

Edge detection methods based on RBF interpolation

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Why RBFs?

- meshless method
- work on non-gridded data
- easily extended to multivariate problems

Kernels and RBFs

A symmetric kernel

$$K : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R},$$

is **translation invariant** on \mathbb{R}^d if there exists a function $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that

$$K(x, y) = \Phi(x - y) \quad \forall \quad x, y \in \mathbb{R}^d.$$

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If K is also **radial**, i.e. there exists a scalar function $\varphi : [0, +\infty) \longrightarrow \mathbb{R}$ such that

$$K(x, y) = \varphi(\|x - y\|), \quad \text{where } \|\cdot\| \text{ is a norm,}$$

the function φ is called **Radial Basis Function (RBF)**.

Scaled kernels and Variably Scaled Kernels

Kernels K on \mathbb{R}^d can be **scaled** by a positive factor ε in the following way

$$K(x, y; \varepsilon) = K\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \quad \forall x, y \in \mathbb{R}^d.$$

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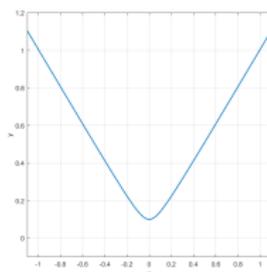
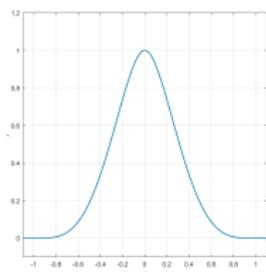
Definition

Let K be a kernel on \mathbb{R}^{d+1} . If a scale function $c : \mathbb{R}^d \rightarrow (0, \infty)$ is given, we define a **Variably Scaled Kernel** or **VSK** on \mathbb{R}^d by

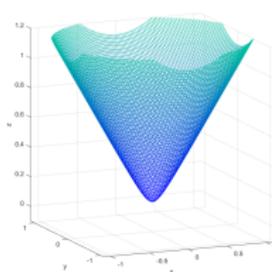
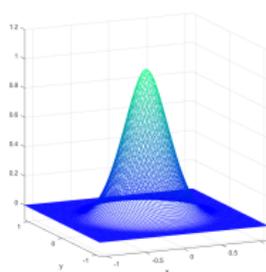
$$K_c(x, y) = K((x, c(x)), (y, c(y))) \quad \forall x, y \in \mathbb{R}^d.$$

Popular RBFs: Multiquadric and Wendland

Multiquadric	$\sqrt{1 + r^2}$
C^2 -Wendland	$(1 - r)_+^4(4r + 1)$
Scaled multiquadric	$\sqrt{\varepsilon^2 + r^2}$
Scaled C^2 -Wendland	$(1 - \varepsilon r)_+^4(4\varepsilon r + 1)$



Left: C^2 -W. Right: MQ with $\varepsilon = 0.1$.



Left: C^2 -W. Right: MQ with $\varepsilon = 0.1$.

To scale the support of a Wendland function we use the basic function $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$, where φ is the one of the definition. Specifically the support radius of φ_ε is $\frac{1}{\varepsilon}$.

Interpolation with RBFs

If φ is a positive definite radial function, the **interpolant**

$$s_{f, X}(x) = \sum_{i=1}^N \lambda_i \varphi(\|x - x_i\|)$$

of a set of data $\{x_i, f_i\}_{i=1}^N$, **exists** and is **unique**

The coefficients λ_i are obtained solving the linear system

$$\mathbf{M}\boldsymbol{\lambda} = \mathbf{f},$$

where \mathbf{M} has elements $M_{ij} = \varphi(\|x_i - x_j\|)$.

Study of coefficients: WORK IN PROGRESS

From ¹ and ² we have a formula for the **cardinal expansion coefficients**

$$\lambda_k = \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} \frac{e^{ik\xi}}{\sum_{j=-\infty}^{\infty} \widehat{\varphi}(|\xi + 2\pi j|)} d\xi, \quad (1)$$

if φ is a positive definite radial function and the expression at the denominator is never null.

