Tchebycheffian B-splines

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Joint work with Carla Manni and Hendrik Speleers

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February 19, 2018

Some applications and names

- splines in tension,
- shape approximation,
- exact representation of conic sections without rationals,
- isogeometric analysis, $\cdots$

Barry, Beccari, Bister, Bracco, Carnicer, Casciola, Chen, Costantini, Dyn, Jerome, Koch, Laurent, L, Mainar, Manni, Mazure, Mühlbach, Nürnberger, Pelosi, Pena, Pottmann, Prautzsch, Roman, Ron, Sampoli, Sanchez-Reyes, Scherer, Schumaker, Schweikert, Sommer, Speleers, Strauss, Sun, Wang, Xu, Zhou,$\cdots$

content

- How to define a B-spline?

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- Can the recurrence relation be generalized?

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- GB-splines

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7. $\cdots$

1 B-spline divided difference, degree p

$$B_{j,p,\boldsymbol{\xi}}(x) := (\xi_{j+p+1} - \xi_j)[\xi_j, \ldots, \xi_{j+p+1}](\cdot - x)_+^p, \quad x \in \mathbb{R}$$

$$(y - x)_+^p := \max((y - x)^p, 0)$$

Popoviciu, Chakalov, 1930's
Curry&Schoenberg,1966

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Trigonometric and Tchebycheffian divided differences can be used to define more general B-splines
Karlin 1966, L&Winther 1978

2 recurrence relation

$$B_{j,p,\xi}(x) := \frac{x-\xi_j}{\xi_{j+p}-\xi_j} B_{j,p-1,\xi}(x) + \frac{\xi_{j+p+1}-x}{\xi_{j+p+1}-\xi_{j+1}} B_{j+1,p-1,\xi}(x)$$

$$B_{i,0,\xi}(x) := \begin{cases} 1, & \text{if } x \in [\xi_i \xi_{i+1}), \\ 0, & \text{otherwise.} \end{cases}$$

Popoviciu, Chakalov, Cox, deBoor, Mansfield

difficult to generalize

3 integrating the differentiation formula

The differentiation formula:

$$DB_{j,p,\xi} = \frac{B_{j,p-1,\xi}}{\gamma_{j,p-1,\xi}} - \frac{B_{j+1,p-1,\xi}}{\gamma_{j+1,p-1,\xi}}$$

$$\gamma_{i,p-1,\xi} := \int_{\xi_i}^{\xi_{i+p}} B_{i,p-1,\xi}(t)\,\mathrm{d}t = \frac{\xi_{i+p} - \xi_i}{p}$$

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Integrating:

$$B_{j,p,\xi}(x) := \int_{\xi_j}^{x} \frac{B_{j,p-1,\xi}(t)}{\gamma_{j,p-1,\xi}}\,\mathrm{d}t - \int_{\xi_{j+1}}^{x} \frac{B_{j+1,p-1,\xi}(t)}{\gamma_{j+1,p-1,\xi}}\,\mathrm{d}t, \quad x \in \mathbb{R}$$

Sommer, Strauss,1988

We use this idea to define more general B-splines

More general spaces

B-splines defined using pieces taken from one space:

$$\mathbb{U} = \langle u_0, \ldots, u_p \rangle := \{\sum_{j=0}^{p} c_j u_j : c_j \in \mathbb{R}\} \subset C^p[a, b]$$

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- **General exponentials**
  $(e^{\alpha_0 x}, \ldots, e^{\alpha_p x}), \quad \alpha_i \in \mathbb{R}, \quad \alpha_i \neq \alpha_j$ for $i \neq j$.

