Saturation Rates for Filtered Back Projection

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3rd IM-Workshop on Applied Approximation, Signals and Images

Bernried, 21st February 2018

Universität Hamburg
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Outline

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   - Approximate Reconstruction Formula
2. Analysis of the Reconstruction Error in the L^2-Norm
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   - Convergence
3. L^2-Error Analysis for $\mathcal{C}^k$-Window Functions
   - Numerical Observations
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4. Asymptotic L^2-Error Analysis

Basic Reconstruction Problem

Problem formulation:

Let $\Omega \subset \mathbb{R}^2$ be bounded. Reconstruct a bivariate function $f \equiv f(x, y)$ with support $\mathrm{supp}(f) \subseteq \Omega$ from given Radon data

$$\{\mathcal{R}f(t, \theta) \mid t \in \mathbb{R},\, \theta \in [0, \pi)\},$$

where the **Radon transform** $\mathcal{R}f$ of $f \in \mathrm{L}^1(\mathbb{R}^2)$ is defined as

$$\mathcal{R}f(t, \theta) = \int_{\{x\cos(\theta) + y\sin(\theta) = t\}} f(x, y)\, \mathrm{d}x\, \mathrm{d}y \quad \text{for } (t, \theta) \in \mathbb{R} \times [0, \pi).$$

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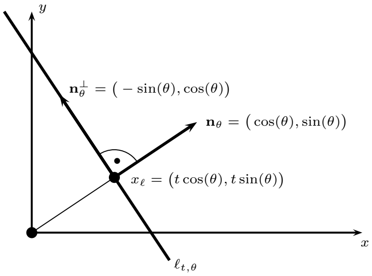

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Analytical solution:

The inversion of $\mathcal{R}$ involves the **back projection** $\mathcal{B}h$ of $h \in \mathrm{L}^1(\mathbb{R} \times [0,\pi))$,

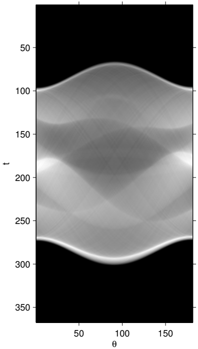
$$\mathcal{B}h(x,y) = \frac{1}{\pi}\int_0^\pi h(x\cos(\theta) + y\sin(\theta), \theta)\,\mathrm{d}\theta \qquad \text{for } (x,y) \in \mathbb{R}^2,$$

and is given, for $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathcal{C}(\mathbb{R}^2)$, by the **filtered back projection formula**

$$f(x,y) = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}[|S|\mathcal{F}(\mathcal{R}f)(S,\theta)]\big)(x,y) \quad \forall\,(x,y) \in \mathbb{R}^2.$$

The Shepp-Logan Phantom and its Radon Transform

The Shepp-Logan phantom f

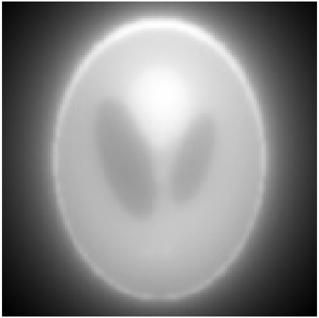


The Radon transform $\mathcal{R}f$

The Back Projection of the Shepp-Logan Phantom

Original Shepp–Logan Phantom

Unfiltered backprojection



The Filtered Back Projection Formula

OBS! The filtered back projection formula

$$f(x, y) = \frac{1}{2}\mathcal{B}\big(\mathcal{F}^{-1}[|S|\mathcal{F}(\mathcal{R}f)(S, \theta)]\big)(x, y)$$

is ugly and unstable.

OBS! The filtered back projection formula

$$f(x,y) = \frac{1}{2}\mathcal{B}\big(\mathcal{F}^{-1}[|S|\mathcal{F}(\mathcal{R}f)(S,\theta)]\big)(x,y)$$

is ugly and unstable.

