Polynomial reproduction of Hermite subdivision schemes

joint work with Costanza Conti

Svenja Hüning

Graz University of Technology
Institute of Geometry

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DOCTORAL PROGRAM
DISCRETE MATHEMATICS
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Outline

- Scalar subdivision scheme $\rightarrow$ algebraic conditions for polynomial reproduction

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- Generalisation

initial data $\boldsymbol{x}^0 = \{x_j^0,\ j \in \mathbb{Z}\}$

mask $\boldsymbol{a} = \{a_j \in \mathbb{R},\ j \in \mathbb{Z}\}$
with finitely many $a_j \neq 0$

Scalar subdivision scheme

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subdivision operator $S_{\boldsymbol{a}}$ with $\boldsymbol{x}^{\ell+1} = S_{\boldsymbol{a}} \boldsymbol{x}^{\ell}$, $\ell \in \mathbb{N}$

Parametrization

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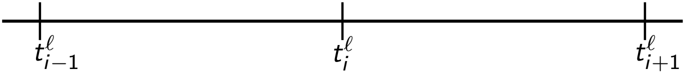

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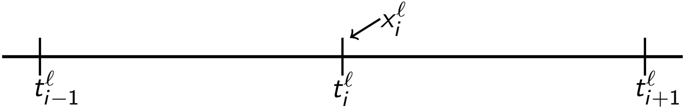

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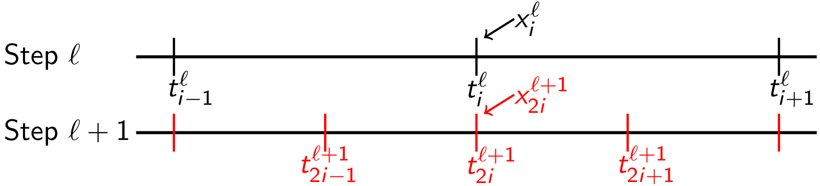

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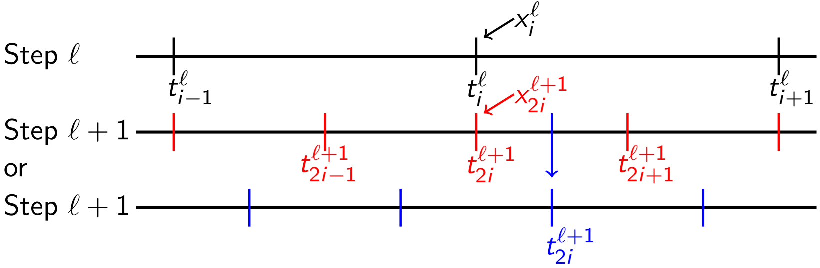

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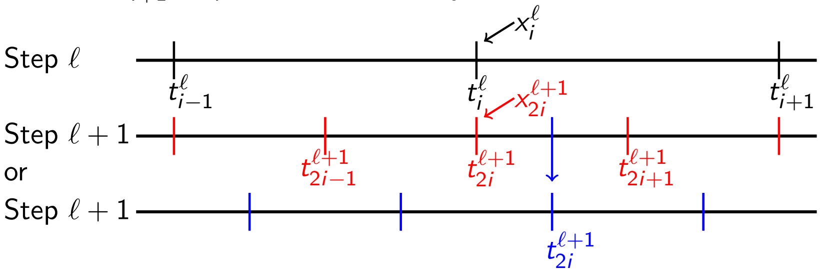

Relative shifts: $\tau_\ell = (t_0^\ell - t_0^{\ell+1})2^{\ell+1}$, $\quad \ell \in \mathbb{N}$

Primal: $\tau = 0$ Dual: $\tau = -\frac{1}{2}$

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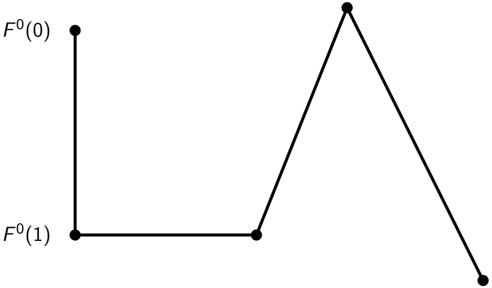

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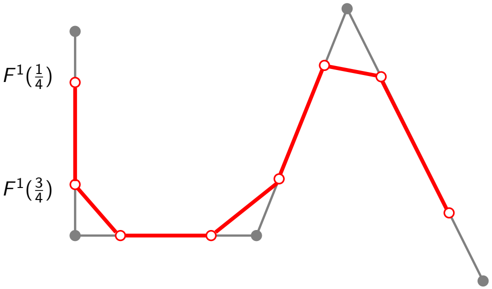

