Hankel operators of finite rank

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The operations

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Definition

1. **Sequences:** $c, d : \mathbb{Z}^s \to \mathbb{R}$.
2. **Convolution:**

$$c * d = \sum_{\alpha \in \mathbb{Z}^s} c(\cdot - \alpha)\, d(\alpha) = \sum_{\alpha \in \mathbb{Z}^s} c(\alpha)\, d(\cdot - \alpha)$$

3. **Correlation:** (not symmetric)

$$c \star d := \sum_{\alpha \in \mathbb{Z}^s} c(\alpha)\, d(\cdot + \alpha)$$

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For $f \in \ell(\mathbb{Z}^s) := \{f : \mathbb{Z}^s \to \mathbb{R}\}$

1. **Hankel operator**

$$H(f) : x \mapsto x \star f = \sum_{\alpha \in \mathbb{Z}^s} f(\cdot + \alpha)\, x(\alpha)$$

2. **Toeplitz operator**

$$T(f) : x \mapsto x * f = \sum_{\alpha \in \mathbb{Z}^s} f(\cdot - \alpha)\, x(\alpha)$$

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Infinite matrix representation.

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Infinite matrix representation. Subdivision: $f(\alpha - \Xi\beta)$.

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Finite segments

1. $A, B \subset \mathbb{Z}^s$.
2. Linear operators $\mathbb{C}^A \to \mathbb{C}^B$:

$$H_{A,B}(f) := \left[f(\alpha+\beta) : \begin{array}{c} \alpha \in A \\ \beta \in B \end{array} \right], \qquad T_{A,B}(f) := \left[f(\alpha-\beta) : \begin{array}{c} \alpha \in A \\ \beta \in B \end{array} \right].$$

3. Multiindexed matrices.

Explicit

$$H_{A,B}(f)\, x = (x|_B \star f)|_A, \qquad T_{A,B}(f)\, x = (x|_B * f)|_A, \qquad x \in \ell(\mathbb{Z}^s).$$

Canonical embedding: $\mathbb{C}^B \ni x|_B \hookrightarrow x \in \ell(\mathbb{Z}^s)$.

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Rank of a Hanke1 or Toeplitz operator:

Nonnegative rank

$$r_H^+(f) := \sup_{A,B \in \mathbb{N}_0^s} \operatorname{rank} H_{A,B}(f).$$

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Hankel rank / Toeplitz rank

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$$r_H^+(f) := \sup_{A,B \in \mathbb{N}_0^s} \operatorname{rank} H_{A,B}(f).$$

Clearly: $r_H^+(f) \leq r_H(f)$.

Shift invariance

Definition

$\mathscr{X} \subset \ell(\mathbb{Z}^s)$ is called **shift invariant** if

$$f \in \mathscr{X} \qquad \Leftrightarrow \qquad \tau^\alpha f \in \mathscr{X}, \quad \alpha \in \mathbb{Z}^s.$$

Definition

Principal shift invariant space of $f \in \ell(\mathbb{Z}^s)$

$$\mathscr{S}(f) := \operatorname{span}\left\{\tau^\alpha f : \alpha \in \mathbb{Z}^s\right\}.$$

Example: exponentials

1. $f(\alpha) = e^{\omega^T \alpha}$.
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Some facts

Theorem

If $f \in \ell(\mathbb{Z}^s)$ is *finitely supported*, then $\dim \mathscr{S}(f) = \infty$.

Theorem

$$r_H(f) = r_T(f) = r_H^+(f), \qquad f \in \ell(\mathbb{Z}^s).$$

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$$\dim \mathscr{S}(f) < \infty \qquad \Rightarrow \qquad \dim \mathscr{S}(f) = r_H(f).$$

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$$\dim \mathscr{S}(f) < \infty \qquad \Leftrightarrow \qquad f = \sum_{\omega \in \Omega} f_\omega \, e^{\omega^T \cdot}, \qquad f_\omega \in \mathbb{C}.$$

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More background

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Terminology

1. Bilinear forms

$$(\cdot,\cdot) : \ell(\mathbb{Z}^s) \times (\mathbb{Z}^s) \to \mathbb{C}, \qquad (f,x) \mapsto \begin{cases} x \star f \\ x * f \end{cases}$$

2. Well defined?

Algebraic formulation

1. **Symbol** $x^\sharp(z) := \sum_{\alpha \in \mathbb{Z}^s} x(\alpha) z^\alpha, \qquad x \in \ell_{00}(\mathbb{Z}^s), z \in (\mathbb{C} \setminus \{0\})^s.$
2. **Difference equations** (algebraic formulation):

$$x \star f = x^\sharp(\tau) f, \qquad x * f = x^\sharp\left(\tau^{-1}\right) f.$$

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2. Well defined? By finite support.