Defining B-splines recursively for more general spaces

B-splines defined using pieces of

$$\mathbb{U} = \langle u_0, \ldots, u_p \rangle := \{\sum_{j=0}^{p} c_j u_j : c_j \in \mathbb{R}\} \subset C^p[a,b]$$

For $k = 1, \ldots, p$

$$B^{\mathbb{U}}_{j,k,\boldsymbol{\xi}}(x) := \frac{s_k(x-\xi_j)}{s_k(\xi_{j+k}-\xi_j)} B^{\mathbb{U}}_{j,k-1,\boldsymbol{\xi}}(x) + \frac{s_k(\xi_{j+k+1}-x)}{s_k(\xi_{j+k+1}-\xi_{j+1})} B^{\mathbb{U}}_{j+1,k-1,\boldsymbol{\xi}}(x),$$

where

$$B^{\mathbb{U}}_{i,0,\boldsymbol{\xi}}(x) := \begin{cases} 1, & \text{if } x \in [\xi_i \xi_{i+1}), \\ 0, & \text{otherwise.} \end{cases}$$

$s_k(t) := t \implies B^{\mathbb{U}}_{j,p,\boldsymbol{\xi}} = B_{j,p,\boldsymbol{\xi}}$ the usual polynomial B-splines

Assume:

- $s_1, \ldots, s_p \in C^{p-1}[0, b-a]$
- $s_k(0) := 0$, $s_k(t) > 0$ for $t > 0$
- $B^{\mathbb{U}}_{j,p,\boldsymbol{\xi}}$ piecewise in $\mathbb{U}$.

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- **trigonometric B-splines**: $s_k(t) := \sin(\alpha t)/\alpha$, all k, $p = 2n$:
  piecewise in $\langle 1, \cos(\alpha x), \sin(\alpha x), \ldots, \cos(\alpha n x), \sin(\alpha n x)\rangle$

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- **exponential B-splines**: $s_k(t) := \sinh(\alpha t)/\alpha$, all k, $p = 2n$:
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Using the recurrence relation it can be shown:

- $B^{\mathbb{U}}_{j,p,\boldsymbol{\xi}} = 0$, $x < \xi_j$, $x > \xi_{j+p+1}$
- $B^{\mathbb{U}}_{j,p,\boldsymbol{\xi}} > 0$, $\xi_j < x < \xi_{j+p+1}$ (if $\xi_{j+p+1} - \xi_j < 2\pi$ in the trigonometric case)
- $B \in C^{p-\mu}(\xi)$ at a knot of multiplicity μ.
- partition of unity if rescaled
- Marsden identity, $\cdots$

L&Winther 1979, Schumaker 1982

Are there other examples?

The continuity property implies that $s_k = s$ all k (independence of k). Thus

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The continuity property also implies

$$2s(h)s'(h) = s'(0)s(2h), \text{ all } h > 0.$$

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- exp since $2\sinh(h)\cosh(h) = \sinh(2h)$

Two recurrence relations for general Tchebycheffian B-splines

$$L1985: B_p^{\mathbb{U}} = \alpha B_{1,p-1}^{\mathbb{U}_1} + (1-\alpha) B_{1,p-1}^{\mathbb{U}_2}, \quad 0 \le \alpha \le 1$$

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$$Dyn, Ron1988 : B_p^{\mathbb{U}} = a_1 B_{1,p-1}^{\mathbb{U}} + a_2 B_{2,p-1}^{\mathbb{U}} + a_3 B_{3,p-1}^{\mathbb{U}} a_4 B_{4,p-1}^{\mathbb{U}}$$

The Zoo of splines

Why Zoo?

Exponential B-splines in Tension

Per Erik Koch and Tom Lyche

Abstract. This paper describes material to appear in [4]. We define B-splines for exponential splines in tension and derive their main properties. Our principal result is a simple and stable way of calculating with these B-splines.

§1. B-splines

Splines in tension, considered first by Schweikert [13], (see also [2,6-12,15,16]), are piecewise in the space $T_4^{\rho_i}$, where for any integer $k \geq 2$ and any positive ρ

$$T_k^\rho = \operatorname{span}\{e^{\rho x}, e^{-\rho x}, 1, x, \cdots, x^{k-3}\}.$$

The parameter ρ_i can be used for shape control on the knot interval $I_i = (t_i, t_{i+1})$. If ρ_i is small, then the spline behaves like a cubic polynomial on I_i, while if ρ_i is large (increased tension), then the spline is almost a straight line on I_i. We note that T_k^ρ is the null space of the linear differential operator $L_k = D^k - \rho^2 D^{k-2}$. Therefore, it is natural to define $T_k^\rho = \pi_{k-1}$ (the set of polynomials of degree less than k) for $\rho = 0$ and $T_k^\rho = \pi_{k-3}$ for $\rho = \infty$, and $k \geq 3$. We also note that if $f \in T_k$ then $Df = df/dx \in T_{k-1}$.

