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Dual pairs of Gabor frames for trigonometric generators without the partition of unity property

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Frames is a strong tool to obtain series expansions in Hilbert spaces under less restrictive conditions than imposed by orthonormal bases. In order to apply frame theory it is necessary to construct a pair of so called dual frames. The goal of the article is to provide explicit constructions of dual pairs of frames having Gabor structure. Unlike the results presented in the literature we do not base the constructions on a generator satisfying the partition of unity constraint.

Keywords: Gabor frames; dual frame

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1. Introduction

Gabor frames provide a convenient way of obtaining series expansions of functions in $L^2(\mathbb{R})$ and a large class of associated Banach spaces. In order for this to be done in practice it is necessary to construct a so called dual frame. In this article we demonstrate how to construct explicit and convenient pairs of a Gabor frame and an associated dual frame based on certain trigonometric functions. Unlike most of the constructions appearing in the literature we avoid the partition of unity constraint discussed below.

The paper is organized as follows. In the rest of this section we introduce frames, in particular, Gabor frames, and set the stage for the results to follow. The new results are presented in Sections 2–3, with a long and technical proof delayed till Section 4.

For a given $a \in \mathbb{R}$ we define the *translation operator*

$$T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (T_a f)(x) := f(x - a).$$

Equally, for any $b \in \mathbb{R}$ we define the corresponding *modulation operator*

$$E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (E_b f)(x) := e^{2\pi ibx} f(x).$$

Both operators are linear, bounded and unitary.

Choose a function $g \in L^2(\mathbb{R})$ and two parameters $a, b > 0$. By the *Gabor system* generated by the function g and the parameters a, b we understand the system of functions consisting of

$$\{(E_{mb}T_{na}g)(x)\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imbx} g(x - na)\}_{m,n\in\mathbb{Z}}.$$

Usually we simply denote the Gabor system by $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$.

We will consider Gabor systems that form a so called *frame*. Frames can be studied in general Hilbert spaces, see, e.g., the books [1], [8], [14], but we will only consider Gabor frames in $L^2(\mathbb{R})$.

Definition 1.1. Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. The system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is called a Gabor frame if there exist constants $A, B > 0$, such that

$$A||f||^2 \le \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 \le B||f||^2,\ \ \forall f \in L^2(\mathbb{R}). \tag{1.1}$$

If the system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ satisfies the upper inequality in (1.1) it is called a Bessel sequence.

A function g that generates a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for appropriate parameters $a, b > 0$ is called a *generator* or *window function*.

An important result for Gabor frames (and frames in general) tells us that they lead to convenient representation formulas, similar to what we know from orthonormal bases:

Theorem 1.2. *Assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame. Then there exists functions $h \in L^2(\mathbb{R})$, such that*

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g,\ \ \forall f \in L^2(\mathbb{R}). \tag{1.2}$$

One can show that if $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for which (1.2) holds, then $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ is also a Gabor frame. The system $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ is called a *dual frame,* and the function h is called a *dual generator* or *dual window*. Gabor frames $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and their dual generators have been characterized by Ron & Shen [12], as well as Janssen [9]. Their result implies the following:

Theorem 1.3. *Let $g, h \in L^2(\mathbb{R})$ be bounded functions with compact support, and let $a, b > 0$ be given. Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ are dual frames if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g(x - n/b - ka)}h(x - ka) = b\delta_{n,0}, \text{ a.e. } x \in [0, a]. \tag{1.3}$$

Explicit constructions of dual Gabor frames have been given by Christensen and Kim [2], [4], as well as Laugesen [11] and Kim [10]. Following [2] and [4] we will assume that $a = 1$ and search for dual windows having the form

$$h(x) = \sum c_k g(x + k)$$

for a certain finite sequence $\{c_k\}$. The advantage of this special form is that many properties of h can be derived directly from the corresponding properties of g, e.g., the regularity and the size of the support. We mention the following result from [2]:

Theorem 1.4. *Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R})$ be a real valued bounded function with supp $g \subseteq [0, N]$, for which*

$$\sum_{k\in\mathbb{Z}} g(x - k) = 1, \ x \in \mathbb{R}. \tag{1.4}$$

Let $b \in]0, \frac{1}{2N-1}]$. Then the function g and the function h given by

$$h(x) = bg(x) + 2b\sum_{k=1}^{N-1} g(x + k) \tag{1.5}$$

generate dual Gabor frames $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Theorem 1.4 can be used to construct Gabor frames generated by the B-splines, which are known to satisfy (1.4). However, in general condition (1.4) is very restrictive. In the following section we construct dual pairs of Gabor frames based on certain trigonometric functions that do not satisfy the partition of unity condition.

For later use we note that condition (1.4) can be replaced by the condition

$$\sum_{k\in\mathbb{Z}} g(x - k) = c \neq 0, \ x \in \mathbb{R}, \tag{1.6}$$

followed by an appropriate normalization. Therefore we will focus on (1.6) in the sequel.

2. Frames generated by $\sin^\eta(\frac{1}{3}\pi x)\chi_{[0,3]}(x)$

In this section we will consider functions of the form

$$g(x) = \sin^\eta(\tfrac{1}{3}\pi x)\chi_{[0,3]}(x), \tag{2.1}$$

where $\eta \in \mathbb{N}$. We note that the choice of the support implies that g is continuous for all $\eta \in \mathbb{N}$, and that the smoothness increases with η. Thus, high values of η leads to a function with good time-frequency localization.