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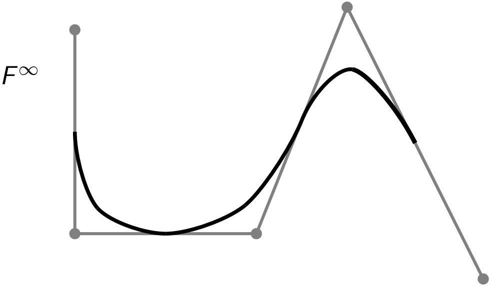

Step-wise polynomial reproduction: $S_{\boldsymbol{a}}$ reproduces polynomials of degree d if for any polynomial p of degree $\leqslant d$ and any initial sequence

$$x_i^0 = p(i) \quad \Rightarrow \quad x_i^\ell = p\Big(\frac{i+\tau}{2^\ell}\Big)$$

for all $\ell \geqslant 1$ and all i.

Symbol

Symbol of subdivision scheme: $a(z) = \sum_{i \in \mathbb{Z}} a_i z^i, \quad z \in \mathbb{C} \setminus \{0\}$

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Goal: Algebraic conditions characterising polynomial reproduction.

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Results

Let $S_{\boldsymbol{a}}$ be a convergent, non-singular subdivision scheme.

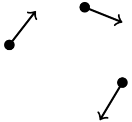
Theorem 1 [Conti, Hormann, 2011]: $S_{\boldsymbol{a}}$ reproduces linear polynomials if and only if $\tau_\ell = \frac{a'(1)}{2}$ for all ℓ and $a'(-1) = 0$.

Theorem 2 [Conti, Hormann, 2011]: $S_{\boldsymbol{a}}$ reproduces polynomials of degree d if and only if

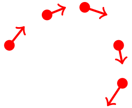
$$a^{(k)}(1) = 2 \prod_{j=0}^{k-1} (\tau - j) \quad \text{and} \quad a^{(k)}(-1) = 0$$

for $k = 0, \ldots, d$.

Hermite data: $\boldsymbol{f}_\ell = \{\mathbf{f}_\ell(j) \in \mathbb{R}^{2\times 1},\ j \in \mathbb{Z}\}$



$\rightarrow$ vector consisting of function value and derivative $\mathbf{f}_\ell(j) = \begin{bmatrix} g(j) \\ g'(j) \end{bmatrix}$



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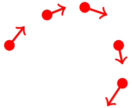
Refinement step:

$$\mathbf{D}^{\ell+1}\mathbf{f}_{\ell+1}(i) = \sum_{j\in\mathbb{Z}} A_{i-2j}\mathbf{D}^{\ell}\mathbf{f}_\ell(j) \quad \forall\ i \in \mathbb{Z}, \quad \ell \geqslant 0,$$

with mask $\boldsymbol{A} = \{A_j \in \mathbb{R}^{2\times 2},\ j \in \mathbb{Z}\}$ and $\mathbf{D} = diag(1,\ \frac{1}{2})$.

Hermite subdivision scheme



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Symbol: $\mathbf{A}(z) = \sum_{j\in\mathbb{Z}} A_j z^j \qquad \mathbf{A}'(z) := \sum_{j\in\mathbb{Z}} jA_j z^{j-1}$

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for all j, $\ell \geqslant 1$ and polynomials p of degree $\leqslant d$.

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We are interested in polynomial reproduction because...

- ... it implies spectral condition.
- ... it is related to approximation order.

Result

Theorem: $H_{\boldsymbol{A}}$ reproduces constants if and only if

$$\mathbf{A}(-1)\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\quad \mathbf{A}(1)\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}.$$

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$H_{\boldsymbol{A}}$ reproduces polynomials up to degree $d \geqslant 1$ if and only if it reproduces constants and for all $k = 1, \ldots, d$ we have

$$\mathbf{A}^{(k)}(-1)\begin{bmatrix}1\\0\end{bmatrix}+\sum_{\ell=1}^{k}\alpha_{k,\ell}\cdot\mathbf{A}^{(k-\ell)}(-1)\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},$$

$$\mathbf{A}^{(k)}(1)\begin{bmatrix}1\\0\end{bmatrix}+\sum_{\ell=1}^{k}\tilde{\alpha}_{k,\ell}\cdot\mathbf{A}^{(k-\ell)}(1)\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2q_{k,2\tau}(-\frac{\tau}{2})\\ \tilde{q}_{k,2\tau}(-\frac{\tau}{2})\end{bmatrix}.$$

