THE commutative diagram of signal processing from a Clifford perspective

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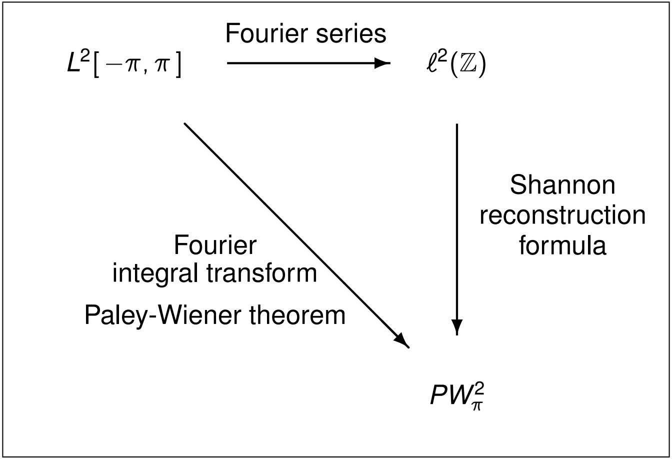
THE commutative diagram of signal processing



Outline

1. The commutative diagram of signal processing

Paley–Wiener Theorem

Let $f : \mathbb{R} \to \mathbb{C}$. Then the following are equivalent:

(i) $f \in L^2(\mathbb{R})$ and f can be extended to an entire function of exponential type $\pi > 0$; i.e.,

$$|f(z)| \leq \text{const.}\, e^{\pi |z|} \quad \text{for all } z \in \mathbb{C}.$$

(ii) There exists a function $F \in L^2([-\pi, \pi])$ such that

$$f(z) = \int_{-\pi}^{\pi} F(t) e^{izt}\, dt \quad \text{for all } z \in \mathbb{R}.$$

1. The commutative diagram of signal processing

Paley–Wiener space

$f \in PW_\pi^2$, iff $f \in L^2(\mathbb{R})$ and there exists a function $F \in L^2([-\pi, \pi])$ such that

$$f(z) = \int_{-\pi}^{\pi} F(t) e^{izt}\, dt \quad \text{for all } z \in \mathbb{R}. \tag{1.1}$$

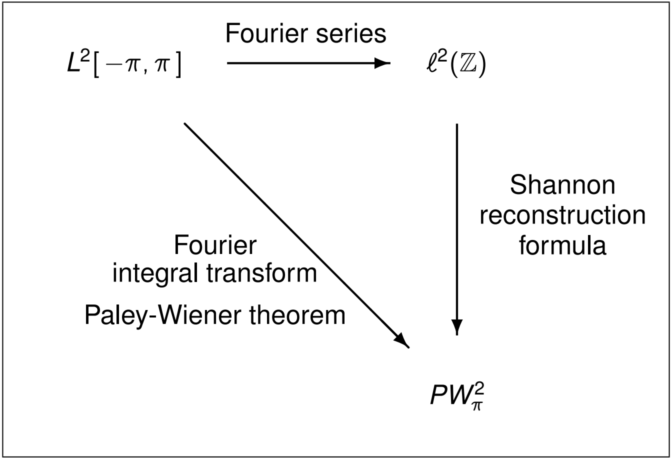
I.E., $f : \mathbb{R} \to \mathbb{C}$ satisfies (ii).

- In signal processing speech:

  Band-limited signals of finite energy.

- **Bernstein space** $B_2([-i\pi, i\pi])$ consists entire functions of exponential type at most π which are square-integrable on the real axis, i.e., (i).
- Paley-Wiener theorem: $\boxed{PW_\pi^2 \cong B_2([-i\pi, i\pi])}$

1. The commutative diagram of signal processing



1. The commutative diagram of signal processing

The other two transforms:

Fourier transform

$$\begin{aligned} \widehat{\ } \quad &: \quad L^2([-\pi,\pi]) \to l^2(\mathbb{Z}), \\ \widehat{F}(n) \quad &= \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)e^{-int}\,dt \quad \forall n \in \mathbb{Z}. \end{aligned}$$

Sampling theorem – Shannon reconstruction formula

Every $f \in PW_\pi^2(\mathbb{R})$ can be reconstructed from its samples $\{f(k), k \in \mathbb{Z}\}$ via the cardinal series

$$f(t) = \sum_{k\in\mathbb{Z}} f(k)\frac{\sin \pi(t-k)}{\pi(t-k)} \quad \forall t \in \mathbb{Z}.$$

$$\widehat{F}(n) = f(n) \quad \text{for all } n \in \mathbb{Z}.$$

2. Extension to irregular sampling

The sampling theorem is related to Lagrange interpolation:

$$\begin{aligned} f(t) &= \sum_{k\in\mathbb{Z}} f(k)\,\mathrm{sinc}\,(t-k) \quad = \quad \sum_{k\in\mathbb{Z}} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)} \\ &= \sin(t)\sum_{k\in\mathbb{Z}} f(k)\frac{1}{(t-k)\pi\cos(\pi k)} \end{aligned}$$

2. Extension to irregular sampling

Replace sin by a **sine-type function** S.