For simplicity we restrict attention to the cubic case, $k = 4$ in this paper. With $\Delta_i = t_{i+1} - t_i$ we define

$$\begin{aligned} u_i(x) &= \phi(x - t_i, \rho_i, \Delta_i), \\ v_i(x) &= \phi(t_{i+1} - x, \rho_i, \Delta_i), \\ w_i(x) &= u_i(x) - v_{i+1}(x), \end{aligned} \tag{1}$$

where for real x, $r \geq 0$, and $\Delta > 0$,

$$\phi(x, r, \Delta) = \begin{cases} x^3/(6\Delta), & \text{if } r = 0, \\ (\sinh rx - rx)/(r^2 \sinh r\Delta), & \text{if } 0 < r < \infty, \\ 0, & \text{if } r = \infty. \end{cases} \tag{2}$$

The function ϕ plays the role of the monomial x^3.

Approximation Theory VI: Volume 2
C. K. Chui, L. L. Schumaker and J. D. Ward (eds.), pp. 361-364.

ISBN 0-12-174587-2.

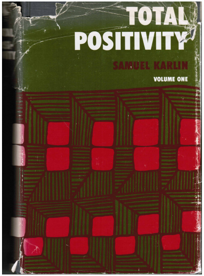
Some species (Schumaker 2002)

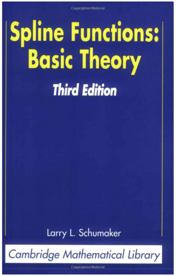
The Zoo of Splines

- Analytic splines
- Arc splines
- Beta splines
- B-splines
- Bernoulli splines
- Box splines
- Cardinal splines
- Circular splines
- Complete splines
- Complex splines
- Confined splines
- Deficient splines
- D^m splines
- Discrete splines
- Euler splines
- Exponential splines
- Gamma splines
- GB-splines
- HB-splines
- Hyperbolic Splines
- Monosplines
- Nu-splines
- Natural splines
- L-splines
- Lg-splines
- Nonlinear splines
- One-sided splines
- Parabolic splines
- Perfect splines
- Periodic splines
- Poly-splines
- Rational splines
- Simplex splines
- Spherical splines
- Taut splines
- Tchebycheffian splines
- Tension splines
- Thin plate splines
- Trigonometric splines
- Whittaker splines

Submit abstracts before March 4.

Extended Tchebycheff space (ET)
Extended Complete Tchebycheff space (ECT)





Karlin/Studden 1966, Karlin 1968, Schumaker 1981/2007

ET space

- I interval on the real line.
- p integer, degree.
- $\mathbb{T}_p(I) \subset C^p(I)$ linear space of dimension $p+1$

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- $\mathbb{T}_p(I) \subset C^p(I)$ linear space of dimension $p+1$
- $\mathbb{T}_p$ is an **ET space** on I if any Hermite interpolation problem with $p+1$ data on I has a unique solution in $\mathbb{T}_p(I)$.
- Equivalently: any nonzero element of $\mathbb{T}_p$ has at most p zeros in I counting multiplicities.
- $\langle \sin x, \cos x \rangle := \{a \sin x + b \cos x : a, b \in \mathbb{R}\}$ is an ET-space on $[0, \pi)$.