¹M.D. Buhmann, *Radial Basis Functions*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press (2004)

²B. Fornberg, N. Flyer, S. Hovde, C. Piret, Localization Properties of Rbf Expansion Coefficients for Cardinal Interpolation. I. Equispaced Nodes, *IMA J. Num. Anal.* 28 (2007) 121-142.

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This is relevant for **studying of the behaviour of interpolation coefficients** near jump discontinuities, in fact we can express any kind of discrete data as a linear combination of cardinal functions.

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This is relevant for studying of the behaviour of interpolation coefficients near jump discontinuities, in fact we can express any kind of discrete data as a linear combination of cardinal functions.

Especially, we would like to have a closed expression for the coefficient of the Wendland cardinal interpolant. For φ_ε

$$\lambda_k = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(k\xi)}{\sum_{h=1}^{\lfloor \frac{1}{\varepsilon} \rfloor} (2\varphi_\varepsilon(h) \cos(h\xi)) + \varphi_\varepsilon(0)} d\xi.$$

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Literature: Iterative adaptive method (IA)

An **iterative adaptive edge detection method** based on **MQ RBF interpolation** has been proposed in [Jung et al., 2011]³.

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- 3 change to zero the correspondent shape parameter

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Our method with variably scaled kernel (VSK1D)

The first method we propose for the jump detection in 1D is the iterative adaptive method with VSKs.

Instead of adapting the bases used for the interpolation with vanishing scale parameters, we use **variably scaled kernels with step scale functions**.

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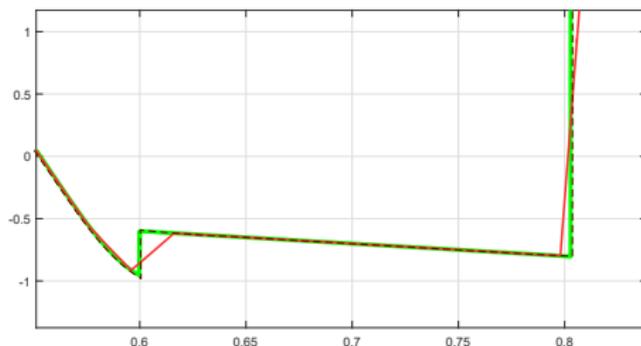
- 1 interpolate the data
- 2 find the coefficients greater than a certain tolerance level in absolute value
- 3 add a step function to the kernel in the points found at the previous step

Advantages of VSK1D method

- **Any radial kernel** can be used with our algorithm. C^2 -Wendland RBF can conveniently replace MQ RBF as it is compactly supported and improves the stability of interpolation.

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- The reconstruction for **steep jumps** is **qualitatively better**, as the VSKs reproduce the exact discontinuities instead of doing linear interpolation.



MQ interpolant with vanishing parameters at the jump points (solid red). VSK interpolant with step scale function (dashed black). Exact function (solid green).

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- The reconstruction for **steep jumps** is **qualitatively better**, as the VSKs reproduce the exact discontinuities instead of doing linear interpolation.
- For each pair (or triplet) of consecutive possible edge points identified in the same iteration, only their centroid is added to the edge, resulting in a **thinner edge**.

Our Non-Iterative (N-I) method

The Non-Iterative algorithm is based on the fact that the **local maxima of the absolute value interpolation coefficients** identify the region where the **edge** is located.

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Our Non-Iterative (N-I) method

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The idea of our algorithm is to

- 1 interpolate the data
- 2 compute the local maxima of the absolute value of coefficient expansion

Advantages of N-I method

- The N-I method is **faster** than the iterative methods, as λ is computed only once.