Example

Hermite scheme introduced by J.-L. Merrien

$$A_{-1} = \begin{bmatrix} \frac{1}{2} & \lambda \\ \frac{1}{2}(1-\mu) & \frac{\mu}{4} \end{bmatrix}, \qquad A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad A_1 = \begin{bmatrix} \frac{1}{2} & -\lambda \\ \frac{1}{2}(\mu-1) & \frac{\mu}{4} \end{bmatrix}.$$

- Reproduction of cubic polynomials if and only if $\lambda = -\frac{1}{8}$ and $\mu = -\frac{1}{2}$.

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Goal: Construction of a 'nice' Hermite scheme which reproduces polynomials of higher degree by using algebraic conditions.

Start with

$$\bar{A}_{-3} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \quad \bar{A}_{-1} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad \bar{A}_0 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix},$$

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- Reproduces polynomials up to degree 5 if $b_3 = 0$, $b_2 = \frac{1}{384}$, $a_1 = \frac{1}{2}$, $a_2 = -17/128 \approx -0.13$, $a_3 = 135/176 \approx 0.77$ and $a_4 = -189/1408 \approx -0.13$.

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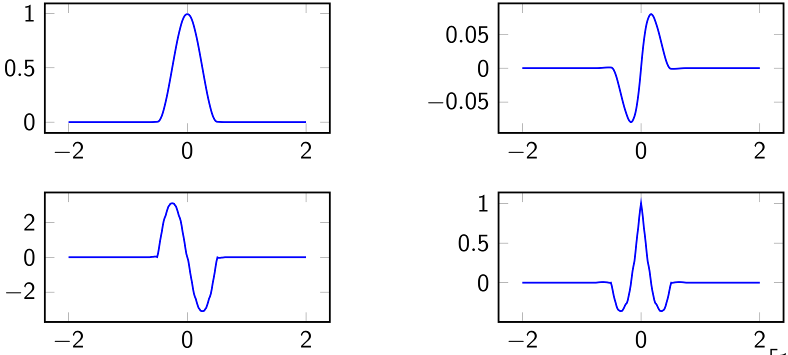

Figure: Column by column: Limit function and derivative for initial data $\begin{bmatrix}1\\0\end{bmatrix}$ $\left(\text{resp. } \begin{bmatrix}0\\1\end{bmatrix}\right)$ at 0 and $\begin{bmatrix}0\\0\end{bmatrix}$ else.

Generalisation

Theorem: $H_{\boldsymbol{A}}$ of order $m = 2, 3$ reproduces constants if and only if

$$\mathbf{A}(-1)\mathbf{e}_{1,m} = \mathbf{0}_m, \quad \mathbf{A}(1)\mathbf{e}_{1,m} = 2\mathbf{e}_{1,m}.$$

$H_{\boldsymbol{A}}$ reproduces polynomials up to degree $d \geqslant 1$ if and only if it reproduces constants and for all $k = 1, \dots, d$ we have

$$\mathbf{A}^{(k)}(-1)\mathbf{e}_{1,m} + \sum_{s=2}^{m} \Big(\sum_{\ell=s-1}^{k} \alpha_{k,\ell}^{m} \cdot \mathbf{A}^{(k-\ell)}(-1)\mathbf{e}_{s,m} \Big) = \mathbf{0}_m,$$

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Conjecture: $H_{\boldsymbol{A}}$ of order $m = 2, 3, \ldots$ reproduces constants if and only if

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Thank you for your attention!

References

- C. Conti and K. Hormann. *Polynomial reproduction for univariate subdivision schemes of any arity*, Journal of Approximation Theory, 163(4), 413-437, 2011.
- C. Conti, L. Romani and J. Yoon. *Approximation order and approximate sum rules in subdivision*, Journal of Approximation Theory, 380-401, 2016.
- J.-L. Merrien. *A family of Hermite interpolants by bisection algorithms*, Numerical Algorithms, 2, 187-200, 1992.
- J.-L. Merrien and T. Sauer. *A generalized Taylor factorization for Hermite subdivision schemes*, J. Comput. Appl. Math., 236(4), 565-574, 2011.
- J.-L. Merrien and T. Sauer. *Extended Hermite subdivision schemes*, J. Comput. Appl. Math., 317, 343-361, 2017.