ECT space- and system

- $\mathbb{T}_p(I) \subset C^p(I)$ is an **ECT space** if there exists a basis $\{u_0, \ldots, u_p\}$ for $\mathbb{T}_p(I)$ such that every subspace

$$\langle u_0, \ldots, u_k \rangle := \{\sum_{j=0}^{k} c_j u_j : c_j \in \mathbb{R}\}$$

  is an ET-space on I for $k = 0, \ldots, p$.
- $(u_0, u_1, \ldots, u_p)$ is called an **ECT-system.**

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- $(u_0, u_1, \ldots, u_p)$ is called an **ECT-system.**
- $\langle \sin x, \cos x \rangle := \{a \sin x + b \cos x : a, b \in \mathbb{R}\}$ is an ECT-space on $(0, \pi)$, but not on on $[0, \pi)$.

ECT examples

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- **exponential polynomials** $(e^{\alpha_0 x}, \dots, e^{\alpha_p x})$ on $I = \mathbb{R}$
- **generalized polynomials**:
  $\mathbb{P}_p^{U,V} := \langle 1, x, \dots, x^{p-2}, U(x), V(x) \rangle, \quad x \in I$, $U, V \in C^p(I)$,
  $\langle D^{p-1} U, D^{p-1} V \rangle$ has at most one simple zero on I
  - $U(x) = \cos \alpha x$, $V(x) = \sin \alpha x$, $\quad I = [a, a + \pi)$, $a \in \mathbb{R}$
  - $U(x) = \cosh \alpha x$, $V(x) = \sinh \alpha x$, $\quad I = \mathbb{R}$
  - $U(x) = x^n$, $V(x) = x^{n+1}, n \geq p$, $\quad I = (0, \infty)$

  Costantini, L, Manni 2005

Tchebycheffian B-splines

Karlin 1968, Schumaker 2007, Mazure, Mazure et al ·

Two existing definitions of Tchebycheffian B-splines

- using **Tchebycheffian divided differences**

Two existing definitions of Tchebycheffian B-splines

- using **Tchebycheffian divided differences**
- or by integrating the **differentiation formula**:

$$D\left(\frac{B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}}{w_p}\right) = \frac{B^{\mathbf{w}}_{j,p-1,\boldsymbol{\xi}}}{\gamma^{\mathbf{w}}_{j,p-1,\boldsymbol{\xi}}} - \frac{B^{\mathbf{w}}_{j+1,p-1,\boldsymbol{\xi}}}{\gamma^{\mathbf{w}}_{j+1,p-1,\boldsymbol{\xi}}},$$

where $w_0, \ldots, w_p$ are sufficiently smooth positive functions called "weights",

$$\gamma^{\mathbf{w}}_{i,k,\boldsymbol{\xi}} := \int_{\xi_i}^{\xi_{i+k+1}} B^{\mathbf{w}}_{i,k,\boldsymbol{\xi}}(t)\,\mathrm{d}t,$$

and

$$B^{\mathbf{w}}_{i,0,\boldsymbol{\xi}}(x) := \begin{cases} w_0(x), & \text{if } x \in [\xi_i \xi_{i+1}), \\ 0, & \text{otherwise}, \end{cases}$$

We use the second definition allowing weights defined piecewise.

Integrate the differentiation formula

$$D\left(\frac{B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}}{w_p}\right) = \frac{B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}}{\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}} - \frac{B^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}}{\gamma^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}}$$

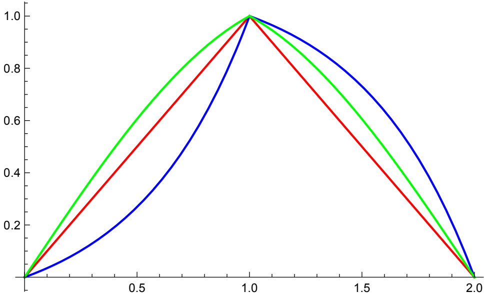
The j-th **Tchebycheffian B-spline** $B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}} : \mathbb{R} \to \mathbb{R}$ of degree p is identically zero if $\xi_{j+p+1} = \xi_j$ and otherwise defined recursively,

$$B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) := w_p(x)\left(\int_{\xi_j}^{x} \frac{B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}}\,\mathrm{d}y - \int_{\xi_{j+1}}^{x} \frac{B^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}}\,\mathrm{d}y\right),$$

$$\gamma^{\boldsymbol{w}}_{i,k,\boldsymbol{\xi}} = 0 \implies \int_{\xi_i}^{x} \frac{B^{\boldsymbol{w}}_{i,k,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{i,k,\boldsymbol{\xi}}}\,\mathrm{d}y := \begin{cases} 1, & \text{if } x \geq \xi_{i+k+1}, \\ 0, & \text{otherwise} \end{cases}$$