Advantages of N-I method

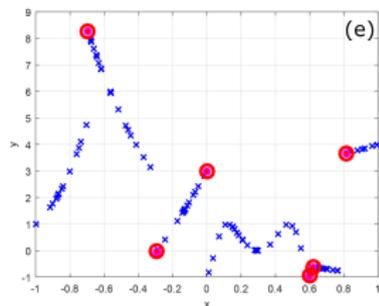
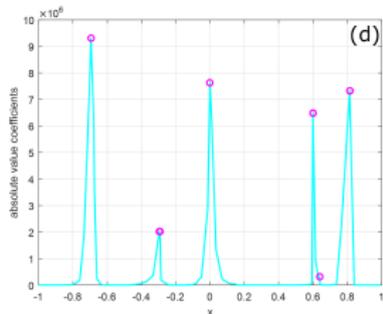
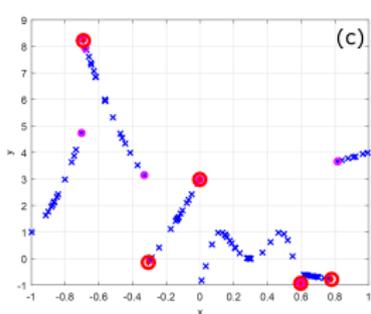
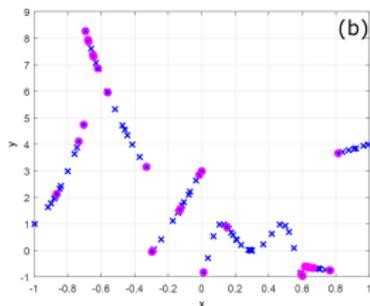
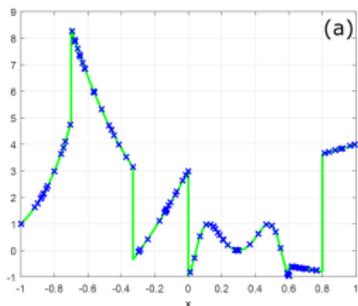
- The N-I method is **faster** than the iterative methods, as λ is computed only once.
- In 1D, the method provides **thinner edges** than IA method, as for each pair (or triplet) of consecutive possible edge points identified in the same iteration, only their centroid is added to the edge.

1D example

For the one dimensional numerical experimentation we use the following function

$$f(x) = \begin{cases} (x + 2)^6 & -1 \leq x \leq -0.7 \\ (1 - x)^4 & -0.7 < x \leq -0.3 \\ (x + 2)^3 - 5 & -0.3 < x \leq 0 \\ \sin((7x - 2.1)^2) & 0 < x \leq 0.6 \\ -x & 0.6 < x \leq -0.8 \\ x^2 + 3 & 0.8 < x \leq 1 \\ 0 & \text{elsewhere.} \end{cases} \quad (2)$$

The RBFs we use are the multiquadric with $\varepsilon = 1$ for the Jung method and C^2 -Wendland and $\varepsilon = 1$ for our methods.



(a) Original function and sampled random data. (b) Edge points obtained by IA1D. (c) Edge points obtained by VSK1D with C^2 -W. (d) Absolute value of interpolation coefficients with C^2 -W, local maxima in magenta. (e) Edge points obtained by N-I with C^2 -W.

