Filter banks for arbitrary dilations

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February 22, 2018

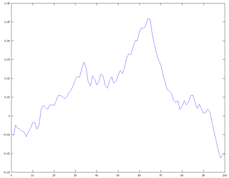
Definition

$\ell(\mathbb{Z}^s)$ denotes the set of all signals of the form

$$c = \left(c(\alpha) \mid \alpha \in \mathbb{Z}^s\right) = \left(c(\alpha_1, \ldots, \alpha_s) \mid \alpha_1, \ldots, \alpha_s \in \mathbb{Z}\right),$$

and $c(\alpha) \in \mathbb{R}$.

- $s = 1$ audio signal



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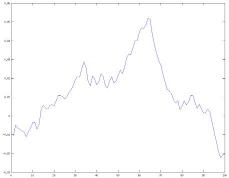
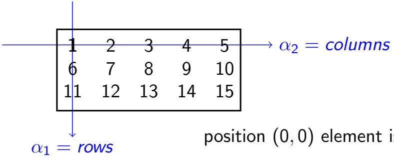
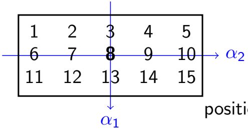


- $s = 2$ example an image



position $(0, 0)$ element is $(1, 1)$



position $(0, 0)$ element is $(2, 3)$

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- any LTI-filter can be written as a *convolution*

$$Fc = f * c = \sum_{\alpha \in \mathbb{Z}^s} f(\cdot - \alpha) c(\alpha)$$

with the impulse response f

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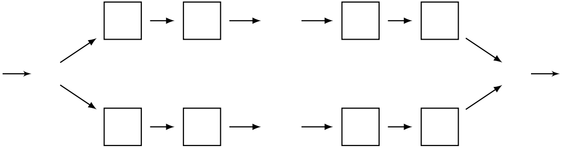
- anisotropic matrices for edge detection

- filter bank 1-dimensional: $M = 2$



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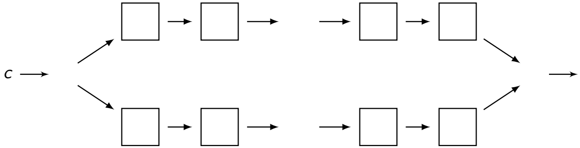

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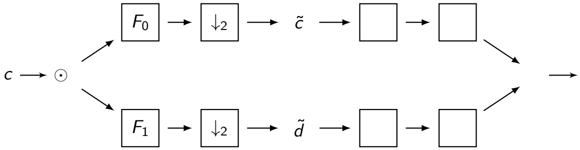

analysis

- F_0 lowpass analysis filter and F_1 highpass analysis filter

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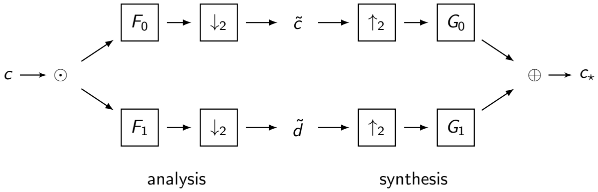

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- G_0 lowpass synthesis filter and G_1 highpass synthesis filter

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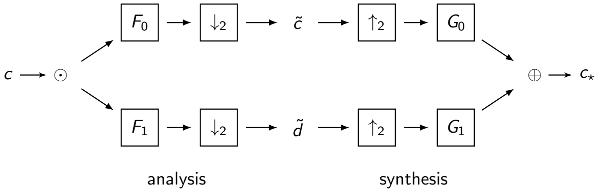

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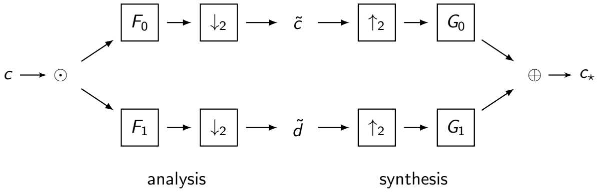