Degree one

$$B^{\boldsymbol{w}}_{j,1,\boldsymbol{\xi}}(x) = w_1(x)\begin{cases} \dfrac{\int_{\xi_j}^{x} w_0(y)\,\mathrm{d}y}{\int_{\xi_j}^{\xi_{j+1}} w_0(y)\,\mathrm{d}y}, & \xi_j \le x < \xi_{j+1}, \\ \dfrac{\int_{x}^{\xi_{j+2}} w_0(y)\,\mathrm{d}y}{\int_{\xi_{j+1}}^{\xi_{j+2}} w_0(y)\,\mathrm{d}y}, & \xi_{j+1} \le x < \xi_{j+2}. \end{cases}$$



$\langle 1, x\rangle$, $\langle \cos(1.2x), \sin(1.2x)\rangle$, $\langle 1, e^{2x}\rangle$

Properties of Tchebycheffian splines

- local support

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Positivity

Using our definition

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the most difficult property to prove is positivity. From positivity the other properties follow easily. Bister, thesis, Karlsruhe, 1996

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Lemma (L, Manni, Speleers)

The function

$$s := \sum_{i=j}^{j+k} c_i B^{\boldsymbol{w}}_{i,p,\boldsymbol{\xi}}$$

has at most k sign changes provided all $\gamma^{\boldsymbol{w}}_{r,\ell,\boldsymbol{\xi}}$ *used to define these* $B^{\boldsymbol{w}}_{i,p,\boldsymbol{\xi}}$ *are positive.*

This lemma can be used to prove that all γ's are really nonzero and that

$$B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) > 0, \quad x \in (\xi_j, \xi_{j+p+1}).$$

Inserting one knot

Let $\check{\boldsymbol{\xi}} = (\check{\xi}_i)_{i=1}^{n+p+2}$ be a knot sequence obtained from $\boldsymbol{\xi} = (\xi_j)_{j=1}^{n+p+1}$ by inserting a knot $\check{\xi} \in [\xi_m, \xi_{m+1})$ at position $m+1$. Let

$$s = \sum_{j=1}^{n} c_j B_{j,p,\boldsymbol{\xi}}^{\boldsymbol{w}} = \sum_{i=1}^{n+1} \check{c}_i B_{i,p,\check{\boldsymbol{\xi}}}^{\boldsymbol{w}}$$

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then

$$\check{c}_i = \begin{cases} c_i, & i \leq m-p \\ \alpha_{i,p,\boldsymbol{\xi}}(\tilde{\xi})c_i + (1-\alpha_{i,p,\boldsymbol{\xi}}(\tilde{\xi}))c_{i-1}, & m-p < i \leq m, \\ c_{i-1}, & i > m. \end{cases}$$

where $0 < \alpha_{i,p,\boldsymbol{\xi}}(\tilde{\xi}) < 1$.

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where $0 < \alpha_{i,p,\boldsymbol{\xi}}(\tilde{\xi}) < 1$. Moreover,

$$\alpha_{i,p,\boldsymbol{\xi}}(\tilde{\xi}) = \begin{cases} 1 & \mu_i = p+1, \\ \frac{\gamma^{\boldsymbol{w}}_{i,p-1,\tilde{\xi}} \cdots \gamma^{\boldsymbol{w}}_{i,\mu_j-1,\tilde{\xi}}}{\gamma^{\boldsymbol{w}}_{i,p-1,\xi} \cdots \gamma^{\boldsymbol{w}}_{i,\mu_j-1,\xi}} & \text{otherwise} \end{cases}.$$

de Boor like algorithm for ECT(L,1985)