Comparison

IA1D

Grid	N	Iter.	Edge pts
U	100	4	13
H	100	6	23
R	100	15	33

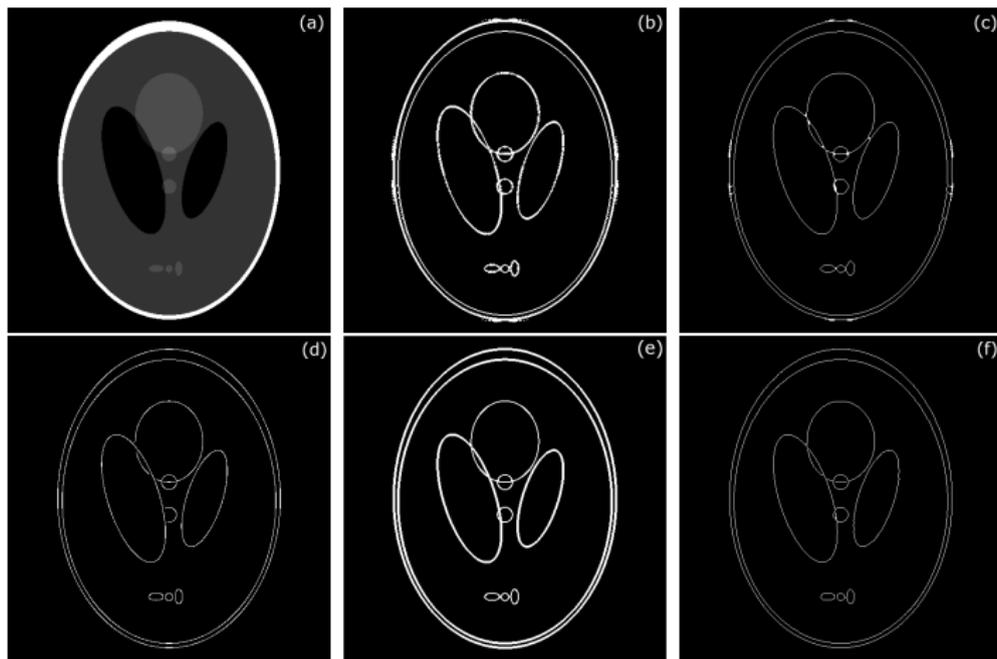
VSK1D with C^2 -W

Grid	N	Iter.	Edge pts
U	100	2	10 (5)
H	100	3	12 (5)
R	100	3	12 (5)

N-I1D with C^2 -W

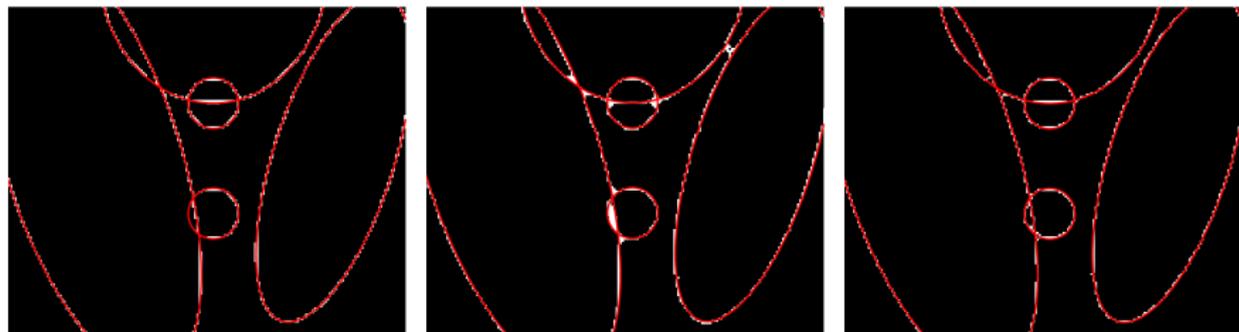
Grid	N	Edge pts
U	100	10 (5)
H	100	9 (5)
R	100	9 (6)

Shepp-Logan phantom



(a) Original 512×512 pixel image. (b) Edges for IA2D dimension-by-dimension method. (c) Edges for IA2D dimension-by-dimension method with post-processing. (d) Edges for Canny edge detection. (e) Edges for dimension-by-dimension N-I method. (f) Edges for dimension-by-dimension N-I method with post-processing.

