Polynomials Used in Frequency Analysis and Their Zeros

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Topics

The Signal

$$h(x) \ := \ \sum_{j=1}^{m} \lambda_j e^{\omega_j x}, \ x \geq 0 \ , \quad \lambda_j \in \mathbb{C} \backslash \{0\} \ , \quad m < \infty \ ,$$

$\omega_j \in [-\alpha, 0] + \mathrm{i}\,(-\pi, \pi]$, $\omega_i \neq \omega_j$ for $i \neq j$, $\alpha > 0$,
$z_j := e^{\omega_j} \in [[e^{-\alpha}, 1]]$,
Known samples $h_k := h(k),\ k = 0, \ldots, N-1$.
If reconstruction is desired: $N \geq 2m$.
Problem: (Re)Construct/Approximate the signal h from given samples.
Subproblem: Determination of the frequencies ω_j.

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analogously $((a, b]]$, $[[a, b))$, $((a, b))$ for $a, b \in \mathbb{R}$, $a \leq b$;
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Szegő's Approach

Calculate the moments

$$\mu_j := \begin{cases} \sum_{k=0}^{N-j-1} h_k \overline{h}_{k+j} \ , & j = 0, 1, 2, \dots \\ \overline{\mu}_{-j} \ , & j = -1, -2, -3, \dots \end{cases}$$

Use them to define monic Szegő polynomials $\tilde{s}_j$ resp. their recurrence/reflection/Schur coefficients, e. g. by the Wiener-Levinson method.↪ Details

Use these coefficients to build up a Hessenberg matrix whose eigenvalues are the zeros of $\tilde{s}_m$. Use them as approximants for $z_j = e^{\omega_j}$.

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Calculate all zeros z_j of ρ_m,
$\omega_j := \mathrm{Log}\, z_j,\ j = 1, \dots, m$

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- Perturbations=noised samples?
- Calculations sensitive against numerical errors ('inverse problem').

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There exist more stabilized variants of these classical methods (matrix pencil, ESPRIT, optimization and approximation methods, ...). Some methods can also detect and correct noised samples and can be used for sparse approximation, too. $\hookrightarrow$ Basic Ideas

On the Position of the Zeros

Consider 'Prony-like' polynomial

$$\rho_m(z) \quad = \quad \sum_{j=0}^{m} p_j z^j = p_m \prod_{j=1}^{m} (z - z_j) \ , \quad p_m \neq 0,$$

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All zeros of ρ_m in $[[e^{-\alpha}, 1]]$

$$0 \neq \rho_m(0) = p_0 = p_m (-1)^m \prod_{j=1}^{m} z_j$$

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$\Rightarrow |p_0| \leq |p_m|$; equality iff **all** zeros are unimodular.
$|p_m| - |p_0| > 0$ (at least one zero has modulus < 1) $\Rightarrow |p_m|^2 - |p_0|^2 > 0$.

Construction of a Recurrence ($0 \neq |p_0| < |p_m|$)

Reciprocal polynomial $\rho_m^\star$,

$$\rho_m^\star(z) := z^m \overline{\rho}_m(1/z) = \overline{p}_0 z^m + \ldots + \overline{p}_m$$

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Approach:

$$p_{m-1}^{(1)} \rho_m(z) = p_m z \rho_{m-1}(z) + p_0 \frac{p_{m-1}^{(1)}}{\overline{p}_{m-1}^{(1)}} \rho_{m-1}^\star(z)$$

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Find the coefficients $p_\nu^{(1)}$ of ρ_{m-1},

$$\rho_{m-1}(z) = \sum_{\nu=0}^{m-1} p_\nu^{(1)} z^\nu \ , \quad p_{m-1}^{(1)} \neq 0,$$

resp. its reciprocal polynomial $\rho_{m-1}^\star$.

From approach:

$$\overline{p}_{m-1}^{(1)}\rho_m^{\star}(z) = \overline{p}_m\rho_{m-1}^{\star}(z) + z\overline{p}_0\frac{\overline{p}_{m-1}^{(1)}}{p_{m-1}^{(1)}}\rho_{m-1}(z)\ .$$

From approach:

$$\overline{p}^{(1)}_{m-1}\rho^\star_m(z) = \overline{p}_m\rho^\star_{m-1}(z) + z\overline{p}_0\frac{\overline{p}^{(1)}_{m-1}}{p^{(1)}_{m-1}}\rho_{m-1}(z)\ .$$

