Adaptive anisotropic approximation of multivariate functions by piecewise constants

O. Davydov[1], O. Kozynenko[2], D. Skorokhodov[2]
[1]Justus-Liebig-Universität Gießen, Gießen, Germany
[2]Oles Honchar Dnipro National University, Dnipro, Ukraine

IM-Workshop
Bernried, February 19 – 23, 2018

Definitions: convex partition

Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, be a bounded domain.

Partition

A finite collection $\Delta = \{\omega\}$ of subdomains $\omega \subset \Omega$ is called a *partition* of Ω provided that $\omega \cap \omega' = \varnothing$, for any $\omega, \omega' \in \Delta$, $\omega \neq \omega'$, and $\sum_{\omega \in \Delta} |\omega| = |\Omega|$, where $|\cdot|$ is the Lebesgue measure.

Convex partition

A partition Δ is called *convex* if every cell $\omega \in \Delta$ is convex.

For $N \in \mathbb{N}$, denote by $\mathfrak{D}_N$ the set of all convex partitions of Ω comprising N cells.

Definitions: Sobolev space

For $1 \leqslant q \leqslant \infty$ and $k \in \mathbb{N}$, by $W_q^k(\Omega)$ we denote the standard Sobolev space of functions $f : \Omega \to \mathbb{R}$ endowed with the norm

$$\|f\|_{W_q^k(\Omega)} = \sum_{\alpha \in \mathbb{Z}_+^d \,:\, |\alpha| \leqslant k} \|D^\alpha f\|_{L_q(\Omega)},$$

where $\alpha \in \mathbb{Z}_+^d$ is the multi-index.

Definitions: Error of the best L_p-approximation

For $1 \leqslant p \leqslant \infty$, we define the error of the best L_p-approximation of a function $f : \Omega \to \mathbb{R}$ by picewise constant functions on N cells:

$$E_N(f)_p := \inf_{\Delta \in \mathfrak{D}_N} \inf_{s \in \mathcal{S}(\Delta)} \|f - s\|_{L_p(\Omega)},$$

where $\mathcal{S}(\Delta)$ is the space of functions $s : \Omega \to \mathbb{R}$ constant on every $\omega \in \Delta$.

History

- Birman M. S., Solomyak M. Z. (1967): for $f \in W_q^1(\Omega)$, $E_N(f)_p = O\left(N^{-1/d}\right)$ as $N \to \infty$ provided that $\frac{1}{d} + \frac{1}{p} - \frac{1}{q} > 0$
- R. A. DeVore (1990) established that for f from Besov space $B_{q,\sigma}^{\alpha}(\Omega)$, the quantity $E_N(f)_p$ behaves as $O\left(N^{-\alpha/d}\right)$ as $N \to \infty$ provided that $\frac{\alpha}{d} + \frac{1}{p} - \frac{1}{q} > 0$
- O. Davydov (2012) : $E_N(f)_p = O\left(N^{-2/(d+1)}\right)$ as $N \to \infty$ for functions $f \in W_p^2(\Omega)$
- O. Davydov (2012) : proved that the saturation order of piecewise constant approximation in L_p norm on convex partitions is $N^{-2/(d+1)}$, and achievable for any $f \in W_p^2(\Omega)$.

Algorithm proposed by Professor Davydov for $f \in W_p^2(\Omega)$

Assume $f \in W_p^2(\Omega)$, $\Omega = (0,1)^d$. Split Ω into $N_1 = m^d$ cubes $\omega_1, ..., \omega_{N_1}$ of edge length $h = \frac{1}{m}$. Then split each ω_i into N_2 slices ω_{ij} , $j = 1, ..., N_2$, by equidistant hyperplanes orthogonal to the average gradient $h_i := |\omega_i|^{-1} \int_{\omega_i} \nabla f$ on ω_i. Set $\Delta = \left\{\omega_{ij} : i = 1, ..., N_1, j = 1, ..., N_2\right\}$, and define the piecewise constant approximation $s_\Delta(f) := \sum_{\delta \in \Delta} f_\delta \chi_\delta$. Clearly, $|\Delta| = N_1 N_2$ and each ω_{ij} is a convex polyhedron with at most $2(d+1)$ facets.

Illustration of the algorithm

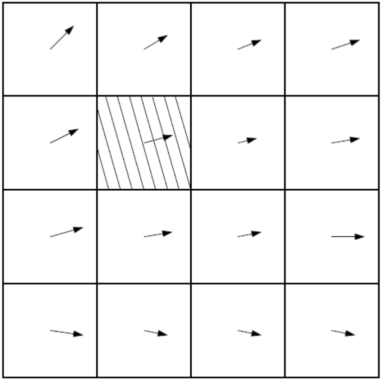

Fig. 1: Demonstration of the algorithm when d=2, m=4. The arrows stand for the average gradients h_i on the cubes ω_i. The cells ω_{ij} are shown only for one cube.