For functions of the type (2.1) we search for dual generators h of the form seen in Theorem 1.4, that is,

$$h(x) = Cg(x) + Dg(x+1) + Eg(x+2) \tag{2.2}$$

for appropriate $C, D, E \in \mathbb{R}$.

We also note that the condition (1.3) with $n = 0$ would remain unchanged if we also added a term $Fg(x+3)$ in the expression for h. However, the support of h would increase, and hereby eventually force us to choose a smaller value of the parameter b. For this reason we do not include a term of the type $Fg(x+3)$.

Our goal is to show that functions g of the type (2.1) will generate Gabor frames for certain values of b, having duals of the desired form, but without necessarily to satisfy the partition of unity property. We first show that g actually generates a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for certain parameters a, b:

Proposition 2.1. *Let $a = 1$ and $b \in]0, 1/3]$. For all $\eta \in \mathbb{N}$ the function $g(x) = \sin^\eta(\pi x/3)\chi_{[0,3]}(x)$ generates a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.*

Proof. The Gabor system $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ forms a Bessel sequence for any bounded and compactly supported function $g \in L^2(\mathbb{R})$. Thus we only need to show that the lower frame condition is satisfied. Let $b \in]0, 1/3]$. Since supp $g = [0, 3]$,

$$\sum_{k\neq 0} \Big| \sum_{n\in\mathbb{Z}} g(x-n)g(x-n-k/b) \Big| = 0$$

and therefore we have

$$\begin{aligned} A &:= \tfrac{1}{b} \inf_{x\in[0,1]} \left[\sum_{n\in\mathbb{Z}} |g(x-n)|^2 - \sum_{k\neq 0} \Big| \sum_{n\in\mathbb{Z}} g(x-n)g(x-n-k/b) \Big| \right] \\ &= \tfrac{1}{b} \inf_{x\in[0,1]} \left[\sum_{n\in\mathbb{Z}} |g(x-n)|^2 \right] \\ &\geq \tfrac{1}{b} \inf_{x\in[0,1]} |g(x+1)|^2 = \tfrac{1}{b}(\tfrac{3}{4})^\eta > 0. \end{aligned}$$

By Theorem 9.1.5 in [1], $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is a frame for any $\eta \in \mathbb{N}$. $\square$

We now analyze the structure of dual generators for frames generated by functions g of the form (2.1). Interestingly, dual generators of the form (2.2) only exist for sufficiently small values of η :

Theorem 2.2. *Let $\eta \in \mathbb{N}$, $b \in]0, 1/5]$ and $C, D, E \in \mathbb{R}$ and consider the functions*

$$g(x) = \sin^\eta\left(\tfrac{1}{3}\pi x\right)\chi_{[0,3]}(x) \tag{2.3}$$

and

$$h(x) = \left(\tfrac{4}{3}\right)^\eta b\big(Cg(x) + Dg(x+1) + Eg(x+2)\big). \tag{2.4}$$

Then the following holds:

(i) *The function g generates a Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ with a dual generator h of the form* (2.4) *if and only if $\eta < 6$.*
(ii) *For $\eta < 6$ the dual generators h of the form* (2.4) *are characterized by the following:*

$$\begin{aligned}
\eta = 1: &\ C \in \mathbb{R},\ D = 1 - 2C, E = 2C - 1;\\
\eta = 2: &\ C \in \mathbb{R},\ D = 1 - 2C, E = 1 - 2C;\\
\eta = 3: &\ C = 1/3, D = 1/3, E = -1/3;\\
\eta = 4: &\ C = 1/4, D = 1/2, E = 1/2;\\
\eta = 5: &\ C = 1/5, D = 3/5, E = -3/5.
\end{aligned}$$

The proof needs some preparation and is quite lengthy - see section 4.

The structure of the solutions in Theorem 2.2 is interesting. For instance, the solutions have the same form for $\eta = 1$ and $\eta = 2$, except a change of sign in the coefficient E. It is also striking that for $\eta = 1$ and $\eta = 2$ we can choose the parameter C arbitrary, while for $\eta = 3, 4, 5$ there is a unique solution of the desired form. Fig. 1 shows examples of pairs of dual generators.

Note that the range of the modulation parameter b is $]0, 1/5]$ in Theorem 2.2, while it was $]0, 1/3]$ in Proposition 2.1. Thus, to get a nice structure of the dual window is more restrictive than just to obtain the frame property. The technical reason for this is that the function h in general has larger support than g, so in order for (1.3) to hold for $n \neq 0$ we assume that $b \in]0, 1/5]$.

Corollary 2.3. *The duals described in Theorem 2.2(ii) can be rewritten as follows:*

$$\begin{aligned}
\eta = 1: \ h(x) &= \left(\tfrac{4}{3}\right)^{\eta} b \left[\big(g(x) - 2g(x+1) + 2g(x+2)\big)C \right.\\
&\qquad\qquad\qquad\qquad \left. +\ g(x+1) - \ g(x+2)\right];\\
\eta = 2: \ h(x) &= \left(\tfrac{4}{3}\right)^{\eta} b \left[\big(g(x) - 2g(x+1) - 2g(x+2)\big)C \right.\\
&\qquad\qquad\qquad\qquad \left. +\ g(x+1) + \ g(x+2)\right];\\
\eta = 3: \ h(x) &= \left(\tfrac{4}{3}\right)^{\eta} b\, \tfrac{1}{3}\big(g(x) + \ g(x+1) - \ g(x+2)\big);\\
\eta = 4: \ h(x) &= \left(\tfrac{4}{3}\right)^{\eta} b\, \tfrac{1}{4}\big(g(x) + 2g(x+1) + 2g(x+2)\big);\\
\eta = 5: \ h(x) &= \left(\tfrac{4}{3}\right)^{\eta} b\, \tfrac{1}{5}\big(g(x) + 3g(x+1) - 3g(x+2)\big).
\end{aligned}$$

Concerning g and the partition of unity condition, we have the following result.