Combination:

$$\begin{aligned}
&\overline{p}_m p^{(1)}_{m-1}\rho_m(z) - p_0\overline{p}^{(1)}_{m-1}\rho^\star_m(z) \\
&= \left(|p_m|^2 - |p_0|^2\frac{\overline{p}^{(1)}_{m-1}}{p^{(1)}_{m-1}}\right) z\rho_{m-1}(z) + p_0\overline{p}_m\left(\frac{p^{(1)}_{m-1}}{\overline{p}^{(1)}_{m-1}} - 1\right)\rho^\star_{m-1}(z) \\
&= p_0\overline{p}_m(p^{(1)}_{m-1} - \overline{p}^{(1)}_{m-1}) + z\sum_{\nu=0}^{m-1}\left(\overline{p}_m p^{(1)}_{m-1}p_{\nu+1} - \overline{p}^{(1)}_{m-1}p_0\overline{p}_{m-\nu-1}\right)z^\nu\ .
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\end{aligned}$$

Normalization $p^{(1)}_{m-1} \in \mathbb{R}\backslash\{0\}$: For $p_m^2 - |p_0|^2 > 0$

$$p^{(1)}_\nu = \underbrace{\frac{p^{(1)}_{m-1}}{p_m^2 - |p_0|^2}}_{=:c_1}\underbrace{\left(p_m p_{\nu+1} - p_0\overline{p}_{m-\nu-1}\right)}_{=:\beta^{(1)}_\nu}\ ,\quad \nu = 0,\dots,m-1\ .$$

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Otherwise restart with $\rho_{m-1}^{(1)}$ (Normalization $p_{m-2}^{(2)} \in \mathbb{R}\backslash\{0\}$) etc.
We get $(\rho_{m-\mu})_{\mu=0}^{\delta}$, $\delta \leq m$,

$$\rho_{m-\mu}(z) \quad = \quad \sum_{\nu=0}^{m-\mu} p_\nu^{(\mu)} z^\nu \ ,$$

where $\rho_{m-\delta}$ has only unimodular zeros for $\delta \leq m-1$.

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where $\rho_{m-\delta}$ has only unimodular zeros for $\delta \leq m-1$.
Arbitray chosen leading coefficients $\kappa_{m-j} := p_{m-j}^{(j)} \in \mathbb{R}\backslash\{0\}$ of ρ_{m-j},
$a_{m-j} := \rho_{m-j}(0) = p_0^{(j)}$, especially

$$\left| \frac{a_{m-\delta}}{\kappa_{m-\delta}} \right| = 1$$

Calculation of monomial coefficients

($a_{m-1},\ c_1,\ \beta_\nu^{(1)}$ known from before)

For $\mu = 1, \ldots, m-\delta-1$ **do**

$$c_{\mu+1} \quad := \quad c_\mu \frac{\kappa_{m-(\mu+1)}}{\kappa_{m-\mu}^2 - |a_{m-\mu}|^2} \ ; \quad \text{(normalization of } \rho_{m-(\mu+1)}\text{)}$$

For $\nu = 0, \ldots, m-\mu$ **do**

$$\beta_\nu^{(\mu+1)} \quad := \quad \kappa_{m-\mu}\beta_{\nu+1}^{(\mu)} - a_{m-\mu}\overline{\beta}_{m-\mu-\nu-1}^{(\mu)} \ ; \quad \text{(from the recurrence approach)}$$

$$p_\nu^{(\mu+1)} \quad := \quad \beta_\nu^{(\mu+1)} c_{\mu+1} \ ; \quad a_{m-(\mu+1)} := p_0^{(\mu+1)};$$

Recurrence Relations

From approach:

$$\rho_\mu(z) = \frac{\kappa_\mu}{\kappa_{\mu-1}} z \rho_{\mu-1}(z) + \frac{a_\mu}{\kappa_{\mu-1}} \rho^\star_{\mu-1}(z) \ ,$$

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$\rho_{m-\delta}$ is a common divisor of each $\rho_{m-\nu}$: $\rho_{m-\nu} = \rho_{m-\delta}s_{\delta-\nu}$ for $\nu = 0, \ldots, \delta$,

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$$\begin{aligned} s_j(z) &= \frac{\kappa_{j+m-\delta}}{\kappa_{j+m-\delta-1}} z s_{j-1}(z) + \frac{a_{j+m-\delta}}{\kappa_{j+m-\delta-1}} \frac{\kappa_{m-\delta}}{a_{m-\delta}} s^\star_{j-1}(z) , \\ s^\star_j(z) &= \frac{\kappa_{j+m-\delta}}{\kappa_{j+m-\delta-1}} s^\star_{j-1}(z) + z\frac{\bar{a}_{j+m-\delta}}{\kappa_{j+m-\delta-1}} \frac{\kappa_{m-\delta}}{\bar{a}_{m-\delta}} s_{j-1}(z) , \quad j = 1, \dots, \delta , \end{aligned}$$

where $s_0 := s^\star_0 := 1$.