S is an an entire function of exponential type satisfying

$$0 < c < |S(z)|e^{-\pi|\mathrm{Im}\, z|} < C < \infty \quad (\text{when } |\mathrm{Im}\, z| > K)$$

for some constants $c, C, K > 0$.
Λ set of zeros of S.

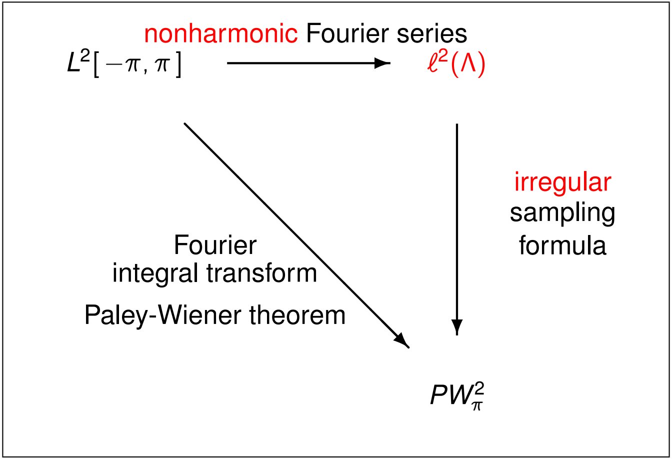
When Λ is a separated set, then

$$f(t) = S(t) \sum_{\lambda \in \Lambda} f(\lambda) \frac{1}{(t-\lambda)S'(\lambda)}.$$

Convergence as for the sampling theorem for $f \in PW_\pi^2$.

[Levin & Ljubarskii, 1975]

2. Extension to irregular sampling



3. Case $1 < p < \infty$

How about $L^p(\mathbb{R})$ spaces?

The Hausdorff-Young / Hardy / Titchmarsh / Babenko-Beckner Theorem

$$\|\mathcal{F}(f)\|_q \leq \text{const.}\|f\|_p$$

holds for $1 < p \leq 2$ and fails for $p > 2$. ($\frac{1}{p} + \frac{1}{q} = 1$).

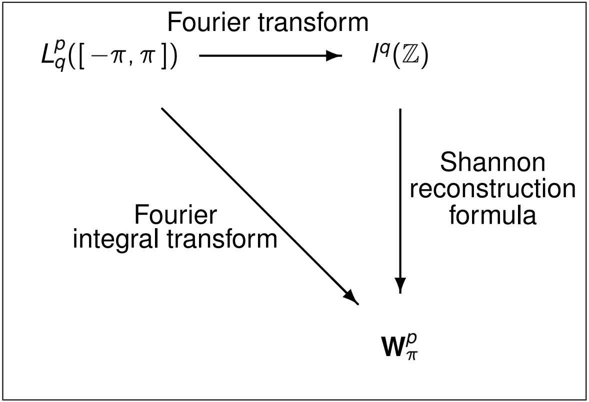
Theorem (Maergoiz 2006)

Let $1 < p < 2$ and $A > 0$.
Denote

- *$L^q_p([-A, A])$ the class of functions $F \in L^q([-A, A])$ whose Fourier coefficients are in ℓ^p, $\frac{1}{p} + \frac{1}{q} = 1$.*
- *$\mathbf{W}^p_A$ denote the class of entire functions of exponential type at most $A > 0$ that are L^p on the real axis.*

Then the spaces $\mathbf{W}^p_A$ and $L^q_p([-A, A])$ are isomorphic.
The isomorphism is the Fourier transform.

3. Case $1 < p < 2$


Fourier transform
$L_q^p([-\pi, \pi])$
$l^q(\mathbb{Z})$
Fourier integral transform
Shannon reconstruction formula
$\mathbf{W}_\pi^p$

4. Multidimensional Extension

How to extend to higher dimensions?

- Tensoring – no interaction between coordinates
- Clifford algebras – Cauchy-Riemann-type interaction

Ingredients

- Multidimensional structure
- Extension of holomorphy
- Extension of the Fourier transform
- Extension of the exponential growth condition

4. Multidimensional Extension

Clifford algebra

- Start with $e_1, \ldots, e_n$ orthonormal basis of $\mathbb{R}^n$.
- Just-do-it multiplication:

$$e_j e_k = -e_k e_j \quad \text{and} \quad e_j^2 = -1 \quad (\text{and} \quad \overline{e_j} = -e_j).$$

yields a Clifford algebra $\mathbb{R}_n$ or $\mathbb{C}_n$ of dimension 2^n.