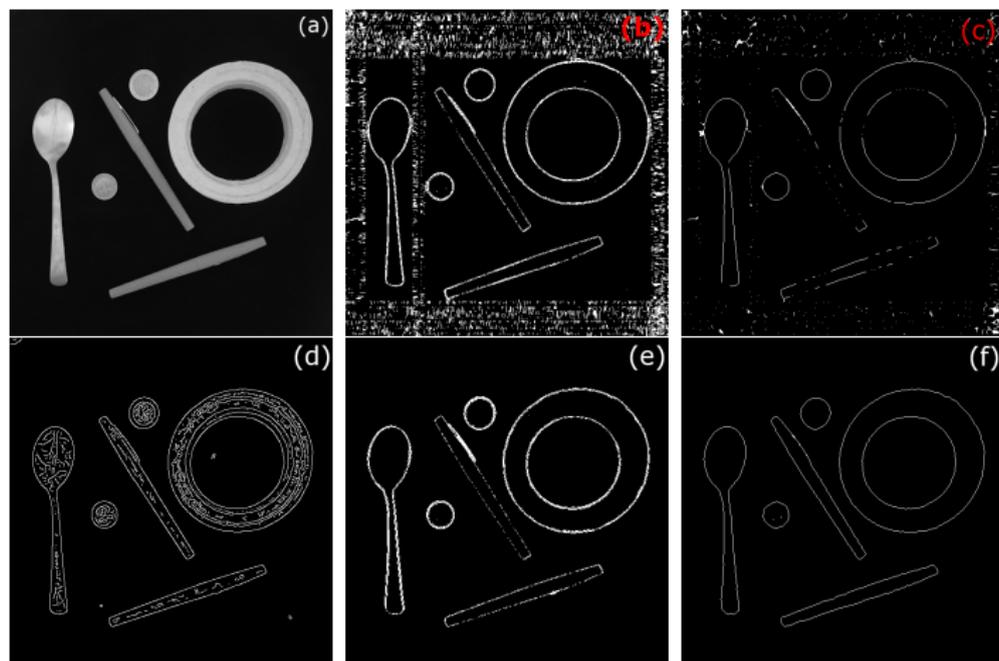
Shepp-Logan phantom: zoom



From left to right: Edges for Canny edge detection. Edges for IA2D dimension-by-dimension method with post-processing. Edges for dimension-by-dimension N-I method with post-processing.

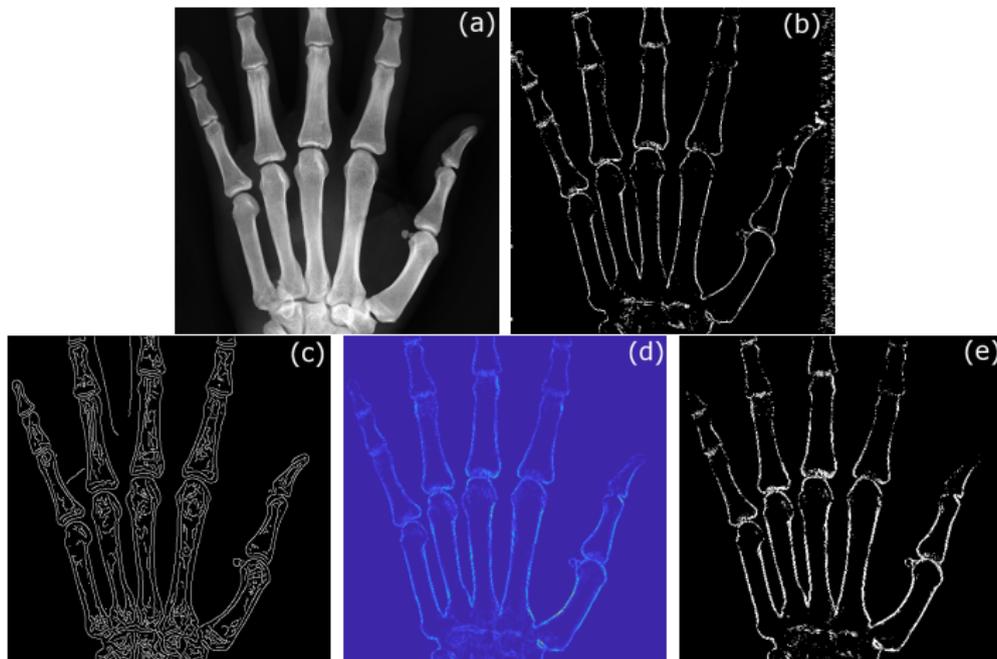
We can compare the results with the SSIM, an index of structural similarity for binary images, where 1 means that the two images are identical. If we compare Canny and our method we obtain 0.9606. If we compare Canny to the IA method we obtain 0.9221.