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All zeros of s_j *are in the interior of the unit circle. Furthermore,* s_j *and* $s_{j-1},\ j = 1, \ldots, \delta$*, have no common zeros.*

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Theorem

In our model the Prony-like polynomial ρ_m can be factorized as

$$\rho_m(z) = \rho_{m-\delta}(z) s_\delta(z) \ ,$$

where all zeros of $\rho_{m-\delta}$ lie on $[[1, 1]]$ and a Szegő polynomial s_δ, whose zeros are in $[[e^{-\alpha}, 1))$.

Zeros in the Interior of the Unit Circle

Abbreviations $\alpha_j := \frac{a_{j+m-\delta}}{\kappa_{j+m-\delta-1}} \frac{\kappa_{m-\delta}}{a_{m-\delta}}$, $\beta_0 := 1$,
$\beta_j := \prod_{\mu=1}^{j} \left(\gamma_\mu - \frac{|\alpha_\mu|^2}{\gamma_\mu} \right) = \left(\gamma_j - \frac{|\alpha_j|^2}{\gamma_j} \right) \beta_{j-1} > 0$, $\gamma_j := \frac{\kappa_{j+m-\delta}}{\kappa_{j+m-\delta-1}}$ and $m - \delta =: \omega$, $0 \leq \omega \leq m$.

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$m - \delta =: \omega, \ 0 \le \omega \le m$.
For $j = 1, \ldots, \delta$:

$$z s_{j-1}(z) \ = \ \frac{1}{\gamma_j} s_j(z) - \frac{\alpha_j}{\gamma_j} s_{j-1}^{\star}(z)$$

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For $j = 1, \ldots, \delta$:

$$\begin{aligned} z s_{j-1}(z) &= \frac{1}{\gamma_j} s_j(z) - \frac{\alpha_j}{\gamma_j} s_{j-1}^{\star}(z) \\ &= \frac{1}{\gamma_j} s_j(z) - \frac{\alpha_j}{\gamma_j} \frac{\overline{\alpha}_{j-1}}{\gamma_{j-1}} s_{j-1}(z) - \frac{\alpha_j \beta_{j-1}}{\gamma_j} \sum_{\nu=0}^{j-2} \frac{\overline{\alpha}_\nu}{\gamma_\nu \beta_\nu} s_\nu(z) \; . \end{aligned}$$

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$$z \begin{pmatrix} s_0(z) \\ s_1(z) \\ \vdots \\ s_{\delta-1}(z) \end{pmatrix} = \mathbf{A} \begin{pmatrix} s_0(z) \\ s_1(z) \\ \vdots \\ s_{\delta-1}(z) \end{pmatrix} + \frac{1}{\gamma_\delta} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ s_\delta(z) \end{pmatrix} .$$

Hence, the zeros of s_δ are the eigenvalues of $\mathbf{A} \in \mathbb{C}^{\delta,\delta}$.

$$\mathbf{A} \;=\; \mathbf{D}\mathbf{B}\mathbf{D}^{-1}$$

where $\mathbf{D} := \operatorname{diag}(\beta_0, \dots, \beta_{\delta-1})$, $\mathbf{D}^{-1} = \operatorname{diag}\left(\frac{1}{\beta_0}, \dots, \frac{1}{\beta_{\delta-1}}\right)$ and