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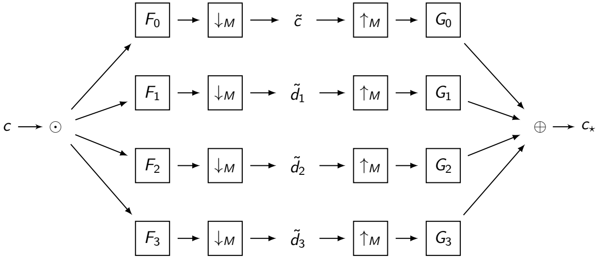

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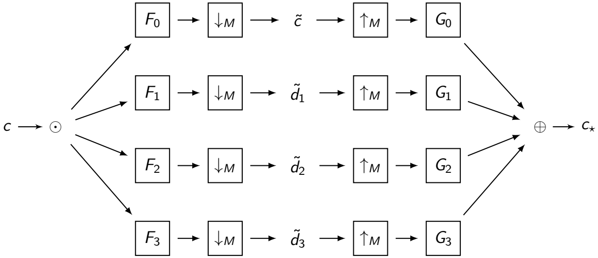

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Theorem

$M \in \mathbb{Z}^{s \times s}$ can be decomposed as

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Let $M \in \mathbb{Z}^{s \times s}$ *and* $i \in \{1, \ldots, s\}$*. Define* f_i *to be the greatest common divisor of all the determinants of* $i \times i$ *minors of* M*. These* f_i *are called* determinantal divisors *of* M*.*
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For the decomposition $M = PDQ$ we can find the elements d_i as $d_i = \frac{f_{i+1}}{f_i}$ where f_i is defined as before. Then we call D the Smith normal form *of M.*

- $D_1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$

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- Smith normal form is unique

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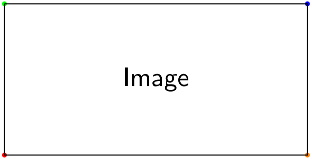
iv) perform a matrix transformation Q^{-1}, that means $D^{-1}P^{-1}\mathbb{Z}^s \to Q^{-1}D^{-1}P^{-1}\mathbb{Z}^s$

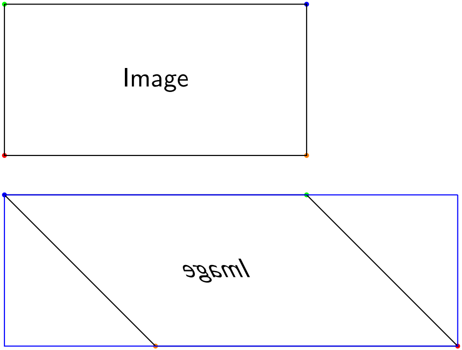
i) compute Smith decomposition

$$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

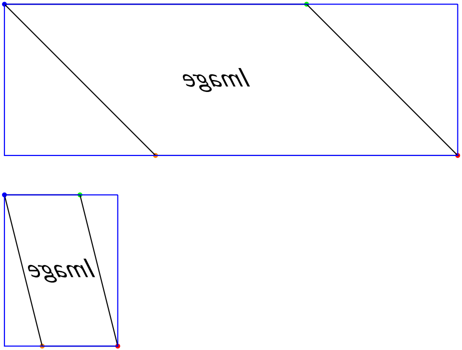


ii) perform a matrix transformation

$$P^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

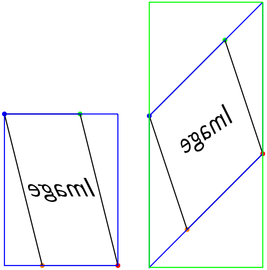


iii) $\downarrow_D,\ D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$



iv) perform a matrix transformation

$$Q^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$



Note that P and Q are not unique:

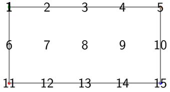
$$\begin{aligned} M = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} \end{aligned}$$

Does it make a difference for the result?

