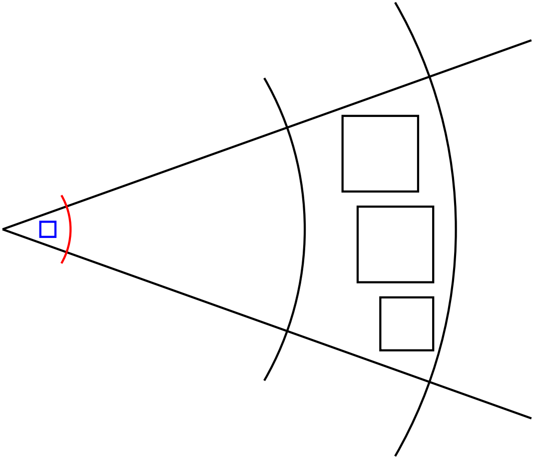
$$\Lambda_{\iota,0} := \{(i,j,k) \in \Lambda_\iota : 0 < \operatorname{dist}(Q_{j,k}, 0) < 2^{-\iota}\}.$$



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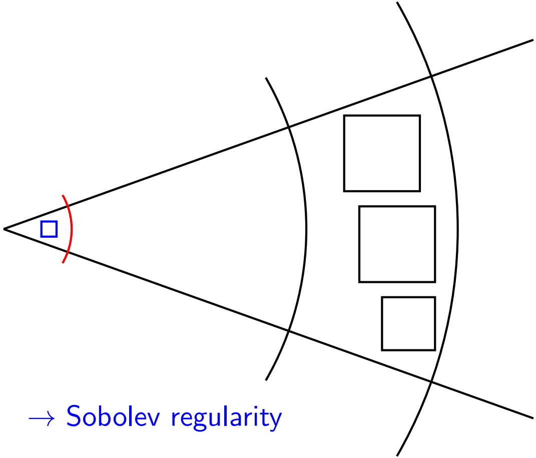

$\rightarrow$ Sobolev regularity

Sketch of the proof III

Estimate the coefficients $|\langle v, \psi_{i,j,k}\rangle|$ in three steps:

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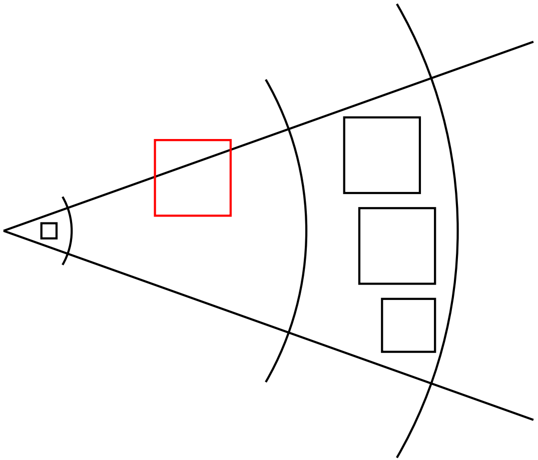
$$\Lambda_\iota^{\#} := \{(i,j,k) : Q_{j,k} \cap \partial\mathcal{K}_0 \neq \emptyset, 2^{-3\iota} \leq 2^{-j} \leq 2^{-3\iota+2}\}.$$



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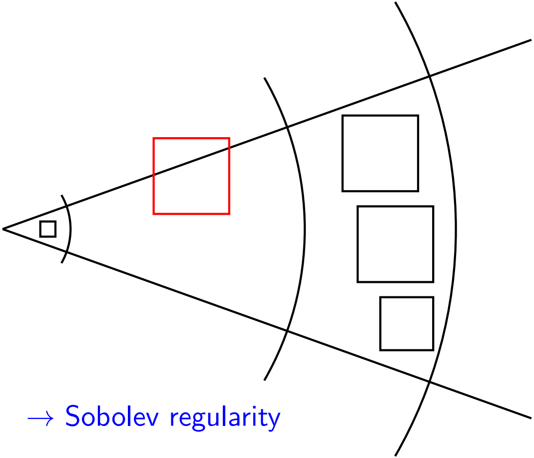

→ Sobolev regularity

Norm estimate I

We obtain

- $||u||_{H^{\alpha_0+1}(\mathcal{K}_0)^3} + ||p||_{H^{\alpha_0}(\mathcal{K}_0)} \lesssim ||f||_{L_2(\mathcal{K}_0)^3} + ||g||_{H^{\alpha_0}(\mathcal{K}_0)}.$

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- Define

$$\mathcal{H}_\beta := \left\{ u \in W^{l,2}_{\beta,0}(\mathcal{K})^3 : u = 0 \text{ on } \Gamma_j,\ j = 1, \ldots, d \right\}.$$

The functional

$$F(v) := \int_{\mathcal{K}} (f + \nabla g) \cdot v \, \mathrm{d}\, x$$

defines a linear and continuous mapping on $\mathcal{H}_{l-1-\beta}$.

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  defines a linear and continuous mapping on $\mathcal{H}_{l-1-\beta}$.
- The solution (u, p) found in the Proposition fulfills

$$||u||_{W^{l,2}_{\beta,\delta}(\mathcal{K})^3} + ||p||_{W^{l-1,2}_{\beta,\delta}(\mathcal{K})} \lesssim$$

$$\left(||F||_{\mathcal{H}^*_{l-1-\beta}} + ||g||_{V^{0,2}_{\beta-l+1}(\mathcal{K})} + ||f||_{W^{l-2,2}_{\beta,\delta}(\mathcal{K})} + ||g||_{W^{l-1,2}_{\beta,\delta}(\mathcal{K})} \right).$$

Norm estimate II

- We show for $\beta < l - 1$

$$||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0))} \lesssim$$

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- All in all this leads to

$$||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0))} \lesssim$$

$$||F||_{H^*_{l-1-\beta}} + ||g||_{V^{0,2}_{\beta-l+1}(\mathcal{K})} + ||g||_{W^{l-1,2}_{\beta,\delta}(\mathcal{K})} + ||g||_{H^{\alpha_0}(\mathcal{K}_0)} +$$

$$||f||_{W^{l-2,2}_{\beta,\delta}(\mathcal{K})^3} + ||f||_{L_2(\mathcal{K}_0)^3}$$