Smoothness of non-linear Lane-Riesenfeld algorithms refining numbers

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Outline of the Talk

1. The linear Lane-Riesenfeld (LR) algorithms

2. Non-linear averages of numbers

3. LR algorithms with non-linear averages

4. Conjecture about smoothness equivalence and a counter example

5. Sufficient conditions for smoothness equivalence

6. A conjecture based on simulations

7. Method of analysis

The linear Lane-Riesenfeld algorithms (Lane, Riesenfeld, 1980)

The data: $m,\ \mathbf{f^0} = \{f_i^0 \in \mathbb{R} :\ i \in \mathbb{Z}\}$

for level $k = 1, 2, \ldots,$ do:

for $i \in \mathbb{Z}$ do: $f_{2i}^{k,0} \leftarrow f_i^{k-1}, \quad f_{2i+1}^{k,0} \leftarrow f_i^{k-1}$
(defining $\mathbf{f}^{k,0}$ by an elementary refinement of $\mathbf{f}^{k-1}$)

for round $\ell = 1, 2, \ldots, m$ do:

for $i \in \mathbb{Z}$ do: $f_i^{k,\ell} \leftarrow \frac{f_i^{k,\ell-1} + f_{i+1}^{k,\ell-1}}{2}$
(defining $\mathbf{f}^{k,\ell}$ by averaging)

for $i \in \mathbb{Z}$ do: $f_i^k \leftarrow f_i^{k,m}$
(defining $\mathbf{f}^k$—the data at refinement level k)

On the linear LR algorithm of order m (LR_m)

1. The algorithm converges to a C^{m-1} limit function

2. The limit is the spline function

$$\sum_{i\in\mathbb{Z}} f_i^0 B_m(t-i)$$

B_m is a B-spline of degree m with integer knots and support $(0, m+1)$

Non-linear LR algorithms are obtained by replacing the arithmetic average in the linear LR algorithm by Non-lineaer symmetric binary averages

Non-linear symmetric binary averages

$A(u,v),\ u,v \in \mathbb{R}_+$ is a symmetric binary average if it satisfies

1. $A(u,v) = A(v,u)$ and $A(u,u) = u$

2. $\min\{u,v\} < A(u,v) < \max\{u,v\}$ whenever $u \neq v$

Examples:

1. geometric average $A(u,v) = (uv)^{\frac{1}{2}}$

2. harmonic average $A(u,v) = \frac{1}{\frac{1}{2}\left(\frac{1}{u}+\frac{1}{v}\right)}$

3. p-average $A_p(u,v) = \left(\frac{u^p+v^p}{2}\right)^{\frac{1}{p}},\ p \neq 0,1$

The only linear symmetric binary average is the arithmetic mean $\frac{u+v}{2}$

For a smooth (C^M) average $A(u,v)$ Property 1. of $A(u,v)$ implies

$$A(u,v) = \frac{u+v}{2} + (u-v)^2 \sum_{i=0}^{M-2} c_i (u-v)^i$$

If the non-linear LR algorithm of order m (NLR_m) is converging, then for k large the non-linear averages operate on $\mathbf{f^k}$ with $|f_{i+1}^k - f_i^k|$ small, so that $A(f_i^k, f_{i+1}^k)$ is close to the arithmetic mean of f_i^k and f_{i+1}^k. Therefore the conjector that the smoothness of the limits generated by NLR_m equals that of the limits of the linear LR_m (the smoothness equivalence property) is reasonable

Previous results strengthening the equivalence conjecture

• Goldman, Schaefer, Vouga (2008) showed that if the arithmetic average in a linear LR algorithm is replaced everywhere by the SAME non-linear average, then the algorithm converges to a limit which is a function of a limit of the corresponding linear scheme.
Under certain mild conditions on the non-linear average used, properties of the limit can be deduced from those of the limit of the linear scheme, in particular the smoothness

In that work the smoothness equivalence, when DIFFERENT non-linear averages replace the arithmetic means, is conjectured, based on simulations

• Dyn, Goldman (2011) considered a more general class of NLR_m algorithms, obtained by replacing the arithmetic means in a LR_m algorithm by DIFFERENT non-linear averages, all C^2 functions with a uniform bound on their second pure partial derivatives