Stabilization: Replace $|S|$ by a **low-pass filter**

$$A_L(S) = |S|W(S/L)$$

with *finite* **bandwidth** $L > 0$ and *even* **window** W of *compact* $\operatorname{supp}(W) \subseteq [-1,1]$.

Approximate Reconstruction

Stabilization: Replace the factor $|S|$ by a **low-pass filter** $A_L : \mathbb{R} \longrightarrow \mathbb{R}$,

$$A_L(S) = |S| W(S/L)$$

with *finite* **bandwidth** $L > 0$ and an *even* **window function** $W : \mathbb{R} \longrightarrow \mathbb{R}$ with *compact support* $\mathrm{supp}(W) \subseteq [-1, 1]$.

Approximate reconstruction formula:
We can express the resulting *approximate FBP reconstruction* f_L as

$$f_L = \frac{1}{2} \mathcal{B}\big(\mathcal{F}^{-1} A_L * \mathcal{R}f\big) = f * K_L,$$

where we rely for $f \in \mathrm{L}^1(\mathbb{R}^2)$ and $g \in \mathrm{L}^1(\mathbb{R} \times [0, \pi))$ on the standard relation

$$\mathcal{B}g * f = \mathcal{B}(g * \mathcal{R}f)$$

and define the **convolution kernel** $K_L : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as

$$K_L(x, y) = \frac{1}{2} \mathcal{B}\big(\mathcal{F}^{-1} A_L\big)(x, y) \quad \text{for } (x, y) \in \mathbb{R}^2.$$

Three Popular Examples for Low-Pass Filters

The Ram-Lak filter.

$$A_L(S) = |S| \cdot \sqcap_L(S) = \begin{cases} |S| & \text{iff } |S| \leq L; \\ 0 & \text{iff } |S| > L. \end{cases}$$

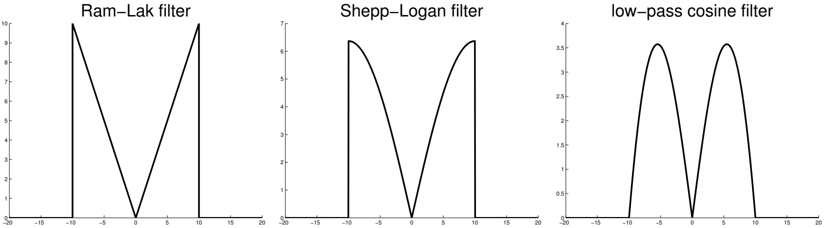
The Shepp-Logan filter.

$$A_L(S) = |S| \cdot \left(\frac{\sin(\pi S/(2L))}{\pi S/(2L)} \right) \cdot \sqcap_L(S) = \begin{cases} \frac{2L}{\pi} \cdot |\sin(\pi S/(2L))| & \text{iff } |S| \leq L; \\ 0 & \text{iff } |S| > L. \end{cases}$$

The low-pass cosine filter.

$$A_L(S) = |S| \cdot \cos(\pi S/(2L)) \cdot \sqcap_L(S) = \begin{cases} |S| \cdot \cos(\pi S/(2L)) & \text{iff } |S| \leq L; \\ 0 & \text{iff } |S| > L. \end{cases}$$

Three Popular Examples for Low-Pass Filters



Analysis of the Reconstruction Error

Aim

Analyse the FBP reconstruction error

$$e_L = f - f_L$$

depending on the window function W and the bandwidth $L > 0$.

Previous results:

- Pointwise and L^∞-error estimates by [Munshi et al., 1991, Munshi, 1992]
- L^p-error estimates in terms of L^p-moduli of continuity by [Madych, 1990]

Our approach:

- L^2-error estimates for target functions f from Sobolev spaces of fractional order, i.e.,

  $$f \in \mathrm{H}^\alpha(\mathbb{R}^2) = \left\{g \in \mathcal{S}'(\mathbb{R}^2) \mid \|g\|_\alpha < \infty\right\} \qquad \text{for } \alpha > 0,$$

  where

  $$\|g\|_\alpha^2 = \frac{1}{2\pi} \int_\mathbb{R} \int_\mathbb{R} \left(1 + x^2 + y^2\right)^\alpha |\mathcal{F}g(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y$$