Proposition 2.4. *Let $\eta \in \mathbb{N}$ and consider the function g in* (2.1). *Then the following holds:*

(i) *g satisfies condition* (1.6) *if and only if $\eta \in \{2, 4\}$.*
(ii) *For $\eta = 2$,*

$$\sum_{k\in\mathbb{Z}} g(x-k) = \tfrac{3}{2}.$$

(iii) For $\eta = 4$,

$$\sum_{k\in\mathbb{Z}} g(x-k) = \tfrac{9}{8}.$$

Proof. The proof consists of two parts. For $\eta < 6$, direct calculation shows that g only satisfies (1.6) for $\eta \in \{2,4\}$. For $\eta \geq 6$, Theorem 2.2 shows that there does not exist a dual of the form (1.5). By Theorem 1.4 this implies that the function g does not satisfy condition (1.4) or condition (1.6). $\square$

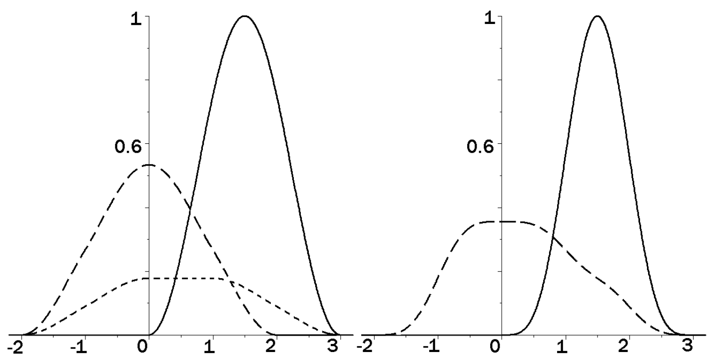

Fig. 1. Left: Plot of $g(x) = \sin^2(\pi x/3)\chi_{[0,3]}(x)$ (solid) with duals generated by Theorem 2.2 for $C = 0$ (dashed) and $C = 1/3$ (dotted). Right: Plot of $g(x) = \sin^4(\pi x/3)\chi_{[0,3]}(x)$ (solid) with its unique dual given by Theorem 2.2 (dashed).

By Proposition 2.4 we are able to satisfy the partition of unity condition for $\eta = 2, 4$ by rescaling of g. In the case $\eta = 4$ the unique dual given by Theorem 2.2 equals the one given by Theorem 1.4.

We note that the dual exhibited in Theorem 1.4 is just a special case of a class of duals, see Theorem 3.1 in [4] by Christensen and Kim. For $\eta = 2$ Theorem 2.2 exhibits duals that are not covered by the results in [4].

Observe that Laugesen [11] has reported another approach of construction of dual pairs of Gabor frames, based on computer algebraic programs. That approach and the subsequent result reported in [10] do not need partition of unity condition either.

3. Frames generated by $\sin(\frac{\eta}{3}\pi x)\chi_{[0,3]}(x)$

We will now consider functions of the form

$$g(x) = \sin(\tfrac{\eta}{3}\pi x)\chi_{[0,3]}(x), \tag{3.1}$$

where $\eta \in \mathbb{N}$. Note that in contrast with the function in (2.1), this is not a bump-like function, but rather an oscillation with high frequency for large values of η.

Interestingly, the functions in (3.1) behave differently from the functions in (2.1) in the context considered here. This is the main reason for including the analysis here.

Proposition 3.1. *Let $\eta \in \mathbb{N}$ and consider the function g in* (3.1)*. Then the following holds:*

(i) Let $b \in]0, 1/3]$. If $\eta \in \mathbb{N} \setminus 3\mathbb{N}$ then $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is a tight Gabor frame for $L^2(\mathbb{R})$ with frame bound $A = 3/2b$.
(ii) For $\eta \in 3\mathbb{N}$, $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is not a Gabor frame for any $b \in \mathbb{R}$.

Proof. We first show (i). Let $\eta \in \mathbb{N} \setminus 3\mathbb{N}$. Similar considerations as in the proof of Proposition 2.1 show that

$$\begin{aligned} A &:= \tfrac{1}{b} \inf_{x\in[0,1]} \left[\sum_{n\in\mathbb{Z}} |g(x-n)|^2 - \sum_{n\in\mathbb{Z}} g(x-n)g(x-n-k/b)| \right] \\ &= \tfrac{1}{b} \inf_{x\in[0,1]} \left[|g(x)|^2 + |g(x+1)|^2 + |g(x+2)|^2 \right]. \end{aligned}$$

We also have

$$\begin{aligned} B &:= \tfrac{1}{b} \sup_{x\in[0,1]} \left[\sum_{n\in\mathbb{Z}} |g(x-n)|^2 + \sum_{n\in\mathbb{Z}} g(x-n)g(x-n-k/b)| \right] \\ &= \tfrac{1}{b} \sup_{x\in[0,1]} \left[|g(x)|^2 + |g(x+1)|^2 + |g(x+2)|^2 \right]. \end{aligned}$$

Using trigonometric identities one can show that $A = B = \frac{3}{2b}$. By Theorem 9.1.5 in [1], $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is a tight frame.