Photographic image (1)



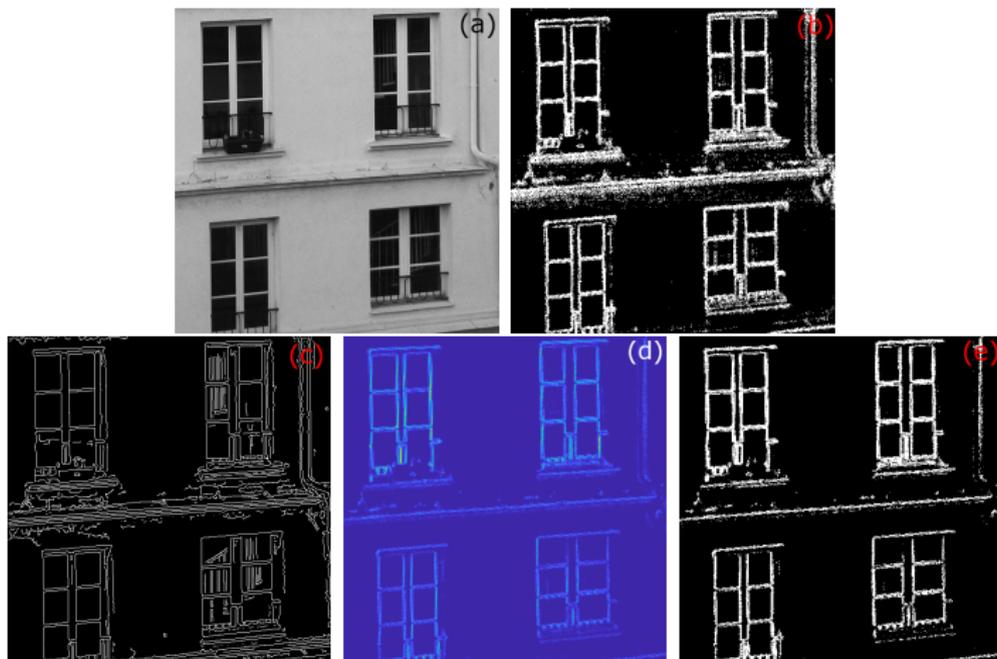
- (a) Original 512×512 pixel image. (b) Edges for IA2D dimension-by-dimension method. (c) Canny edge detection. (d) Absolute value of coefficients for dimension-by-dimension N-I method. (e) Edges for dimension-by-dimension N-I method: sum of rows and columns results. (f) Edges for dimension-by-dimension N-I method: sum of rows and columns results.

Medical image



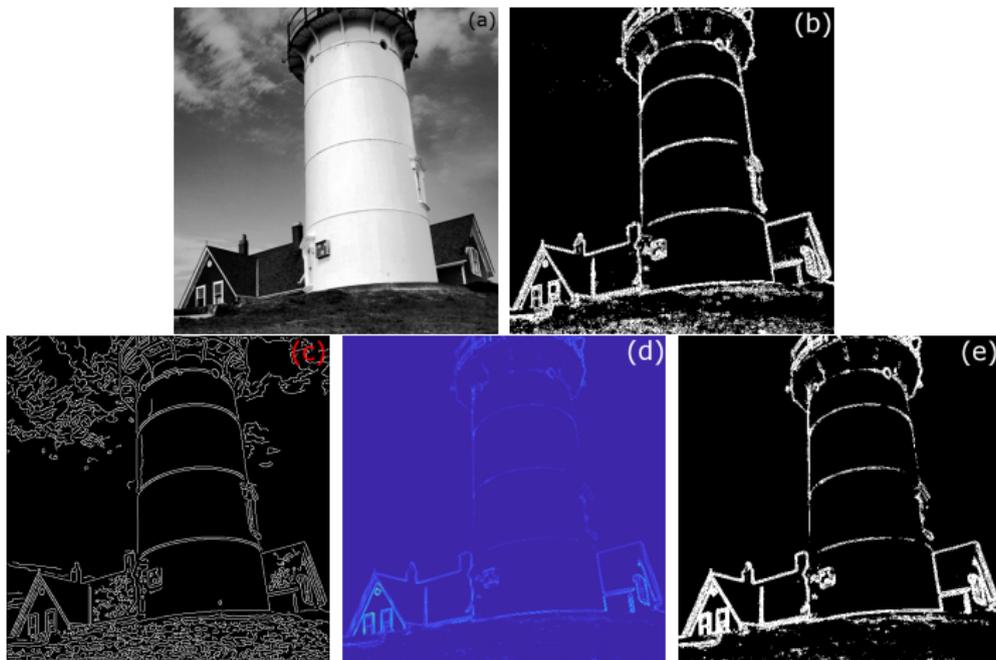
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Photographic image (2)