It is proved there that converging NLR_m schemes with $m \geq 2$, and with ANY CHOICE of such averages generate C^1 limits

Conditions on the initial data guaranteeing convergence are also derived

A counter example to the smoothness equivalence conjecture in its most generality

Duchamp, Xie, Yu (2016): The algorithm NLR_3, where the nonlinear average

$$A(u,v) = \frac{u+v}{2} + c(u-v)^p, \quad p = 2$$

replaces the arirhmetic means only in every third round at odd locations (definition of $f^{k,3}_{2i+1}$). The limits of this scheme are only C^1, as concluded from their general theory and asserted by simulations, while the limits of LR_3 are C^2

It is also shown there that if $p = 4$ in the definition of the non-linear average, then the limit is C^2

The theory of Duchamp, Xie, Yu (2016) is based on their technique termed Differential Proximity Conditions. With this technique they proved (2017) that the smoothness of NLR_m is equal to the smoothness of LR_m, when in any round the same average is used, but in different rounds different averages can be used

Their technique applies to uniform non-linear schemes, while NLR_m with different averages in the same round is a non-uniform scheme (different refinements at different locations) Yet, the example with $p = 4$ indicates that it is possible to have C^2 limits of NLR_3 with DIFFERENT averages in the same round

The next slides are concerned with sufficient conditions on the averages used in the same round for the NLR_m to have C^ℓ limits, with $\ell \leq m-1$.

Smoothness results

Recently Dyn, Levin, Goldman proved that a converging NLR_m can have C^ℓ limits (smoothness ℓ) with $\ell \leq m-1$, if consecutive averages used in the same round vary smoothly with the location

More precisely

Denote by $A_i^{k,r}(u,v)$ the average applied in the NLR_m algorithm at location i in level k and round r.

Theorem 1: Consider a converging NLR_m algorithm. If

$$A_{i+1}^{k,r}(u,v) - A_i^{k,r}(u,v) = (u-v)^{2\ell} C_i(u,v)$$

with $C_i(u,v)$ a smooth function, and $0 \leq \ell \leq m-1$, then the limits generated by this NLR_m are C^ℓ.

Theorem 2: For $k \in \mathbb{Z}_0$ and $r \in \{1, ..., m\}$, let $A^{k,r}(u, v, p)$ be a family of averages depending smoothly on u, v, p. Consider the NLR_m algorithm with $A_i^{k,r}(u, v) = A^{k,r}(u, v, i2^{-k})$. If the algorithm converges then its smoothness (the smoothness of its limits) equals that of the linear LR_m algorithm.

Two families of smooth averages depending on a parameter

(i) The p-averages

$$A(u, v, p) = \left(\frac{u^p + v^p}{2}\right)^{\frac{1}{p}}$$

(ii) The q averages

$$A(u, v, p) = \frac{u^p + v^p}{u^{p-1} + v^{p-1}}$$

Conclusion from simulations

The conditions in Theorem 1 are only sufficient

It seems that to get smoothness $\ell > 1$ of a converging NLR_m algorithm with $m \geq \ell + 1$, any average can be used in the first two rounds, while in round $r \in [3, \ell]$ the averages used should satisfy

$$A_{i+1}^{k,r}(u,v) - A_i^{k,r}(u,v) = (u-v)^{2(r-1)} C_i^{k,r}(u,v)$$

with $C_i^{k,r}(u,v)$ a smooth function. Moreover in the rest of the rounds $r \in [\ell + 1, m]$ the averages used should satisfy

$$A_{i+1}^{k,r}(u,v) - A_i^{k,r}(u,v) = (u-v)^{2\ell} C_i^{k,r}(u,v)$$

The Analysis method

(i) For given initial data $\mathbf{f^0}$ the values generated by a NLR_m, is also generated by a non-uniform linear scheme with masks coefficients depending on $\mathbf{f^0}$ and on the non-linear averages used.

(ii) The smoothness of the limit generated by the NLR_m from $\mathbf{f^0}$ is equal to the smoothness of the above non-uniform linear scheme

(iii) This smoothness is derived with the analysis tools developed in Dyn, Levin, Yoon (2014) for non-uniform linear subdivision schemes