It remains to show that $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is not a Gabor frame for $\eta \in 3\mathbb{N}$. By Proposition 8.3.2 in [1] a necessary condition for $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ to be a frame is that $\inf_{x\in[0,1]} \sum_{k\in\mathbb{Z}} |g(x-k)|^2 > 0$. However, using trigonometric identities one can easily show that $\sum_{k\in\mathbb{Z}} |g(x-k)|^2 = 0$ for $x = 0$. $\square$

Concerning g and the partition of unity condition, we have the following negative result.

Proposition 3.2. *Let $\eta \in \mathbb{N}$ and consider the function g in* (3.1)*. Let*

$$H(x) := \sum_{k\in\mathbb{Z}} g(x-k).$$

Then the following holds:

(i) The function H is constant if and only if $\eta \in 2\mathbb{N} \setminus 6\mathbb{N} = \{2, 4, 8, 10, \dots\}$.
(ii) If $\eta \in 2\mathbb{N} \setminus 6\mathbb{N}$, then

$$H(x) = 0. \tag{3.2}$$

Proof. Note that $\sum_{k\in\mathbb{Z}} g(x-k)$ is 1-periodic, so we only need to consider the interval $x \in [0,1]$. On this interval we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} g(x-k) = \sin(\tfrac{\eta}{3}\pi x) + \sin(\tfrac{\eta}{3}\pi(x+1)) + \sin(\tfrac{\eta}{3}\pi(x+2))\\ = \quad \sin(\tfrac{\eta}{3}\pi x)\big[1+\cos(\tfrac{\eta}{3}\pi)+\cos(\tfrac{2\eta}{3}\pi)\big]\\ + \cos(\tfrac{\eta}{3}\pi x)\big[\sin(\tfrac{\eta}{3}\pi)+\sin(\tfrac{2\eta}{3}\pi)\big].\end{aligned}$$

Since $\{1, \sin(\frac{\eta}{3}\pi x), \cos(\frac{\eta}{3}\pi x)\}$ is a set of linear independent functions, the above expression is constant if and only if

$$\begin{cases} 0 = 1+\cos(\frac{\eta}{3}\pi)+\cos(\frac{2\eta}{3}\pi), \\ 0 = \sin(\frac{\eta}{3}\pi)+\sin(\frac{2\eta}{3}\pi). \end{cases}$$

The sine and cosine functions are 2π-periodic, so it is enough to look at these two equations for $\eta = 1, \dots, 6$. It turns out that the equations are satisfied for $\eta = 2, 4$ only, in which case a direct computation yields that $\sum_{k\in\mathbb{Z}} g(x-k) = 0$. $\square$

Thus, in the sense of (1.6), the functions g of the type (2.1) never satisfy the partition of unity condition. Nevertheless we now show that the generated Gabor frames do have the desired structure:

Theorem 3.3. *Let $\eta \in \mathbb{N}\setminus 3\mathbb{N}$, $b \in]0,1/5]$ and $C, D, E \in \mathbb{R}$ and consider the functions*

$$g(x) = \sin\left(\tfrac{\eta}{3}\pi x\right)\chi_{[0,3]}(x) \tag{3.3}$$

and

$$h(x) = \tfrac{4}{3}\, b\big(Cg(x) + Dg(x+1) + Eg(x+2)\big). \tag{3.4}$$

The function g generates a Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ with a dual generator h of the form (3.4) *characterized by*

$$C \in \mathbb{R},\ D = (2C-1)(-1)^{\eta},\ E = 2C-1. \tag{3.5}$$

Proof. Let $\eta \in \mathbb{N}\setminus 3\mathbb{N}$. We want to characterize all functions $h \in \operatorname{span}\{T_{-k}g\}_{k=0}^{2}$ such that $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$. A complete representation of all functions $h \in \operatorname{span}\{T_{-k}g\}_{k=0}^{2}$ is given by

$$h(x) = r\,g(x) + p\,g(x+1) + q\,g(x+2)\,, \quad p, q, r \in \mathbb{R}. \tag{3.6}$$

Both functions g and h are bounded, have compact support and belong to $L^2(\mathbb{R})$. According to Theorem 1.3 the two functions g and h need to satisfy (1.3) in order for $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ to be dual frames for $L^2(\mathbb{R})$.

Since $\operatorname{supp} g = [0,3]$ and $\operatorname{supp} h = [-2,3]$, is (1.3) satisfied for $n \neq 0$ if $1/b \geq 5$, i.e. $b \in]0,1/5]$. Note that if g and h satisfy

$$\sum_{k\in\mathbb{Z}} g(x-k)h(x-k) = c \neq 0,\ a.e.\ x \in [0,1], \tag{3.7}$$

where c is an arbitrary constant, then the function $\frac{b}{c}h$ will satisfy (1.3) for $n = 0$. We will now find the functions h that satisfy (3.7). First, note that since $\operatorname{supp} g = [0, 3]$ and $x \in [0, 1]$, is

$$\sum_{k\in\mathbb{Z}} g(x-k)h(x-k) = \sum_{k=0}^{2} g(x+k)h(x+k). \tag{3.8}$$

Plugging the expression for g and h into (3.8) yields

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} g(x-k)h(x-k) \\ &= r\sin^2(\tfrac{\eta}{3}\pi x) + r\sin^2\big(\tfrac{\eta}{3}\pi(x+1)\big) + r\sin^2\big(\tfrac{\eta}{3}\pi(x+2)\big) \\ &\quad + p\sin(\tfrac{\eta}{3}\pi x)\sin\big(\tfrac{\eta}{3}\pi(x+1)\big) + q\sin(\tfrac{\eta}{3}\pi x)\sin\big(\tfrac{\eta}{3}\pi(x+2)\big) \\ &\quad + p\sin\big(\tfrac{\eta}{3}\pi(x+1)\big)\sin\big(\tfrac{\eta}{3}\pi(x+2)\big). \end{aligned}$$