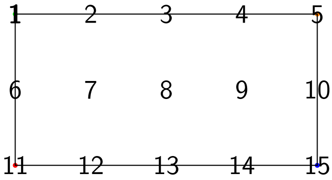
Our image

$$I = \begin{pmatrix} \mathbf{1} & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{pmatrix}$$

the number in boldface is the $(0, 0)$ element.



$$M_Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad \det(M_Q) = 2$$



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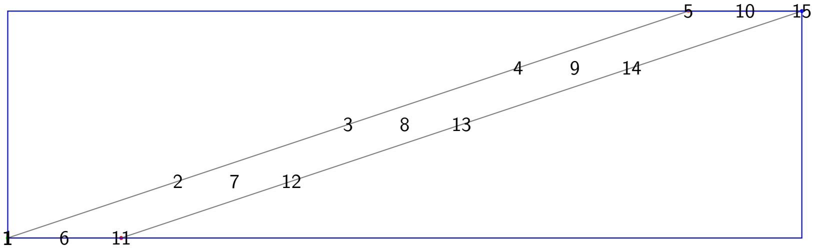

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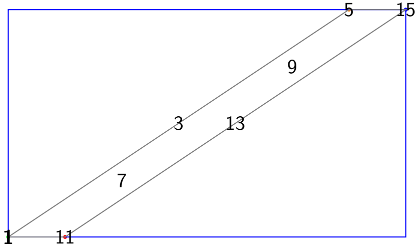

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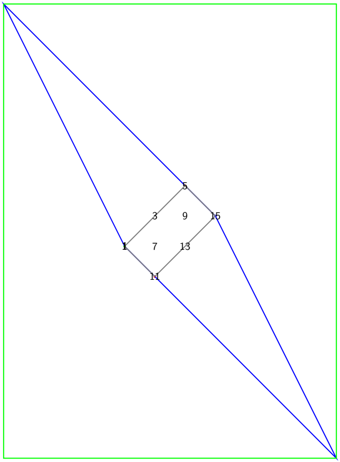

$$
\downarrow_{M_Q} I = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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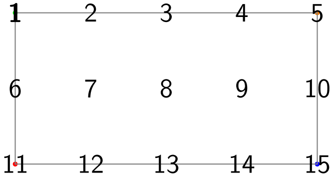
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 9 & 15 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & 7 & 13 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 11 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

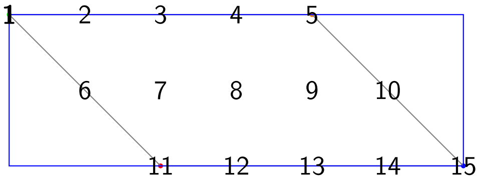
and with the decomposition

$$M_Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



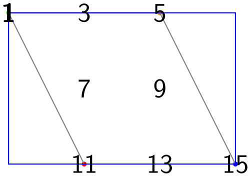
and with the decomposition

$$M_Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



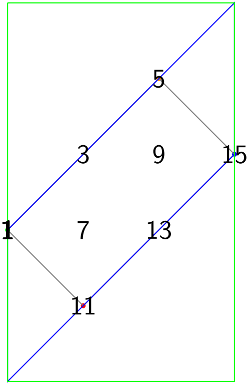
and with the decomposition

$$M_Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



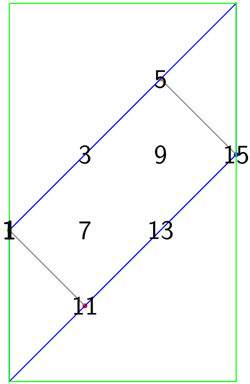
and with the decomposition

$$M_Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

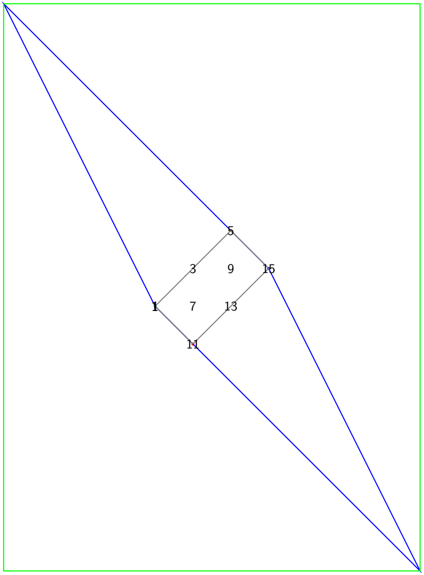


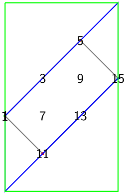
and with the decomposition

$$M_Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



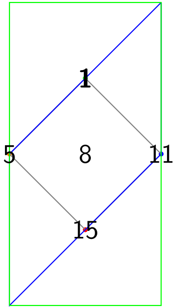
$$\downarrow_{M_Q} I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 3 & 9 & 15 \\ \mathbf{1} & 7 & 13 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