- L^2-convergence rates ($L \longrightarrow \infty$) in terms of bandwidth L and smoothness α

A first L^2 Error Estimate

Theorem (A first L^2-error estimate; Beckmann & I., 2015)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for some $\alpha > 0$, $W \in \mathrm{L}^\infty(\mathbb{R})$ and $K_L \in \mathrm{L}^1(\mathbb{R}^2)$. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \|1 - W\|_{\infty,[-1,1]} \|f\|_{\mathrm{L}^2(\mathbb{R}^2)} + L^{-\alpha} \|f\|_\alpha.$$

Proof: For $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$ and by the Rayleigh–Plancherel theorem, we get

$$\begin{aligned}
\|e_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} &= \|f - f * K_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} = \frac{1}{2\pi} \|\mathcal{F}f - \mathcal{F}f \cdot \mathcal{F}K_L\|^2_{\mathrm{L}^2(\mathbb{R}^2;\mathbb{C})} \\
&= \frac{1}{2\pi} \|\mathcal{F}f - W_L \cdot \mathcal{F}f\|^2_{\mathrm{L}^2(\mathbb{R}^2;\mathbb{C})} = I_1 + I_2,
\end{aligned}$$

where we used $W_L(\|(x,y)\|_2) = \mathcal{F}K_L(x,y)$ for $K_L \in \mathrm{L}^1(\mathbb{R}^2)$, and where we let

$$\begin{aligned}
I_1 &:= \frac{1}{2\pi} \int_{\|(x,y)\|_2 \leq L} |(\mathcal{F}f - W_L \cdot \mathcal{F}f)(x,y)|^2 \, \mathrm{d}(x,y), \\
I_2 &:= \frac{1}{2\pi} \int_{\|(x,y)\|_2 > L} |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y).
\end{aligned}$$

A first L^2 Error Estimate

Proof (continued):

- For $W \in \mathrm{L}^\infty(\mathbb{R})$, the first integral

$$I_1 = \frac{1}{2\pi} \int_{\|(x,y)\|_2 \leq L} |(\mathcal{F}f - W_L \cdot \mathcal{F}f)(x,y)|^2 \,\mathrm{d}(x,y),$$

can be bounded above by

$$I_1 \leq \frac{1}{2\pi} \|1 - W_L\|^2_{\infty,[-L,L]} \|\mathcal{F}f\|^2_{\mathrm{L}^2(\mathbb{R}^2;\mathbb{C})} = \|1 - W\|^2_{\infty,[-1,1]} \|f\|^2_{\mathrm{L}^2(\mathbb{R}^2)}.$$

- For $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$, with $\alpha > 0$, the second integral

$$I_2 = \frac{1}{2\pi} \int_{\|(x,y)\|_2 > L} |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y)$$

can be bounded above by

$$I_2 \leq \frac{1}{2\pi} \int_{\|(x,y)\|_2 > L} \left(1 + x^2 + y^2\right)^\alpha L^{-2\alpha} |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y) \leq L^{-2\alpha} \|f\|^2_\alpha.$$

□

Refined L^2-Error Analysis

Theorem (Refined L^2-error estimate; Beckmann & I., 2016)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$, let $W \in \mathrm{L}^\infty(\mathbb{R})$ and $K_L \in \mathrm{L}^1(\mathbb{R}^2)$. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(\Phi_{\alpha,W}^{1/2}(L) + L^{-\alpha}\right)\|f\|_\alpha,$$

where

$$\Phi_{\alpha,W}(L) = \sup_{S\in[-1,1]} \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha} \quad \text{for } L > 0.$$

Proof: For $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$, with $\alpha > 0$, we bound integral I_2 as before and I_1 by

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_{\|(x,y)\|_2\leq L} \frac{|1-W_L(\|(x,y)\|_2)|^2}{(1+x^2+y^2)^\alpha} \left(1+x^2+y^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y) \\ &\leq \left(\sup_{S\in[-L,L]} \frac{(1-W_L(S))^2}{(1+S^2)^\alpha}\right) \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \left(1+x^2+y^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Refined L^2-Error Analysis