This can also be written as

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} g(x-k)h(x-k) \\ &= \tfrac{1}{2}\sin(\tfrac{2}{3}\eta\pi x)\,\big[p\sin(\tfrac{\eta}{3}\pi) + q\sin(\tfrac{2}{3}\eta\pi)\big] \\ &\quad - \tfrac{1}{2}\cos(\tfrac{2}{3}\eta\pi x)\,\big[r + (-1)^{\eta}\,p + \big(2(-1)^{\eta}r + p\big)\cos(\tfrac{\eta}{3}\pi) \\ &\qquad\qquad\qquad\qquad\qquad\qquad + q\cos(\tfrac{2}{3}\eta\pi)\big] \\ &\quad + \tfrac{3}{2}r + p\,\cos(\tfrac{\eta}{3}\pi) + \tfrac{1}{2}\,q\cos(\tfrac{2}{3}\eta\pi). \end{aligned} \tag{3.9}$$

The right hand side in (3.9) is a linear combination of the three *linear independent* functions $1, \cos(\frac{2}{3}\eta\pi x)$ and $\sin(\frac{2}{3}\eta\pi x)$. Thus $\sum_{k\in\mathbb{Z}} g(x-k)h(x-k)$ is constant if and only if

$$\begin{cases} 0 = p\sin(\tfrac{\eta}{3}\pi) + q\sin(\tfrac{2}{3}\eta\pi), & (3.10\text{a}) \\ 0 = r + (-1)^{\eta}p + \big(2(-1)^{\eta}r + p\big)\cos(\tfrac{\eta}{3}\pi) + q\cos(\tfrac{2}{3}\eta\pi). & (3.10\text{b}) \end{cases}$$

The sine and cosine functions are 2π-periodic, so we can restrict ourselves to solve (3.10) for $\eta = 1, 2, 4, 5$. We now consider each of these cases separately.

Case $\boldsymbol{\eta} = \mathbf{1}$. The equations in (3.10) become

$$\begin{cases} 0 = \tfrac{\sqrt{3}}{2}p + \tfrac{\sqrt{3}}{2}q, & (3.11\text{a}) \\ 0 = -\tfrac{1}{2}p - \tfrac{1}{2}q. & (3.11\text{b}) \end{cases}$$

The solutions are characterized by $q = -p$. From (3.9) it follows that

$$\sum_{k\in\mathbb{Z}} g(x-k)h(x-k) = \tfrac{3}{4}(2r+p). \tag{3.12}$$

Hence (1.3) has the solutions

$$h(x) = \tfrac{4}{3}b(2r+p)^{-1}\big(rg(x) + pg(x+1) - pg(x+2)\big), \tag{3.13}$$

where $p, r \in \mathbb{R}$, $p \neq -2r$.

In a similar way we handle the cases $\eta = 2, 4, 5$. We skip the details. The calculations show that

$$h(x) = \tfrac{4}{3}b\,(2r - (-1)^\eta p)^{-1}\big(rg(x) + pg(x+1) + (-1)^\eta pg(x+2)\big), \tag{3.14}$$

where $p, r \in \mathbb{R}$, $p \neq (-1)^\eta 2r$. To rewrite these functions as presented in the Theorem, we define $C := r(2r - (-1)^\eta p)^{-1}$.

Note that C can attain any value on the real line and that

$$(2C - 1)(-1)^\eta = \frac{p}{2r - (-1)^\eta p}\,. \tag{3.15}$$

The result now follows easily. $\square$

Note that the choice of $C = 1/2$ results in the tight frame mentioned in Proposition 3.1. Fig. 2 shows examples of dual generators h for $\eta = 2, 4$.

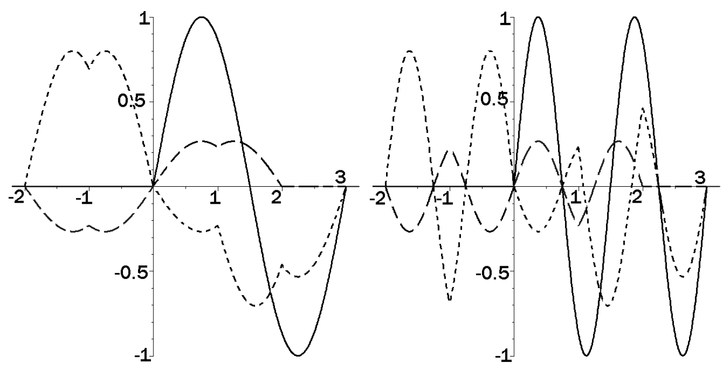

Fig. 2. Plot of $g(x) = \sin(2\pi x/3)\chi_{[0,3]}(x)$ and $g(x) = \sin(4\pi x/3)\chi_{[0,3]}(x)$ (solid) respectively to the left and right with duals generated by Theorem 3.3 for $C = 0$ (dashed) and $C = 2$ (dotted).

