# Intrinsic Elliptic PDE on Submanifolds

## Approximate Ambient Solutions

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In euclidean space, a common partial differential equation is

$$\Delta u - \lambda u = f \qquad \text{in } \Omega$$

such that $\lambda \geq 0$ and

$$u = 0 \qquad \text{in } \partial\Omega$$

Therein, $\Delta f = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial^2 x_i}$ and $\Omega$ a suitable domain.

Now consider a hypersurface $\mathbb{M} \subseteq \mathbb{R}^d$. What does $\Delta f$ in the above version mean there?

Take for example $F : \mathbb{R}^2 \to \mathbb{R}$ to be given as $F(x, y) = x$.

$\implies \Delta F = 0$.

Now choose $\mathbb{M}$ as the unit circle, $\mathbb{M} = \{(x, y) = (\cos t, \sin t), t \in [0, 2\pi]\}$

$\implies F_{|_{\mathbb{M}}}(x, y) = f(t) = \cos t$.

From this point of view, clearly the *intrinsic* second derivative does not vanish.

Now consider a hypersurface $\mathbb{M} \subseteq \mathbb{R}^d$. What does $\Delta f$ in the above version mean there?

Now choose $\mathbb{M}$ as the unit circle, $\mathbb{M} = \{(x, y) = (\cos t, \sin t), t \in [0, 2\pi]\}$
$\Longrightarrow F_{|\mathbb{M}}(x, y) = f(t) = \cos t$.

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$$(\Delta_{\mathbb{M}} f)(t) = -\cos t.$$

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This is achieved if you replace euclidean by tangential derivatives:

DEFINITION (Tangential Derivative)

Let $F : \mathbb{R}^d \to \mathbb{R}$ and $f = F_{|\mathbb{M}}$. The **Tangential Derivative Operator $\mathbf{d}_{\mathbb{M}}$** is given as

$$\mathbf{d}_{\mathbb{M}} f := \mathbf{d}F - \pi_N(\mathbf{d}F)$$

and independent of the choice of $F$. $\pi_N$ is the projection on the pointwise normal space.

One would expect to have something like

$$(\Delta_{\mathbb{M}} f)(t) = -\cos t.$$

This is achieved if you choose the *intrinsic* Laplacian or *Laplace – Beltrami* as:

DEFINITION (Tangential Laplacian)

$$\Delta_{\mathbb{M}} f := \operatorname{div}_{\mathbb{M}} \nabla_{\mathbb{M}} f.$$

Therein, $\operatorname{div}_{\mathbb{M}}, \nabla_{\mathbb{M}}$ are given by tangential derivatives.

So in the hypersurface setting, the PDE reads like

$$\Delta_{\mathbb{M}} u - \lambda u = f \qquad \text{in } \Omega \subseteq \mathbb{M}$$

such that $\lambda \geq 0$ and

$$u = 0 \qquad \text{in } \partial\Omega.$$

There, $\Omega$ is a suitable subdomain of $\mathbb{M}$.

For the sake of simplicity, we restrict ourselves to some $\Omega = \mathbb{M}$ closed and without boundary, and $\lambda = 1$. It still remains

$$\Delta_{\mathbb{M}} u - u = f \qquad \text{in } \mathbb{M}$$

How can this be solved?

## New Attempt (M., Reif)

- Use some function space in a suitable ambient neighbourhood of $\mathbb{M}$: RBF, Splines...
- Transfer intrinsic properties into extrinsic (ambient) properties — approximately.
- Solve approximately intrinsic problem with extrinsic methods.

Pro's:

- Extrinsic function spaces are well understood.
- Extrinsic function space are applicable to any submanifold.
- Easily understood and implemented even for non-mathematicians.

## SOLUTION IDEA: BACKGROUND

THEOREM (e.g. [Dziuk/Elliot, 2013])

Let $f \in C^2(\mathbb{M})$, and $\bar{f}$ an extension that is **constant in normal directions** of $\mathbb{M}$.

Then the $1^{st}$ and $2^{nd}$ **tangential** derivatives of $f$ coincide with the **euclidean** $1^{st}$ and $2^{nd}$ derivatives of $\bar{f}$ on the tangent space.

$\Longrightarrow$ $\Delta$ and $\Delta_{\mathbb{M}}$ coincide!

# SOLUTION IDEA: BACKGROUND

THEOREM (M. 2015)

Let $f \in C^2(\mathbb{M})$, and $F$ an arbitrary extension. Then the deviations of the $1^{st}$ and $2^{nd}$ **tangential** derivatives of $f$ from the **euclidean** $1^{st}$ and $2^{nd}$ derivatives of $F$ on the tangent space are **Lipschitz functions** of the first normal directional derivatives.