Proof (continued): Since

$$\sup_{S\in[-L,L]} \frac{(1-W_L(S))^2}{(1+S^2)^\alpha} = \sup_{S\in[-L,L]} \frac{(1-W(S/L))^2}{(1+S^2)^\alpha} = \sup_{S\in[-1,1]} \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha},$$

and with letting

$$\Phi_{\alpha,W}(L) = \sup_{S\in[-1,1]} \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha} \quad \text{for } L>0$$

we find

$$I_1 \leq \left(\sup_{S\in[-1,1]} \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha}\right) \|f\|_\alpha^2 = \Phi_{\alpha,W}(L)\,\|f\|_\alpha^2$$

and so

$$\|e_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(\sup_{S\in[-1,1]} \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha} + L^{-2\alpha}\right) \|f\|_\alpha^2 = \left(\Phi_{\alpha,W}(L) + L^{-2\alpha}\right) \|f\|_\alpha^2.$$

□

Refined L^2-Error Analysis

Theorem (Convergence of $\Phi_{\alpha,W}$; Beckmann & I., 2016)

Let W be continuous on $[-1,1]$ and satisfy $W(0)=1$. Then, for all $\alpha>0$,

$$\Phi_{\alpha,W}(L) = \max_{S\in[0,1]} \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha} \longrightarrow 0 \quad \textit{for} \quad L\longrightarrow\infty.$$

Proof (sketch): Let $S^*_{\alpha,W,L}\in[0,1]$ be the smallest maximizer in $[0,1]$of

$$\Phi_{\alpha,W,L}(S) := \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha} \quad \text{for } S\in[0,1].$$

Case 1: $S^*_{\alpha,W,L}$ uniformly bounded away from 0, $S^*_{\alpha,W,L}\geq c\equiv c_{\alpha,W}>0$. Then,

$$0\leq\Phi_{\alpha,W,L}(S^*_{\alpha,W,L}) = \frac{\left(1-W(S^*_{\alpha,W,L})\right)^2}{\left(1+L^2(S^*_{\alpha,W,L})^2\right)^\alpha} \leq \frac{\|1-W\|^2_{\infty,[-1,1]}}{(1+L^2c^2)^\alpha} \xrightarrow{L\to\infty} 0.$$

Case 2: $S^*_{\alpha,W,L}\longrightarrow 0$ for $L\longrightarrow\infty$. Then,

$$0\leq\Phi_{\alpha,W,L}(S^*_{\alpha,W,L}) = \frac{\left(1-W(S^*_{\alpha,W,L})\right)^2}{\left(1+L^2(S^*_{\alpha,W,L})^2\right)^\alpha} \leq \left(1-W(S^*_{\alpha,W,L})\right)^2 \xrightarrow{L\to\infty} 0. \qquad \square$$

Convergence of the FBP Reconstruction

Corollary (L^2 convergence of FBP reconstruction; Beckmann & I., '16)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ *for some* $\alpha > 0$, $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ *and* $W \in \mathcal{C}([-1,1])$ *with* $W(0) = 1$*. Then, the* L^2*-norm of the FBP reconstruction error* $e_L = f - f_L$ *satisfies*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} = o(1) \quad \text{for} \quad L \longrightarrow \infty.$$

□

Basic Assumption

Let $S^*_{\alpha,W,L}$ be uniformly bounded away from 0, i.e., there is $c_{\alpha,W} > 0$ satisfying

$$S^*_{\alpha,W,L} \geq c_{\alpha,W} \quad \forall\, L > 0.$$

OBS! Under the basic assumption we have

$$\Phi_{\alpha,W}(L) \leq c_{\alpha,W}^{-2\alpha} \, \|1 - W\|^2_{\infty,[-1,1]} \, L^{-2\alpha} = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