Corollary 3.4. *Under the assumptions in Theorem 3.3(ii) the dual generators in* (3.5) *can be written as*

$$h(x) = h_1(x) + Ch_2(x),\ \ C \in \mathbb{R}, \tag{3.16}$$

where

$$h_1(x) = \tfrac{4}{3}\,b\,\big((-1)^{\eta+1}g(x+1) - g(x+2)\big) \tag{3.17}$$

and

$$h_2(x) = \tfrac{4}{3}\,b\,(g(x) + 2(-1)^\eta g(x+1) + 2g(x+2)). \tag{3.18}$$

Proof. We can write the dual generators in (3.5) as

$$\begin{aligned} h(x) &= \tfrac{4}{3}\,b\,(Cg(x) + (2C-1)(-1)^\eta g(x+1) + (2C-1)g(x+2)) \\ &= \tfrac{4}{3}\,b\,\big((-1)^{\eta+1}g(x+1) - g(x+2)\big) \\ &\qquad + \tfrac{4}{3}\,b\,(g(x) + 2(-1)^\eta g(x+1) + 2g(x+2))\,C \\ &= h_1(x) + Ch_2(x). \end{aligned}$$

$\square$

Note the similarities between h_2 and the dual generators given by Theorem 1.4. Interestingly, for even η we get (except for a factor of $\frac{4}{3}$) the same expression for h_2 as we have for the function h in (1.5).

4. Proof of Theorem 2.2

Before we state the proof of Theorem 2.2 we need a result concerning trigonometric functions.

Lemma 4.1. *Let $n \in \mathbb{N}$. The set of functions given by*

$$\left\{1, \, \sin x \cos x, \, \sin^2 x, \, \sin^3 x \cos x, \, \sin^4 x, \, \dots, \, \sin^{2n-1} x \cos x, \, \sin^{2n} x\right\}$$

is a set of linear independent functions.

Proof. We need to show that if

$$\begin{aligned} &a_0 \cdot 1 + a_1 \cdot \sin^2 x + a_2 \cdot \sin^4 x + \; \dots \; + a_n \cdot \sin^{2n} x \\ &\qquad + b_1 \cdot \sin x \cos x + b_2 \cdot \sin^3 x \cos x + \; \dots \; + b_n \cdot \sin^{2n-1} x \cos x = 0, \end{aligned} \tag{4.1}$$

for all $x \in \mathbb{R}$, then $a_0 = 0$, $a_k = b_k = 0$, for all $k = 1, \dots, n$.

Let $x = 0$. Then (4.1) implies $a_0 = 0$. Thus

$$\begin{aligned} &\sin x \, \big(a_1 \cdot \sin x + a_2 \cdot \sin^3 x + \; \dots \; + a_n \cdot \sin^{2n-1} x \\ &\qquad + b_1 \cdot \cos x + b_2 \cdot \sin^2 x \cos x + \; \dots \; + b_n \cdot \sin^{2n-2} x \cos x\big) = 0 \end{aligned} \tag{4.2}$$

for all $x \in \mathbb{R}$. Due to the continuity of any trigonometric polynomial, this implies

$$\begin{aligned} &a_1 \cdot \sin x + a_2 \cdot \sin^3 x + \; \dots \; + a_n \cdot \sin^{2n-1} x \\ &+ b_1 \cdot \cos x + b_2 \cdot \sin^2 x \cos x + \; \dots \; + b_n \cdot \sin^{2n-2} x \cos x = 0. \end{aligned} \tag{4.3}$$

Taking $x = 0$, we see that necessarily $b_1 = 0$. The same step as before now results in

$$\begin{aligned} &a_1 + a_2 \cdot \sin^2 x + \; \dots \; + a_n \cdot \sin^{2n-2} x \\ &+ b_2 \cdot \sin x \cos x + \; \dots \; + b_n \cdot \sin^{2n-3} x \cos x = 0. \end{aligned} \tag{4.4}$$

Again, for $x = 0$ this implies that $a_1 = 0$. Continuing in the same fashion finally leads to $a_k = b_k = 0$, for all $k = 1, \dots, n$. □

Lemma 4.1 will be used in the following proof of Theorem 2.2.

Proof. Let $\eta \in \mathbb{N}$. We want to characterize all functions $h \in \operatorname{span}\{T_{-k} g\}_{k=0}^2$ such that $\{E_{mb} T_n g\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb} T_n h\}_{m,n \in \mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$. A complete representation of all functions $h \in \operatorname{span}\{T_{-k} g\}_{k=0}^2$ is given by

$$h(x) = r \, g(x) + p \, g(x+1) + q \, g(x+2) \, , \quad p, q, r \in \mathbb{R}. \tag{4.5}$$

Both functions g and h are bounded, have compact support and belong to $L^2(\mathbb{R})$. According to Theorem 1.3 the two functions g and h need to satisfy (1.3) in order for $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ to be dual frames for $L^2(\mathbb{R})$.