- Basis elements of the Clifford algebra:

$$e_{a_1,\ldots,a_d} := e_{a_1} \cdots e_{a_d}$$

with $\{a_1, \ldots, a_d\} \subset \mathcal{P}(\{1, \ldots, n\})$, $a_1 < a_2 < \ldots < a_d$.

Monogenicity extending holomorphy

- $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}_n$ is called left monogenic, if for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$:

$$\left(\sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial t} \right) u(x+t) = 0.$$

- $f : \mathbb{R}^n \to \mathbb{C}_n$ has a left monogenic extension u, if $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}_n$ is monogenic and $u|_{\mathbb{R}_n} = f$.
- Monogenic extension via the Fourier transform:

$$f(x+t) := u(x+t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} E(x+t,\xi)\, \mathcal{F}(f)(\xi)\; d\xi,$$

with the monogenic extension E of the Fourier kernel $\exp(i\langle x, \xi \rangle)$:

$$E(x+t,\xi) = e^{i\langle x,\xi\rangle} e^{-t|\xi|} \tfrac{1}{2} \left(1 + i \tfrac{\xi}{|\xi|}\right) + e^{i\langle x,\xi\rangle} e^{t|\xi|} \tfrac{1}{2} \left(1 - i \tfrac{\xi}{|\xi|}\right).$$

4. Multidimensional Extension

Growth conditions

- $f \in W_A^p$ iff
  - f possesses a left monogenic extension $u(x+t)$, and
  - for all $\varepsilon > 0$, for all $t \in \mathbb{R}$:

$$\int_{\mathbb{R}^n} |u(x+t)|^p \, dx \leq C(n,p,\varepsilon)\, e^{p|t|(A+\varepsilon)} \, \|f\|_p^p.$$

- $f \in \mathbf{W}_A^p$ iff
  - f possesses a left monogenic extension $u(x+t)$, and
  - for all $\varepsilon > 0$, for all $t \in \mathbb{R}$:

$$|u(x+t)| \leq C(n,\varepsilon)\, e^{|x+t|\,(A+\varepsilon)} \quad \text{for all } x+t \in \mathbb{R}^{n+1}.$$

    and
  - has bounded p-norm.

$$W_A^p = \mathbf{W}_A^p?$$

True in case $n = 1$.

Paley-Wiener space

Let PW_A^2 denote the class of all functions $f \in L^2(\mathbb{R}^n \to \mathbb{C}_n)$ with Clifford Fourier transform supported in the closed ball $B_A(0)$.

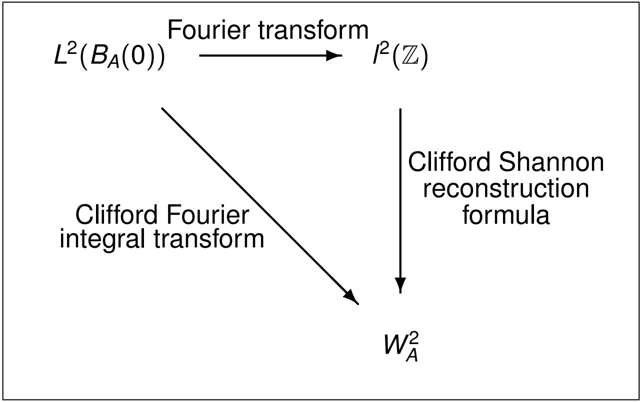
Theorem (Kou, Qian 2002; Franklin, Hogan, Larkin 2017)

$$PW_A^2 \cong W_A^2 \cong \mathbf{W}_A^2.$$

Extending the sinc **for sampling**

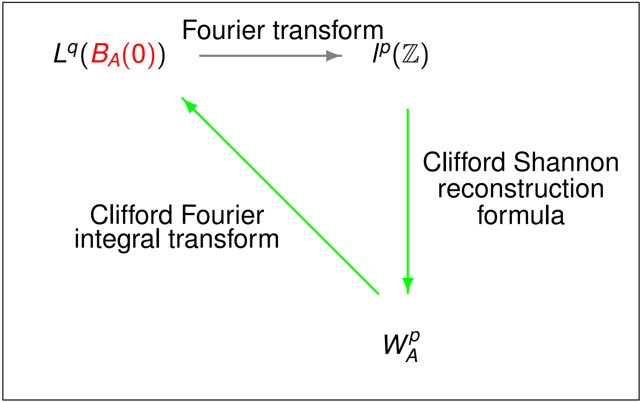
$$\operatorname{sinc}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} E(x, \xi)\, \chi_{[-\pi,\pi]^n}(\xi)\, d\xi, \quad x \in \mathbb{R}^n.$$