Rate of Convergence

Theorem (Rate of convergence; Beckmann & I., 2016)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ *for some* $\alpha > 0$*, let* $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ *and* $W \in \mathcal{C}([-1,1])$ *with* $W(0) = 1$*. Further, let the basic assumption be satisfied for a constant* $c_{\alpha,W}$*. Then, the* L^2*-norm of the FBP reconstruction error* $e_L = f - f_L$ *is bounded by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(c_{\alpha,W}^{-\alpha} \|1 - W\|_{\infty,[-1,1]} + 1\right) L^{-\alpha} \|f\|_\alpha.$$

Therefore,

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} = \mathcal{O}(L^{-\alpha}) \quad \text{for} \quad L \longrightarrow \infty,$$

i.e., the decay rate is determined by the smoothness α *of the target function* f*.* □

OBS! Let the window function $W \in \mathcal{C}([-1,1])$, $W \not\equiv \chi_{[-1,1]}$, satisfy

$$W(S) = 1 \quad \forall\, S \in (-\varepsilon, \varepsilon)$$

for some $0 < \varepsilon < 1$. Then, the basic assumption is fulfilled with $c_{\alpha,W} = \varepsilon$.

Numerical Observations

We investigated the behaviour of $S^*_{\alpha,W,L}$ *numerically* for standard low-pass filters:

- Shepp-Logan filter: $W(S) = \operatorname{sinc}\left(\frac{\pi S}{2}\right) \cdot \chi_{[-1,1]}(S)$,
- Cosine filter: $W(S) = \cos\left(\frac{\pi S}{2}\right) \cdot \chi_{[-1,1]}(S)$,
- Hamming filter (for $\beta \in \left[\frac{1}{2}, 1\right]$): $W(S) = (\beta + (1-\beta)\cos(\pi S)) \cdot \chi_{[-1,1]}(S)$,
- Gaussian filter (for $\beta > 1$): $W(S) = \exp\left(-(\pi S/\beta)^2\right) \cdot \chi_{[-1,1]}(S)$.

- For $\alpha < 2$, we found that the basic assumption

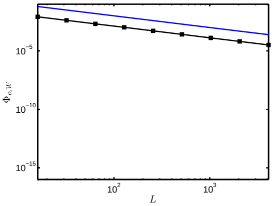
$$\exists\, c_{\alpha,W} > 0 \,\forall\, L > 0 : \; S^*_{\alpha,W,L} \geq c_{\alpha,W}$$

is fulfilled whereby

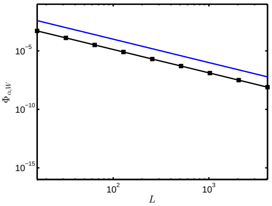
$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

- But for $\alpha \geq 2$, we observed $S^*_{\alpha,W,L} \longrightarrow 0$ for $L \longrightarrow \infty$.
  Moreover, in this case the convergence rate of $\Phi_{\alpha,W}$ stagnates at

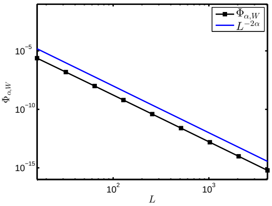
$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-4}) \quad \text{for} \quad L \longrightarrow \infty.$$



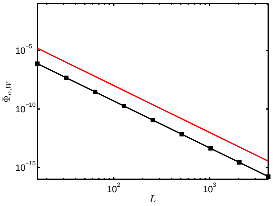
(a) $\alpha = 0.5$



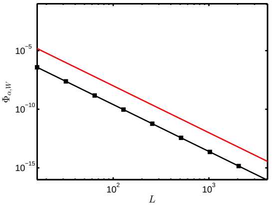
(b) $\alpha = 1$



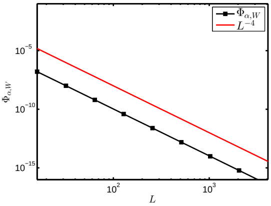
(c) $\alpha = 2$



(d) $\alpha = 2.5$



(e) $\alpha = 3$



(f) $\alpha = 4$

Fig.: Decay rate of $\Phi_{\alpha,W}$ for the Shepp-Logan filter

L^2-Error Analysis for $\mathcal{C}^k$-Windows

Theorem (Convergence rate of $\Phi_{\alpha,W}$ for $W \in \mathcal{C}^k$; Beckmann & I., '16)