Knot insertion was derived earlier using Tchebycheffian divided differences and to give an evaluation algorithm for Tchebycheffian splines

If $s(x) := \sum_{j=1}^{n} c_j^{[0]} B_{j,p,\boldsymbol{\xi}}^{\boldsymbol{w}}(x)$ and $x \in [\xi_m, \xi_{m+1})$ then

$$s(x) = c_m^{[p]}, \quad \text{where}$$

$$\begin{array}{cccc} c_{m-p}^{[0]} & c_{m-p+1}^{[0]} & \cdots & c_m^{[0]} \\ & c_{m-p+1}^{[1]} & \cdots & c_m^{[1]} \\ & & \ddots & \vdots \\ & & & c_m^{[p]} \end{array}$$

$$c_j^{[k+1]} = \alpha_{j,k,\boldsymbol{\xi}}(x) c_j^{[k]} + (1 - \alpha_{j,k,\boldsymbol{\xi}}(x)) c_{j-1}^{[k]},$$

GB-splines

Space

- partition $\Delta := \{\eta_0 < \eta_1 < \cdots < \eta_{\ell+1}\}$
- Space:

$$\mathbb{P}_p^{U,V}(\Delta) := \langle 1, x, \dots, x^{p-2}, U(x), V(x) \rangle, \quad x \in [\eta_0, \eta_{\ell+1}],$$

- $\mathbb{P}_p^{U,V}(\Delta)$ is an ECT-space on each $[\eta_i^+, \eta_{i+1}^-]$

and Weights

- Weights on $[\eta_i^+, \eta_{i+1}^-]$:
  $w_{p,i}(x) = w_{p-1,i}(x) = \cdots w_{2,i}(x) = 1$

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- Global weights: $\boldsymbol{w} := (w_0, \ldots, w_p)$
  $w_j(x) := w_{j,i}(x), \quad x \in [\eta_i, \eta_{i+1}), \quad i = 0, \ldots, \ell, \quad j = 0, \ldots, p.$

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  $w_j(x) := w_{j,i}(x), \quad x \in [\eta_i, \eta_{i+1}), \quad i = 0, \ldots, \ell, \quad j = 0, \ldots, p.$
- $w_j \in C^j([\eta_i^+, \eta_{i+1}^-])$, $i = 0, \ldots, \ell$,
- If $w_{1,i}(\eta_i) = w_{1,i}(\eta_{i+1}) = 1$ then $w_j \in C^{j-1}(\eta_i)$, $i = 1 \ldots, \ell$.

Example: Generalized B-splines of degree p

- $\xi_1, \ldots, \xi_{n+p+1} = \overbrace{\eta_0, \ldots, \eta_0}^{\mu_0}, \ldots, \overbrace{\eta_{\ell+1}, \ldots, \eta_{\ell+1}}^{\mu_{\ell+1}}.$
- $u_i, v_i \in \langle D^{p-1} U, D^{p-1} V \rangle$ on $[\xi_i^+, \xi_{i+1}^-]$
  $u_i(\xi_i) = 1, \quad u_i(\xi_{i+1}) = 0, \quad v_i(\xi_i) = 0, \quad v_i(\xi_{i+1}) = 1.$

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- $p = 1$:

$$B^{\mathbf{w}}_{i,1,\boldsymbol{\xi}}(x) := \begin{cases} v_i(x), & \text{if } x \in [\xi_i, \xi_{i+1}), \\ u_{i+1}(x), & \text{if } x \in [\xi_{i+1}, \xi_{i+2}), \\ 0, & \text{otherwise.} \end{cases}$$

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- Polynomials: $U(x) = x^{p-1}$ and $V(x) = x^p$

$$u_i(x) = \frac{\xi_{i+1} - x}{\xi_{i+1} - \xi_i}, \quad v_i(x) = \frac{x - \xi_i}{\xi_{i+1} - \xi_i}, \quad \xi_i < \xi_{i+1}.$$

Example;