4. Multidimensional Extension



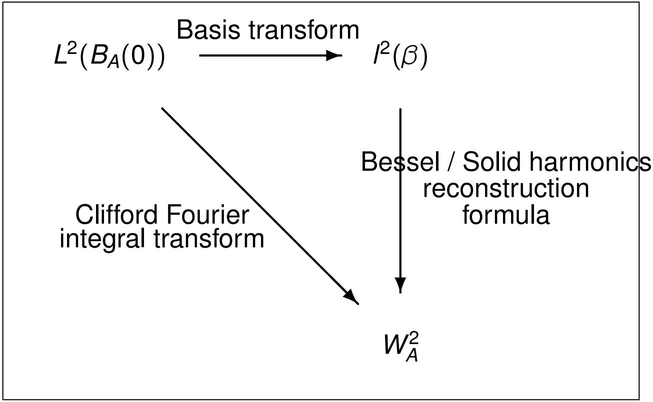
(Kou, Qian 2002; Franklin, Hogan, Larkin 2017)

4. Multidimensional Extension



(1<p<2, F./Hogan)

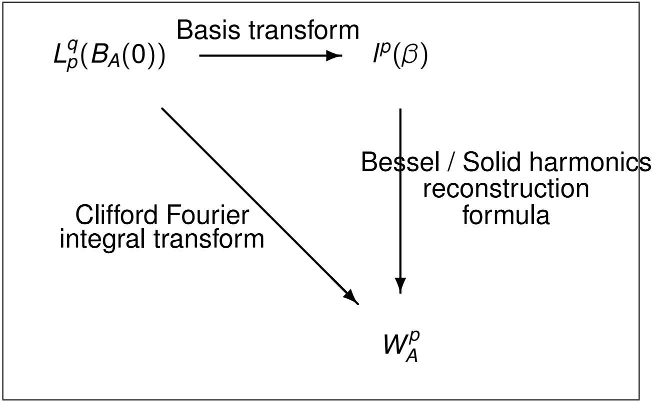
4. Multidimensional Extension



β set of scaled zeros of Bessel functions.
(Sampling: Kou, Qian, Sommen 2007; Basis: F. / Hogan)

4. Multidimensional Extension

Conjecture:



β set of scaled zeros of Bessel functions.

1<p<2.

5. Outlook

- Entire function of exponential type at most τ:

$$|f(z)| \leq \text{const.}\, e^{\tau|z|}.$$

- The smallest possible τ is the called the *exponential type* of f.
- **Indicator function** of f

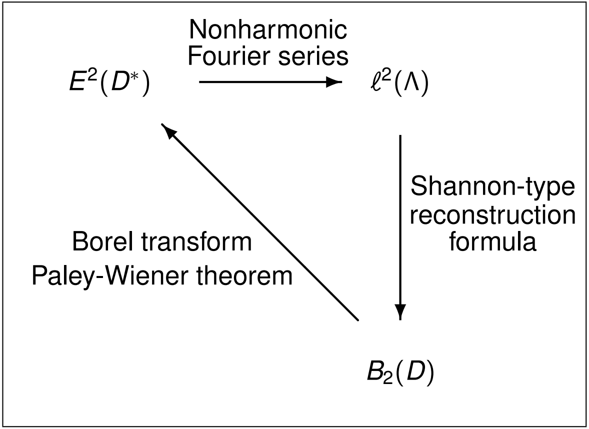
$$h_f(\theta) := \limsup_{r\to\infty} \frac{1}{r} \ln|f(re^{i\theta})|.$$

represents the growth exponent in direction θ .

- **Indicator function** is the supporting function of a convex set D

$$k_K(\theta) := \max_{z\in K} \operatorname{Re}(ze^{-i\theta}) = h_f(\theta).$$

5. Outlook



(Lewin, Ljubarski, Sedletskii, ..., F./Semmler for $1 < p < 2$)

How to extend growth with directions to the multivariate / monogenic Clifford setting? Clifford algebras are known to be very unforgiving when it comes to nonradial features.

Conclusions

- THE commutative diagram of signal processing has many aspects
- For $1 < p < 2$, ℓ^p-summability of Fourier coefficients required
- Multivariate extension via monogenic functions
- Case $1 < p < 2$ needs further characterization of the Fourier series preimage
- Open question: Directional growth for monogenic functions
- Another open question: Irregular sampling for monogenic functions