Since $\operatorname{supp} g = [0,3]$ and $\operatorname{supp} h = [-2,3]$, equation (1.3) satisfied for $n \neq 0$ if $1/b \geq 5$, i.e. $b \in]0, 1/5]$. Note that if g and h satisfy

$$\sum_{k\in\mathbb{Z}} g(x-k)h(x-k) = c \neq 0, \ \ a.e. \ x \in [0,1], \tag{4.6}$$

where c is an arbitrary constant, then the function $\frac{b}{c}h$ will satisfy (1.3) for $n = 0$. We will now find the functions h that satisfy (4.6). First, note that since $\operatorname{supp} g = [0,3]$ and $x \in [0,1]$,

$$\sum_{k\in\mathbb{Z}} g(x-k)h(x-k) = \sum_{k=0}^{2} g(x+k)h(x+k). \tag{4.7}$$

Plugging the expression for g and h into (4.7) yields

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} g(x-k)h(x-k) \\ &= r\sin^{2\eta}(\tfrac{1}{3}\pi x) + r\sin^{2\eta}\big(\tfrac{1}{3}\pi(x+1)\big) + r\sin^{2\eta}\big(\tfrac{1}{3}\pi(x+2)\big) \\ &\quad + p\sin^{\eta}(\tfrac{1}{3}\pi x)\sin^{\eta}\big(\tfrac{1}{3}\pi(x+1)\big) + q\sin^{\eta}\big(\tfrac{1}{3}\pi x\big)\sin^{\eta}\big(\tfrac{1}{3}\pi(x+2)\big) \\ &\quad + p\sin^{\eta}\big(\tfrac{1}{3}\pi(x+1)\big)\sin^{\eta}\big(\tfrac{1}{3}\pi(x+2)\big). \end{aligned} \tag{4.8}$$

Before we continue, let us introduce the following variables.

$$\begin{aligned} &\alpha(m,k) = 2\,(-1)^k\,\tfrac{3^m}{4^\eta}\tbinom{m}{k}\tbinom{2\eta}{2m}, \ \ \beta(m,k) = (-1)^{k+m}\,\tfrac{3^m}{4^\eta}\tbinom{m}{k}\tbinom{\eta}{m}, \\ &\gamma(m,k) = (-1)^k\,\tfrac{3^m}{2^\eta}\tbinom{m}{k}\tbinom{\eta}{2m}, \ \ \delta(m,k) = (-1)^k\sqrt{3}\,\tfrac{3^m}{2^\eta}\tbinom{m}{k}\tbinom{\eta}{2m+1}. \end{aligned}$$

We now rewrite (4.8) as a linear combination of linear independent functions. First, note that

$$\sin\big(\tfrac{1}{3}\pi(x+k)\big) = \sin(\tfrac{1}{3}\pi x)\cos(\tfrac{1}{3}\pi k) + \cos(\tfrac{1}{3}\pi x)\sin(\tfrac{1}{3}\pi k).$$

By use of

$$(a+b)^n = \sum_{m=0}^{n}\tbinom{n}{m}a^{n-m}b^m, \ \text{ for } n\in\mathbb{N}$$

we have

$$\sin^{\eta}\big(\tfrac{1}{3}\pi(x+k)\big) = \sum_{m=0}^{\eta}\tbinom{\eta}{m}\big[\sin(\tfrac{1}{3}\pi x)\cos(\tfrac{1}{3}\pi k)\big]^{\eta-m}\big[\cos(\tfrac{1}{3}\pi x)\sin(\tfrac{1}{3}\pi k)\big]^{m}.$$

Using this expression, one can consider even and odd values of η separately and rewrite (4.8) as

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}} g(x-k)h(x-k) \\
&= \Big[r + \sum_{m=0}^{\eta} \Big(r\alpha(m,m) + (-1)^\eta p\,\beta(m,m)\Big) + \sum_{m=0}^{\lfloor \eta/2\rfloor} \gamma(m,m)\big[p+(-1)^\eta q\big]\Big] \sin^{2\eta}(\tfrac{1}{3}\pi x) \\
&\quad + \sum_{d=1}^{\lfloor \eta/2\rfloor} \Bigg(\Big[\sum_{m=d}^{\eta} \Big(r\alpha(m,m-d) + (-1)^\eta p\,\beta(m,m-d)\Big) \\
&\qquad\qquad + \sum_{m=d}^{\lfloor \eta/2\rfloor} \gamma(m,m-d)\big[p+(-1)^\eta q\big]\Big]\ \sin^{2\eta-2d}(\tfrac{1}{3}\pi x)\Bigg) \\
&\quad + \sum_{d=\lfloor \eta/2\rfloor+1}^{\eta-1} \Bigg(\Big[\sum_{m=d}^{\eta} \Big(r\alpha(m,m-d) + (-1)^\eta p\,\beta(m,m-d)\Big)\Big]\ \sin^{2\eta-2d}(\tfrac{1}{3}\pi x)\Bigg) \\
&\quad + \sum_{d=0}^{\lfloor (\eta-1)/2\rfloor} \Bigg(\Big[\sum_{m=d}^{\lfloor (\eta-1)/2\rfloor} \delta(m,m-d)\ \big[p-(-1)^\eta q\big]\Big]\ \sin^{2\eta-2d-1}(\tfrac{1}{3}\pi x)\cos(\tfrac{1}{3}\pi x)\Bigg) \\
&\quad + r\,\alpha(\eta,0) + (-1)^\eta\, p\,\beta(\eta,0).
\end{aligned} \tag{4.9}
$$