Algebraic formulation

1. **Symbol** $x^\sharp(z) := \sum_{\alpha \in \mathbb{Z}^s} x(\alpha)\, z^\alpha, \qquad x \in \ell_{00}(\mathbb{Z}^s),\ z \in (\mathbb{C} \setminus \{0\})^s.$

2. **Difference equations** (algebraic formulation):

$$x \star f = x^\sharp(\tau) f, \qquad x * f = x^\sharp\left(\tau^{-1}\right) f.$$

More background

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Terminology

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Difference Equations?

Difference equation with constant coefficients

$$Af = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\, \Delta^\alpha f$$

Duality for bilinear form

$$(x, f) \quad \to \quad x \star f$$

1. **Laurent polynomials** $\Lambda = \operatorname{span}\{z^\alpha : \alpha \in \mathbb{Z}^s\}$.
2. Difference equations on $\Lambda \times \ell(\mathbb{Z}^s)$.

Fundamental identity of "adjoint operators"

$$(p, \tau^\alpha f) = ((\cdot)^\alpha p, f), \qquad p \in \Lambda, f \in \ell(\mathbb{Z}^s).$$

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$$Af = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\,(\tau - I)^\alpha f$$

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$$Af = b \star f = b^\sharp(\tau) f = \hat{b}^\sharp(\tau^{-1}) f, \qquad \hat{b} = b(-\cdot).$$

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Invariances & Ideals

Observation

$$0 = (p, f)$$

Theorem

$f \in \ell(\mathbb{Z}^s)$, $p \in \Lambda$,

1. $\{p \in \Lambda : (p,f) = 0\} \subset \Lambda$ is an **ideal**.
2. $\{f \in \ell(\mathbb{Z}^s) : (p,f) = 0\} \subset \mathbb{Z}^s$ is **shift invariant**.

Definition

Ideal $\mathscr{I}$ is called **zero dimensional** if

$$\# \{z \in \mathbb{C}^s : p(z) = 0, p \in \mathscr{I}\} < \infty.$$

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System Theory

Homogeneous solutions of partial differential equations

Given finite set $P \subset \Pi$, characterize solutions f of

$$P(D)f = 0 \qquad \text{i.e.,} \qquad p(D)f = 0, \quad p \in P.$$

Solution

1. Common zeros of $\langle P \rangle = \left\{ \sum_p q_p \, p : q_p \in \Pi \right\}$.
2. $P(\theta) = 0$ yields $\theta^{(\cdot)} \in \ker P(D)$

Theorem[Gröbner]

Any zero dimensional ideal $\mathscr{I} \subset \Pi$ is of the form

$$p \in \mathscr{I} \qquad \Leftrightarrow \qquad q(D)p(\xi) = 0, \quad q \in \mathscr{Q}_\xi, \, \xi \in \Xi.$$

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More Invariances

Polynomial subspaces

A subspace $\mathscr{P} \subset \Pi$ is called

1. **shift invariant**: $p \in \mathscr{P} \Rightarrow p(\cdot +) \in \mathscr{P}$, .
2. ***D*-invariant**: $p \in \mathscr{P} \Rightarrow D^\alpha p \in \mathscr{P}$, $\alpha \in \mathbb{N}_0^s$.

Structural multiplicities

Multiplicity space $\mathscr{Q}_\xi$ in

$$q(D)p(\xi) = 0, \quad q \in \mathscr{Q}_\xi,$$

is ***D*-invariant**.

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A subspace $\mathscr{P} \subset \Pi$ is called

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Counting is not enough: different types of triple zeros . . .

Discrete System Theory

Homogeneous solutions of partial difference equations

Given finite set $P \subset \Lambda$, characterize solutions $f \in \ell(\mathbb{Z}^s)$ of

$$P(\tau)f = 0$$

Remark

1. **System** P is important.
2. Depends only on $\langle P \rangle$, not on P.
3. Any basis works $\Rightarrow$ Computer Algebra.

Prony's problem – yet another déjá–vu

Given $f \in \ell(\mathbb{Z}^s)$ find **basis** P such that $P(\tau)f = 0$.

Discrete System Theory

Homogeneous solutions of partial **difference** equations

Given finite set $P \subset \Lambda$, characterize solutions $f \in \ell(\mathbb{Z}^s)$ of

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3. Any basis works $\Rightarrow$ Computer Algebra.

Prony's problem – yet another déjá–vu

Given $f \in \ell(\mathbb{Z}^s)$ find **basis** P such that $P(\tau)f = 0$.