For $k \geq 2$, let $W \in \mathcal{C}^k[-1,1]$ satisfy

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1$$

Then, we have

$$\Phi_{\alpha,W}(L) \leq \begin{cases} \frac{c_{\alpha,k}^2}{(k!)^2} \, \|W^{(k)}\|_{\infty,[-1,1]}^2 \, L^{-2k} & \text{for } \alpha > k \\ \frac{1}{(k!)^2} \, \|W^{(k)}\|_{\infty,[-1,1]}^2 \, L^{-2\alpha} & \text{for } \alpha \leq k, \end{cases}$$

where the constant

$$c_{\alpha,k} = \left(\frac{k}{\alpha-k}\right)^{k/2} \left(\frac{\alpha-k}{\alpha}\right)^{\alpha/2}$$

is strictly monotonically decreasing in $\alpha > k$. In particular, we have

$$\Phi_{\alpha,W}(L) = \mathcal{O}\left(L^{-2\min\{k,\alpha\}}\right) \quad \text{for} \quad L \longrightarrow \infty.$$

□

L^2-Error Analysis for $\mathcal{C}^k$-Windows

Corollary (L^2-error estimate for $\mathcal{C}^k$-windows; Beckmann & I., 2016)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for some $\alpha > 0$, let $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ and $W \in \mathcal{C}^k([-1,1])$, for $k \geq 2$, with

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded by

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \begin{cases} \left(\frac{c_{\alpha,k}}{k!} \|W^{(k)}\|_{\infty,[-1,1]} L^{-k} + L^{-\alpha}\right) \|f\|_\alpha & \text{for } \alpha > k \\ \left(\frac{1}{k!} \|W^{(k)}\|_{\infty,[-1,1]} L^{-\alpha} + L^{-\alpha}\right) \|f\|_\alpha & \text{for } \alpha \leq k. \end{cases}$$

In particular, we have

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(c\, \|W^{(k)}\|_{\infty,[-1,1]} L^{-\min\{k,\alpha\}} + L^{-\alpha}\right) \|f\|_\alpha = \mathcal{O}\left(L^{-\min\{k,\alpha\}}\right).$$

In conclusion, the rate of convergence saturates at $\mathcal{O}(L^{-k})$. □

Asymptotic L^2-Error Analysis

Theorem (Asymptotic L^2-error estimate; Beckmann & I., 2016)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, *for some* $\alpha > 0$, $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ *and let* $W \in \mathrm{L}^\infty(\mathbb{R})$ *be* k*-times differentiable at the origin,* $k \geq 2$, *with*

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Then, the L^2*-norm of the FBP reconstruction error* $e_L = f - f_L$ *is bounded by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \begin{cases} \left(\frac{\sqrt{2}}{k!}\, c_{\alpha,k}\, |W^{(k)}(0)|\, L^{-k} + L^{-\alpha}\right) \|f\|_\alpha + o(L^{-k}) & \text{for } \alpha > k \\ \left(\frac{\sqrt{2}}{k!}\, |W^{(k)}(0)|\, L^{-\alpha} + L^{-\alpha}\right) \|f\|_\alpha + o(L^{-\alpha}) & \text{for } \alpha \leq k \end{cases}$$

with the strictly monotonically decreasing constant

$$c_{\alpha,k} = \left(\frac{k}{\alpha - k}\right)^{k/2} \left(\frac{\alpha - k}{\alpha}\right)^{\alpha/2} \qquad \text{for } \alpha > k.$$

In particular, we have

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(c\, |W^{(k)}(0)|\, L^{-\min\{k,\alpha\}} + L^{-\alpha}\right) \|f\|_\alpha + o\left(L^{-\min\{k,\alpha\}}\right).$$

□

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