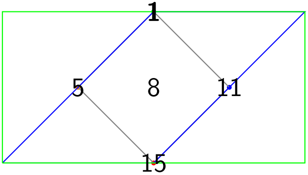

5
3
9
15
1
7
13
11


5
3
9
15
1
7
13
11

$$M_H = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (2)$$

$$= \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{matlab} \quad (3)$$

(2) 

and (3) gives 

$$M_C = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \text{ (matlab)},$$

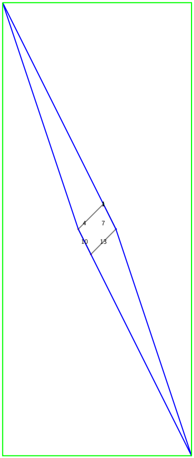
$$\det(M_C) = -3$$

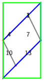
$$\downarrow_{M_{\text{matlab}C}} I = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 10 & 13 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and we get with another decomposition

$$M_C = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\downarrow_{M_C} I = \begin{pmatrix} 0 & \mathbf{1} \\ 4 & 7 \\ 10 & 13 \\ 0 & 0 \end{pmatrix}$$


1
4
7
10
13


1
4
7
10
13

- diagonal matrix can be decomposed into a smith normal form

$$M_D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix}$$

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$$M_D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix}$$

- without Smith decomposition

$$\downarrow_{M_D} I = \begin{pmatrix} \mathbf{1} & 4 \\ 11 & 14 \end{pmatrix}$$

- with Smith decomposition

$$\downarrow_{M_D} I = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&\mathbf{1}&4&0&0&0&0&0\\ 0&0&0&0&11&14&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

With $M = \begin{pmatrix} 2 & -4 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we get

$$\downarrow_M I = \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \\ 0 & 0 \\ 0 & 4 \\ 0 & 13 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and we get with

$$M = \begin{pmatrix} 2 & -4 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & -5 \\ -1 & 6 \end{pmatrix} \quad \text{matlab}$$

(38)

$$\downarrow_{M_{\text{withMat}}} I = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 13 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with size$(\downarrow_{M_{\text{withMat}}} I) = 50 \times 10$ (24 zero rows above the $(0,0)$-element and 22 zero rows at the end) and we get with

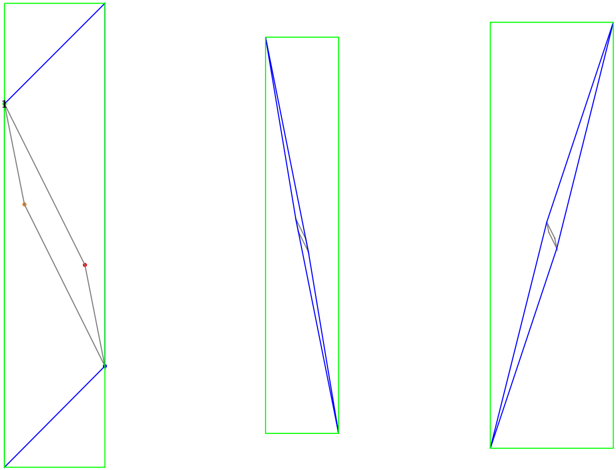
$$M = \begin{pmatrix} 2 & -4 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$$

$$\downarrow_M I = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 13 & 0 & 0 & 0 \end{pmatrix}$$

BUT size($\downarrow_M I$) $= 70 \times 21$, this is too big for the slide

$$M = \begin{pmatrix} 2 & -4 \\ -1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & -5 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$$



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Thank you.