Note that the right hand side in (4.9) is a linear combination of the functions

$$\big\{\, 1,\ \sin x\cos x,\ \sin^2 x,\ \ldots,\ \sin^{2n-1} x\cos x,\ \sin^{2n} x\,\big\}. \tag{4.10}$$

In Lemma 4.1 these functions are shown to be linearly independent. According to (4.6), we need that

$$\sum_{k\in\mathbb{Z}} g(x-k)h(x-k) = c \neq 0,\ \ a.e.\ x\in[0,1].$$

This is possible if and only if

$$
\begin{cases}
0 = r + \sum_{m=0}^{\eta}\Big(r\alpha(m,m) + (-1)^\eta p\beta(m,m)\Big) + \sum_{m=0}^{\lfloor \eta/2\rfloor} \gamma(m,m)\big[p+(-1)^\eta q\big], & \text{(4.11a)} \\
0 = \sum_{m=d}^{\eta}\Big(r\alpha(m,m-d) + (-1)^\eta p\beta(m,m-d)\Big) + \sum_{m=d}^{\lfloor \eta/2\rfloor} \gamma(m,m-d)\big[p+(-1)^\eta q\big], & \text{(4.11b)} \\
0 = \sum_{m=d}^{\eta}\Big(r\alpha(m,m-d) + (-1)^\eta p\beta(m,m-d)\Big), & \text{(4.11c)} \\
0 = \sum_{m=d}^{\lfloor (\eta-1)/2\rfloor} \delta(m,m-d)\big[p-(-1)^\eta q\big]. & \text{(4.11d)}
\end{cases}
$$

Eq. (4.11b), (4.11c) and (4.11d) need to be satisfied for all integers d for which $1 \le d \le \lfloor \frac{\eta}{2}\rfloor$, $\lfloor \frac{\eta}{2}\rfloor + 1 \le d \le \eta - 1$ and $0 \le d \le \lfloor \frac{\eta-1}{2}\rfloor$ respectively. Depending on η (4.11b), (4.11c) and (4.11d) contain a different number of equations. We now solve the system of equations (4.11); we split into the cases $\eta < 6$ and $\eta \geq 6$. Note that the case $\eta = 1$ is already covered by Theorem 3.3.

Case $\boldsymbol{\eta} = \mathbf{3}$. Equation (4.11a) turns into

$$0 = 3r - 2p + q. \tag{4.12}$$

Equation (4.11b) needs to hold for $1 \leq d \leq \lfloor \frac{3}{2} \rfloor$, i.e. for $d = 1$, and yields

$$0 = -\tfrac{9}{2}\, r + \tfrac{27}{8}\, p - \tfrac{9}{8}\, q. \tag{4.13}$$

Equation (4.11c) needs to hold for $\lfloor \frac{3}{2} \rfloor + 1 \leq d \leq 2$, i.e. for $d = 2$, and yields

$$0 = \tfrac{27}{16}\, r - \tfrac{27}{16}\, p. \tag{4.14}$$

Equation (4.11d) needs to holds for $0 \leq d \leq 1$, i.e. for $d = 0$ and $d = 1$, and yields

$$0 = 0\,, \quad 0 = \tfrac{3}{8}\sqrt{3}\,(p + q). \tag{4.15}$$

The five equations have the solution $r \in \mathbb{R},\ p = -q = r$. From (4.9) follows that the function

$$h(x) = rg(x) + rg(x+1) - rg(x+2) \tag{4.16}$$

satisfies

$$\sum_{k \in \mathbb{Z}} g(x-k)h(x-k) = r\,\alpha(3,0) - p\,\beta(3,0) = \left(\tfrac{3}{4}\right)^3 3r\,. \tag{4.17}$$

Thus the function

$$h(x) = \left(\tfrac{4}{3}\right)^3 b\left(\tfrac{1}{3}\, g(x) + \tfrac{1}{3}\, g(x+1) - \tfrac{1}{3}\, g(x+2)\right) \tag{4.18}$$

is the unique solution to (1.3) of the form (4.5).

Similar calculations are done for $\eta = 2, 4$ and 5. We skip the details.

Let us now prove that there are no duals of the form (4.5) for $\eta \geq 6$. Note that if all equations in (4.11) are satisfied, then it follows from (4.9) that

$$\sum_{k \in \mathbb{Z}} g(x-k)h(x-k) = r\alpha(\eta, 0) + (-1)^\eta p\beta(\eta, 0) = \left(\tfrac{3}{4}\right)^\eta (2r + p). \tag{4.19}$$

For $\eta \geq 6$ (4.11c) gives at least two equations. Consider the two equations from (4.11c) for $d = \eta - 1$ and $d = \eta - 2$, that is

$$p = r(\eta - 2) \quad \text{and} \quad p = -\tfrac{1}{6} r(\eta^2 - 11\eta + 12). \tag{4.20}$$

Both equations are satisfied if $r = p = 0$. However, in that case it follows from (4.19) that

$$\sum_{k \in \mathbb{Z}} g(x-k)h(x-k) = 0. \tag{4.21}$$

A scaling of h such that (1.3) is satisfied is in this case impossible. We therefore must assume that

$$\eta - 2 = -\tfrac{1}{6}(\eta^2 - 11\eta + 12).$$

This is only possible for $\eta = 0$ or $\eta = 5$. Thus, it is not possible to find $p, r \in \mathbb{R}$ such that all equations in (4.11) are satisfied for $\eta \geq 6$. This completes the proof. $\square$

$\square$

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