Consider the partition $\Delta = \{0, 1, 2, 3\}$, and

$$U(x) = \begin{cases} x, & \text{if } x \in [0,1), \\ \mathrm{e}^{\alpha x}, & \text{if } x \in [1,2), \\ x, & \text{if } x \in [2,3), \end{cases} \qquad V(x) = \begin{cases} x^2, & \text{if } x \in [0,1), \\ \mathrm{e}^{-\alpha x}, & \text{if } x \in [1,2), \\ x^2, & \text{if } x \in [2,3). \end{cases}$$

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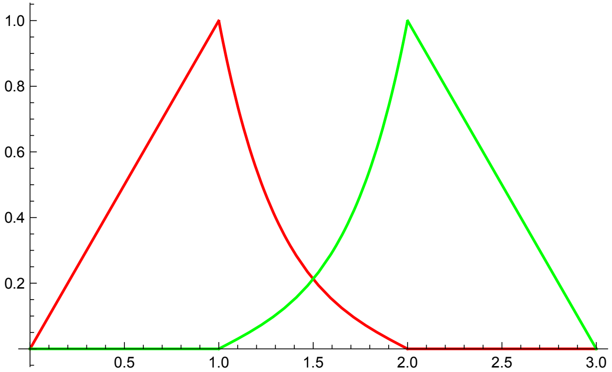
For $p = 1$,

$$B^{w}_{1,1,\xi}(x) = \begin{cases} x, & \text{if } x \in [0,1), \\ \dfrac{\sinh((2-x)\alpha)}{\sinh(\alpha)}, & \text{if } x \in [1,2), \\ 0, & \text{otherwise}, \end{cases}$$

$$B^{w}_{2,1,\xi}(x) = \begin{cases} \dfrac{\sinh((x-1)\alpha)}{\sinh(\alpha)}, & \text{if } x \in [1,2), \\ 3-x, & \text{if } x \in [2,3), \\ 0, & \text{otherwise}, \end{cases}$$

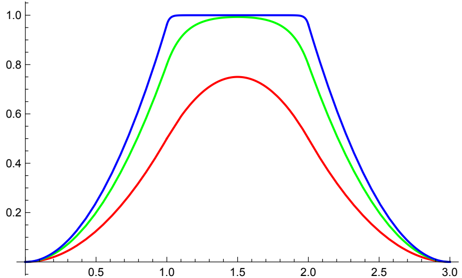
$B^{\boldsymbol{w}}_{1,1,\boldsymbol{\xi}}$ and $B^{\boldsymbol{w}}_{2,1,\boldsymbol{\xi}}$, $\alpha = 3$

$$U(x) = \begin{cases} x, & \text{if } x \in [0,1), \\ \mathrm{e}^{\alpha x}, & \text{if } x \in [1,2), \\ x, & \text{if } x \in [2,3), \end{cases} \qquad V(x) = \begin{cases} x^2, & \text{if } x \in [0,1), \\ \mathrm{e}^{-\alpha x}, & \text{if } x \in [1,2), \\ x^2, & \text{if } x \in [2,3). \end{cases}$$



and for $p = 2$, and $\beta := \alpha/2$

$$B^{\mathbf{w}}_{1,2,\xi}(x) = \frac{1}{1+\dfrac{\sinh(\beta)}{\beta\cosh(\beta)}} \begin{cases} x^2, & \text{if } x \in [0,1), \\ 1+\dfrac{\cosh(\beta)-\cosh((3-2x)\beta)}{\beta\sinh(\beta)}, & \text{if } x \in [1,2), \\ (x-3)^2, & \text{if } x \in [2,3), \\ 0, & \text{otherwise.} \end{cases}$$



$\beta = 0$, $\beta = 4$, $\beta = 25$

Beyond GB

There are other interesting examples than GB-splines

$$\langle 1, x, \ldots, x^{p-4}, U_1(x), V_1(x), U_2(x), V_2(x) \rangle$$

under suitable assumptions on U_1, V_1, U_2, V_2.
Mazure, Xu et al

conclusion

- Can define piecewise Tchebycheffian splines
- "There is always room in the Zoo"

Fine