(a) Original 512×512 pixel image. (b) Canny edge detection. (c) Absolute value of coefficients for dimension-by-dimension N-I method. (d) Edges for dimension-by-dimension N-I method with post-processing. (e) Edges for dimension-by-dimension N-I method with post-processing.

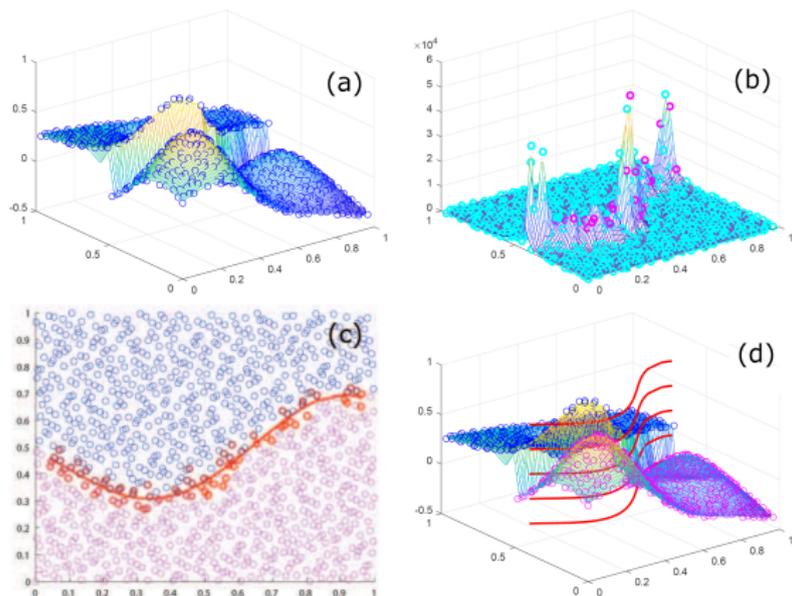
Photographic image (3)



(a) Original 512×512 pixel image. (b) Edges for IA2D dimension-by-dimension method. (c) Canny edge detection. (d) Absolute value of coefficients for dimension-by-dimension N-I method. (e) Edges for dimension-by-dimension N-I method: sum of rows and columns results.

2D example: scattered data

$$f(x, y) = \text{franke}(x, y) - \frac{1}{2}(y \leq -\frac{1}{5} \sin(5x) + \frac{1}{2})$$