$$\mathbf{B} = \begin{pmatrix} -\frac{\overline{a}_\omega a_{\omega+1}}{\kappa_\omega \kappa_{\omega+1}} & 1-\left|\frac{a_{\omega+1}}{\kappa_{\omega+1}}\right|^2 & & & \mathbf{0} \\ -\frac{\overline{a}_\omega a_{\omega+2}}{\kappa_\omega \kappa_{\omega+2}} & -\frac{\overline{a}_{\omega+1} a_{\omega+2}}{\kappa_{\omega+1} \kappa_{\omega+2}} & 1-\left|\frac{a_{\omega+2}}{\kappa_{\omega+2}}\right|^2 & & \\ \vdots & \vdots & \ddots & \ddots & \\ -\frac{\overline{a}_\omega a_{\omega+\delta-1}}{\kappa_\omega \kappa_{\omega+\delta-1}} & -\frac{\overline{a}_{\omega+1} a_{\omega+\delta-1}}{\kappa_{\omega+1} \kappa_{\omega+\delta-1}} & \cdots & -\frac{\overline{a}_{\delta-2} a_{\delta-1}}{\kappa_{\delta-2} \kappa_{\delta-1}} & 1-\left|\frac{a_{\omega+\delta-1}}{\kappa_{\omega+\delta-1}}\right|^2 \\ -\frac{\overline{a}_\omega a_{\omega+\delta}}{\kappa_\omega \kappa_{\omega+\delta}} & -\frac{\overline{a}_{\omega+1} a_{\omega+\delta}}{\kappa_{\omega+1} \kappa_{\omega+\delta}} & \cdots & -\frac{\overline{a}_{\omega+\delta-2} a_{\omega+\delta}}{\kappa_{\omega+\delta-2} \kappa_{\omega+\delta}} & -\frac{\overline{a}_{\omega+\delta-1} a_{\omega+\delta}}{\kappa_{\omega+\delta-1} \kappa_{\omega+\delta}} \end{pmatrix}.$$

Zeros on the Unit Circle

Consider $\rho_\omega(z) = \sum_{\nu=0}^{\omega} p_\nu^{(\omega)} z^\nu$ on $[[1,1]]$.
Simplification: Assume $p_\nu^{(\omega)} \in \mathbb{R}$ (or consider $\rho_\omega \overline{\rho}_\omega$), write $z = e^{\mathrm{i}\,t},\ -\pi < t \leq \pi$, i.e. $\rho_\omega(e^{\mathrm{i}\,t}) = u_\omega(t) + \mathrm{i}\, v_\omega(t)$ with

$$
\begin{aligned}
u_\omega(t) &= \sum_{\nu=0}^{\omega} p_\nu^{(\omega)} \cos \nu t\ , \\
v_\omega(t) &= \sum_{\nu=0}^{\omega} p_\nu^{(\omega)} \sin \nu t\ .
\end{aligned}
$$

Common zeros t_j of u_ω and v_ω in $(-\pi, \pi]$ yield the zeros z_j of ρ_m in $[[1,1]]$ and the frequencies via $\omega_j = \mathrm{i}\, t_j,\ j = 1, \ldots, \omega$.

$\rho_\omega([[1,1]])$ symmetric to the real axis: Consider $\rho_\omega(e^{\mathrm{i}\,t})$ for $t \in [0,\pi]$ instead for $-\pi < t \leq \pi$.

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Transformation $x := \cos t,\ t \in [0,\pi],\ x \in [-1,1]$, is bijective and we have

$$\begin{aligned} \rho_\omega(e^{\mathrm{i}\,t}) &= u_\omega(t) + \mathrm{i}\, \sin t\ \tilde{v}_\omega(t) \\ &= \sum_{\nu=0}^{\omega} p_\nu^{(\omega)} T_\nu(x) + \mathrm{i}\, \sqrt{1-x^2} \sum_{\nu=0}^{\omega-1} p_{\nu+1}^{(\omega)} U_\nu(x) \\ &=: r_\omega(x) + \mathrm{i}\, \sqrt{1-x^2} r_{\omega-1}(x) \end{aligned}$$

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Zeros of ρ_ω on $[[1,1]]$ are either ± 1 or can be obtained from the common zeros of r_ω and $r_{\omega-1}$ in $(-1,1)$.

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- This deflation can be done by the Euclidean algorithm for Chebyshev expansions, too.

Zeros Near the Unit Circle

Distinguishing between ‘interior’ and ‘on the boundary’ of unit circle is numerical fragile. Replace test criterion

$$1 - \left| \frac{a_{m-\mu}}{\kappa_{m-\mu}} \right| > 0 \quad \text{resp.} = 0, \quad 0 \le \mu \le \delta \le m-1,$$

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$$1 - \left| \frac{a_{m-\mu}}{\kappa_{m-\mu}} \right| > \epsilon \ , \quad 0 < \epsilon \ll 1, \quad 0 \le \mu \le \delta \le m-1 \ .$$

May influence $\delta = \delta(\epsilon)$

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- Repeat this e. g. by a bisection strategy on $[[1-\epsilon_\nu, 1-\epsilon_\nu]]$, $\epsilon_\nu := \frac{\epsilon}{2^\nu}$, until all zeros are found.