Consider $\Delta_\tau f := \sum_{i=1}^k \frac{\partial^2 f}{\partial^2 \tau_i}$ with arbitrary ONB $\tau_1, ...\tau_q$ of $T_x\mathbb{M}$.

$\Longrightarrow$ For $\Delta_\tau$ and $\Delta_\mathbb{M}$ differ only in terms of the first order normal directional derivatives!

## Solution idea: Background

What does that mean?

- Euclidean derivatives of $F$ give **approximate** access to tangential derivatives of $f$
- Standard methods can be used to handle intrinsic problems **approximately**
- Intrinsic functionals are easily **approximated** by standard functionals

# New Approach (M., Reif): Solution Idea

Let $U(\mathbb{M})$ be a suitable neighbourhood of $\mathbb{M}$, best such that any point has a unique closest point on $\mathbb{M}$.

Consider a suitable function space $\mathcal{F}(U(\mathbb{M}))$ on $U(\mathbb{M})$.

$$\text{Minimize} \int_{\mathbb{M}} (\Delta_\tau u - u - f)^2$$

such that:

$$\text{Normal derivatives} \to 0$$

## OPTIMIZATION FUNCTIONAL: TENSOR-PRODUCT B-SPLINES

Minimize for grid width $h > 0$ and $\tau_1, ...\tau_q$ an ONB of $T_x\mathbb{M}$

$$\mathbf{E}_{\mathbb{M}}(s_h) := \int_{\mathbb{M}} \left| \sum_{i=1}^{k} \frac{\partial^2 s_h}{\partial^2 \tau_i} - s_h - f \right|^2 + h^{-\sigma} \int_{\mathbb{M}} \left| \frac{\partial s_h}{\partial \nu} \right|^2 + h^{-\sigma} \int_{\mathcal{C}_h(\mathbb{M})} \left| \frac{\partial s_h}{\partial \nu} \right|^2$$

$\mathcal{C}_h(\mathbb{M})$ are the spline cells that intersect $\mathbb{M}$.

$\nu = \nu(x)$ is normal to $x \in \mathbb{M}$.

$\nu = \nu(x)$ for $x \notin \mathbb{M}$ is the normal of the closest point to $x$ on $\mathbb{M}$.

$\sigma$ is a suitable penalty exponent, e.g. $\sigma = 1$.

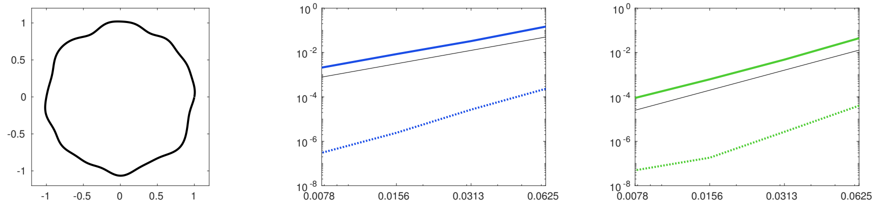
# THEORY: SOLVABILITY AND CONVERGENCE

THEOREM (M. 2017):

Assume that the PDE has a solution $u^*$ in $\mathcal{H}^{2+\ell}(\mathbb{M})$ for $\ell \in \mathbb{N}$. Let the spline order $m$ be sufficiently large.

1. For TP-Splines and sufficiently small $h$, the above problem is uniquely solvable.
2. Restrictions of optimal splines $s_h|_{\mathbb{M}}$ approach unique solution $u^*$ in $\mathcal{H}^2(\mathbb{M})$:

$$\|s_h - u^*\|_{\mathcal{H}^2} \leq Ch^\ell \, \|u^*\|_{\mathcal{H}^{2+\ell}} \, .$$

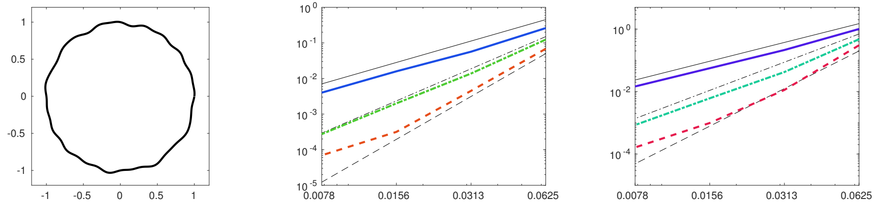



Approximation results for model equation $\Delta_{\mathbb{M}} u - 4u = f$

with solution $u^*(t) \hat{=} \cos(5t)$:

BICUBICS : — energy error (reference $h^2$), $\cdots$ RMS error

BIQUARTICS : — energy error (reference $h^3$), $\cdots$ RMS error




Approximation results for model equation $\Delta_{\mathbb{M}} u - u = f$

with solutions $u_1^*(t) \hat{=} \cos(4t)$, $u_2^*(t) \hat{=} \cos(9t) + \sin(3t)$:

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BIQUARTICS / BIQUARTICS: ·—·—· energy error (reference $h^3$), $\cdots$ RMS error

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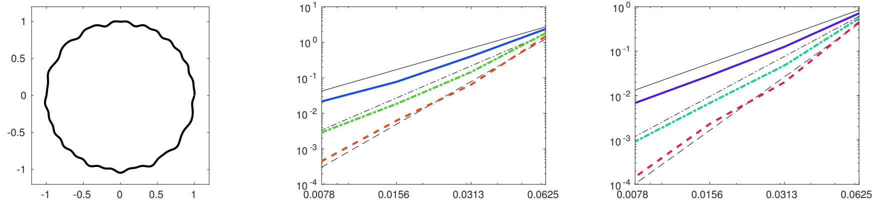


Approximation results for model equation $\Delta_{\mathbb{M}} u - u = f$

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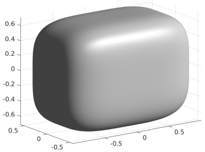
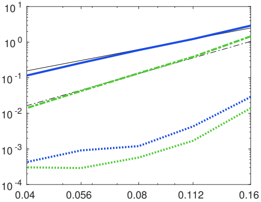
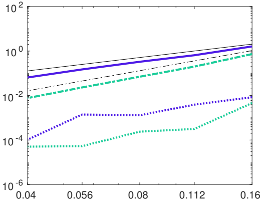
Example functions for surfaces: Restriction of the functions

$$
\begin{aligned}
u_1(x,y,z) :=& \frac{3}{4}\exp((-(9x-2)^2-(9y-2)^2)/4)+\frac{3}{4}\exp(-(9x+1)^2/49-(9y+1)/10) \\
&+\frac{1}{2}*\exp((-(9x-7)^2-(9y-3)^2)/4)-\frac{1}{5}*\exp(-(9x-4)^2-(9y-7)^2) \\
&+\sin(x+y)\exp(xy)\cos(4y+z)
\end{aligned}
$$

$$
\begin{aligned}
u_2(x,y,z) :=& \frac{1}{4}\exp((-(9x-2)^2-(9y-2)^2)/4)+\frac{1}{4}\exp(-(9x+1)^2/49-(9y+1)/10) \\
&+\frac{3}{4}\exp((-(9x-7)^2-(9y-3)^2)/4)-\frac{1}{4}\exp(-(9x-4)^2-(9y-7)^2) \\
&+\cos(x+y)\log(x^2y^2+1)\sin(4y^2+z)
\end{aligned}
$$




Approximation results for model equation $\Delta_{\mathbb{M}} u - u = f$ with solutions $u_1$, $u_2$:

BICUBICS / BICUBICS: — energy / residual error (reference $h^2$), $\cdots$ $L_2$-error

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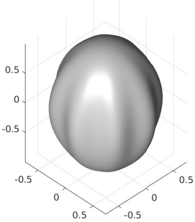


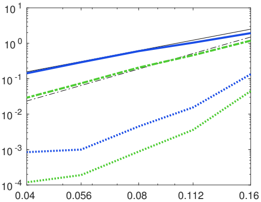


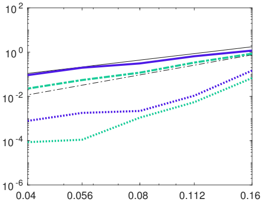


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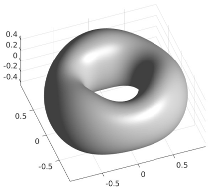


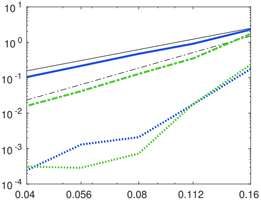


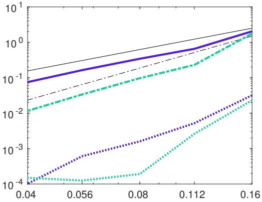


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# SUMMARY

New approach to certain partial differential equations on closed submanifolds:

- Provides satisfactory convergence
- Applies well-known concepts in novel setting
- Easy to implement
- Produces pleasant results