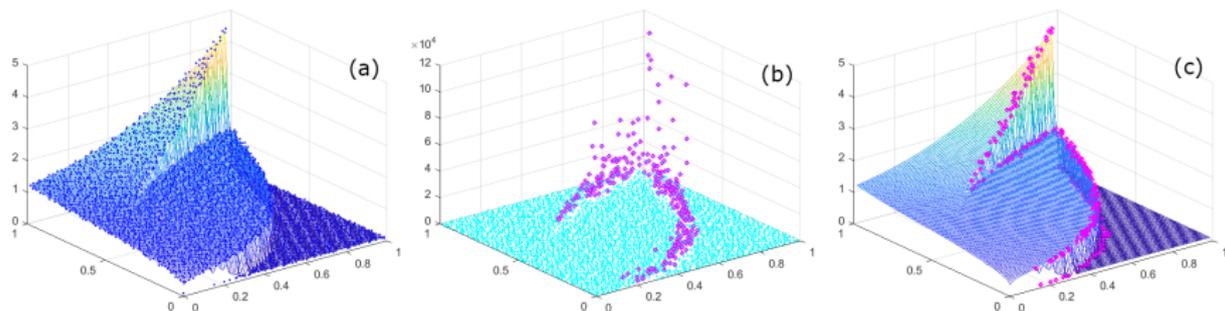


(a) Sampled data ($2^{10} = 1024$) in blue. (b) Coefficients of the Wendland interpolation, edge points in magenta. (c) 2D view of the data, in red the points recognized as edges, the red line is the polynomial interpolation of the edge points. (d) Function with sampled data in blue and magenta, the red lines are the edge.

2D example: scattered data

$$f(x, y) = (x + y - \frac{1}{2})^3(x \leq y^3) + (x + y)^{1/3}(x^3 \leq y) + \frac{1}{10}$$

In this example the coefficients are computed with a partition of the unity method for RBFs.



(a) Function with sampled data ($2^{12} = 4096$) in blue. (b) Coefficients of the RBF interpolation, with the ones corresponding to the edge points in magenta. (c) Function with sampled data corresponding to the edge points in magenta.

Thank you :)

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VSK1D algorithm

input: $\{(x_i, f_i)\}_{i=1}^N$, $c(x) = 0$, $S = \emptyset$, $\eta, \tau > 0$

- 1 compute λ by solving the system $\mathbf{M}\lambda = \mathbf{f}$
- 2 find $S' = \{x_i : |\lambda_i s'_{f,X}(x_i)| / \max_j (|\lambda_j s'_{f,X}(x_j)|) > \eta\}$,
 $S = S \cup S'$
- 3 $c^{\text{new}}(x) = c(x) + \sum_{x_i \in S'} (f_{i+1} - f_i) \mathbf{1}_{(x_i, x_N]}(x)$
- 4 compute λ^{new}
- 5 *if* $\|\lambda - \lambda^{\text{new}}\| > \tau$ *do* repeat 2. -- 4.

output: S (edge points)

N-I algorithm

input: $\{(x_i, f_i)\}_{i=1}^N, \eta \geq 1$

- 1 *if* 2D *do* divide the image in square sub-domains, apply
2. to each, 3. globally
- 2 compute λ and $\max_{\text{loc}}(|\lambda|)$
- 3 $S = \{x_i \in \max_{\text{loc}}(|\lambda|) : |\lambda_i| > \eta \text{ mean}(|\lambda|)\}$

output: S (edge points)

Popular RBFs: Multiquadric and Wendland functions

Definition

The **multiquadric functions** or **MQ** are defined as

$$\Phi_{d,\varepsilon} : \mathbb{R}^d \ni \mathbf{x} \mapsto \varphi_\varepsilon(\|\mathbf{x}\|) \in \mathbb{R}$$

where $\varphi : [0, \infty) \ni r \mapsto \varphi_\varepsilon(r) = \sqrt{\varepsilon^2 + r^2} \in \mathbb{R}$, ε is the **scale parameter**.

Definition

The **C^2 -Wendland functions** are compactly supported radial basis functions defined as

$$\Phi_s : \mathbb{R}^d \ni \mathbf{x} \mapsto \varphi(\|\mathbf{x}\|) \in \mathbb{R},$$

where $\varphi : [0, \infty) \ni r \mapsto \varphi(r) = (1 - r)_+^{\ell+1} [(\ell + 1)r + 1] \in \mathbb{R}$ such that $\ell = \lfloor s/2 \rfloor + 3$. The function Φ_s has its support in $\mathcal{S}^d(\mathbf{0}, 1)$.