Discrete System Theory

Homogeneous solutions of partial **difference** equations

Given finite set $P \subset \Lambda$, characterize solutions $f \in \ell(\mathbb{Z}^s)$ of

$$P(\tau)f = 0 \qquad \text{i.e.,} \qquad (p, f) = 0, \quad p \in P.$$

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A Strange Operator

Taylor formula

1. For $f \in \Pi$
$$f = \sum_{\alpha \in \mathbb{N}_0^s} \frac{1}{\alpha!} D^\alpha f(0) \, (\cdot)^\alpha$$
2. Mixing: $Lf := \sum_{\alpha \in \mathbb{N}_0^s} \frac{1}{\alpha!} \Delta^\alpha f(0) \, (\cdot)^\alpha$,

Theorem

1. $\mathscr{P}$ shift invariant $\Leftrightarrow$ $L\mathscr{P}$ D-invariant.
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$$\mathscr{S}(p) := \operatorname{span}\{p(\cdot + \alpha) : \alpha\}, \qquad \mathscr{D}(p) := \operatorname{span}\{D^\alpha p : \alpha\}.$$

Prony is Dual Filtering

Why shift invariance?

$$f = \sum_{\omega \in \Omega} f_\omega \, e^{\omega^T \cdot}$$

Theorem

For zero dimensional $\langle H(z) \rangle$ are equivalent:

1. $\langle H(z) \rangle = \bigcap_{\theta \in \Theta} \ker \delta_\theta \circ \mathscr{Q}_\theta(D)$.
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- Frequencies = **zeros** of symbol.
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Poor Man's Philosophy

Comparison between differential and difference operators

Structure of *homogeneous* solutions

	differential	**difference**
operator	$p(D)f, \quad p \in P$	$p(\tau)f, \quad p \in P$
solutions	D–invariant	shift invariant
frequencies	zeros of $\langle P \rangle$	zeros of $\langle P \rangle$
polynomials	$\mathscr{Q}_\xi$	$L\mathscr{Q}_\xi$

In both cases: **exponential polynomials**.

Remark

Partially things work also for ideals of general dimension.

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What did we do so far?

Relationship between

1. Hankel- and Toeplitz operators of finite rank.
2. Partial difference equations.
3. Shift invariant spaces.
4. Exponential polynomial sequences.

Observation

1. It's essentially the same.
2. Glue: **Prony's problem**.
3. Key: cancellation filters & ideals.

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A Typical Argument

1. $\mathscr{F} \subset \ell(\mathbb{Z}^s)$ finite dimensional and shift invariant.
2. $f_1, \dots, f_N \in \mathscr{F}$ basis of $\mathscr{F}$.
3. $\mathscr{S}(f_j)$
4. $r_H(f_j) = \operatorname{rank} H_{A,B}(f_j)$ for some A.
5. $\ker H_{A,B}(f_j)$ defines **zero dimensional** Prony ideal $\langle P_j \rangle$.
6. Ideal $\langle P_1, \dots, P_N \rangle$ defines dual space.
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Every finite dimensional shift invariant subspace of $\ell(\mathbb{Z}^s)$ is of exponential polynomial form.

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7. Prony $\Rightarrow f_j$ are exponential polynomials.

Theorem

Every finite dimensional shift invariant subspace of $\ell(\mathbb{Z}^s)$ is of exponential polynomial form.

A Typical Argument

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1. $\mathscr{F} \subset \ell(\mathbb{Z}^s)$ finite dimensional and shift invariant.
2. $f_1, \dots, f_N \in \mathscr{F}$ basis of $\mathscr{F}$.
3. $\dim \mathscr{S}(f_j) < \infty \Rightarrow \operatorname{rank} H_{\mathbb{Z}^s,B}(f_j) = r_H(f_j)$ for some B.
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One More Result

Definition

Vandermonde matrix for $A \subseteq \mathbb{Z}^s$ and $\Xi \subset (\mathbb{C} \setminus \{0\})^s$:

$$V(A, \Xi) := \left[\xi^\alpha : \begin{array}{c} \alpha \in A \\ \xi \in \Xi \end{array} \right].$$

Theorem (Factorization)

If $r_H(f) < \infty$, then there exists finite $\Omega \subset (\mathbb{C} \setminus \{0\})^s$ and nonsingular, diagonal F_Ω such that

$$H(f) = V\left(\mathbb{Z}^s, e^\Omega\right) F_\Omega \, V\left(\mathbb{Z}^s, e^\Omega\right)^T.$$

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Small print: a little bit more complicated for multiplicities, Hermite interpolation, block diagonal ...

Summary

Main message

It’s all the same.

Untold stories

1. Multiplicity details, Stirling numbers, …
2. $\ell_p(\mathbb{Z}^s)$ & stability …
3. Numerical issues …
4. Model reduction …

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\end

Announcement

Next Bernrieds

1. March 4-.8, 2019
2. February 24–28, 2020

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