Adaptive algorithm proposed by Birman and Solomyak

Suppose we are interested in approximation in L_p norm, $1 \leqslant p \leqslant \infty$. Choose $1 \leqslant q < \infty$, such that $\frac{1}{d} + \frac{1}{p} - \frac{1}{q} \geqslant 0$, and assume that $f \in W_q^1(\Omega), \Omega = (0,1)^d$. Set $\Delta_0 = \Omega$. While $|\Delta_k| < N$, obtain Δ_{k+1} from Δ_k by the dyadic subdivision of those cubes $\omega \in \Delta_k$, for which

$$g_\alpha(\omega) \geqslant 2^{-d\alpha} \max_{\omega \in \Delta_k} g_\alpha(\omega),$$

$$g_\alpha(\omega) := |\omega|^\alpha |f|^q_{W_q^1(\omega)}, \alpha = q\left(\frac{1}{d} + \frac{1}{p} - \frac{1}{q}\right).$$

Since $|\Delta_k| < |\Delta_{k+1}|$, the subdivisions terminate at some $\Delta = \Delta_m$ with $|\Delta_m| \geqslant N$ and $|\Delta_m| = O(N)$. The resulting piecewise constant approximation $s_\Delta(f)$ of f is $s_\Delta(f) := \sum_{\delta \in \Delta} f_\delta \chi_\delta$.

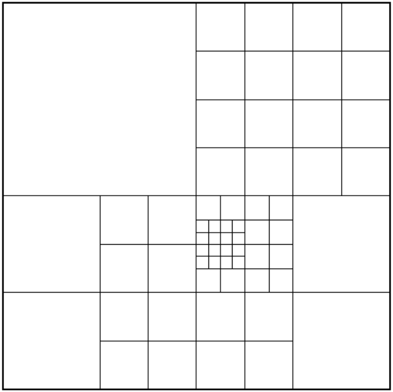

Fig. 2: Dyadic partition.

Theorem 1

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$ be such that $\frac{2}{d+1} + \frac{1}{p} - \frac{1}{q} \geqslant 0$, and let $f \in W_q^2(\Omega)$. Then

$$E_N(f)_p \leqslant C(d,p,q) N^{-\frac{2}{d+1}} \left(|f|^q_{W_q^1(\Omega)} + |f|^q_{W_q^2(\Omega)} \right)^{\frac{1}{q}},$$

with the constant $C(d,p,q)$ independent on f.
Remark. Condition when $W_q^2(\Omega) \subset L_p(\Omega)$ is $\frac{2}{d} + \frac{1}{p} - \frac{1}{q} \geqslant 0$.

Lemma 1

$\Phi(\omega)$ – a nonnegative function of sets $\omega \subset \Omega$, subadditive in sence: if set $\{\omega_i\}$ is such, that $\bigcup_i \omega_i = \omega$ and $\omega_i \cap \omega_j = \varnothing$ then $\sum_i \Phi(\omega_i) \leq \Phi(\omega)$. Set

$$g_\alpha(\omega) := |\omega|^\alpha \Phi(\omega), \quad \omega \subset \Omega.$$

$\{\Delta_k\}_{k=0}^\infty$ – sequence of partitions of bounded domain $\Omega \subset \mathbb{R}^d$ into dyadic cubes. $\Delta_0 = \{\Omega\}$. Obtain Δ_{k+1} from Δ_k by one subdivision into 2^d cubes of those cubes $\omega \in \Delta_k$ for which

$$g_\alpha(\omega) \geqslant 2^{-d\alpha} \max_{\omega \in \Delta} g_\alpha(\omega).$$

For each $k \geqslant 1$ split each $\omega_i \in \Delta_k$ into $N_{i,k} \leqslant |\omega_i|^{-\gamma}$ pieces, where $\alpha \geqslant \gamma$. $N_k := \sum_i N_{i,k}$, then

$$\max_{\omega \in \Delta_k} g_\alpha(\omega) \leqslant C(d,\alpha) N_k^{-\frac{\alpha+1}{\gamma+1}} \Omega^{\frac{\alpha-\gamma}{\gamma+1}} \Phi(\Omega), \quad k = 0, 1, \ldots$$

Algorithm 1

Let $1 \leqslant p \leqslant \infty$, choose $1 \leqslant q < \infty$, such that $\frac{2}{d+1} + \frac{1}{p} - \frac{1}{q} \geqslant 0$, and assume $f \in W_q^2(\Omega), \Omega = (0,1)^d$. Set $\Delta_0 = \Omega$. While $\sum_i |\omega_i|^{-1/d} < N$, obtain Δ_{k+1} from Δ_k by the dyadic subdivision of cubes $\omega \in \Delta_k$, for which

$$g_\alpha(\omega) \geqslant 2^{-d\alpha} \max_{\omega \in \Delta_k} g_\alpha(\omega), \quad g_\alpha(\omega) := |\omega|^\alpha \Phi(\omega),$$

$$\alpha = q\left(\frac{2}{d} + \frac{1}{p} - \frac{1}{q} + \frac{1}{dp}\right), \quad \Phi(\omega) := |f|^q_{W_q^1(\omega)} + |f|^q_{W_q^2(\omega)}.$$

The subdivisions terminate at some $\Delta = \Delta_m$. Each cube $\omega_i \in \Delta_m$ split into $|\omega_i|^{-\frac{1}{d}}$ slices by hyperplanes orthogonal to $h_i := |\omega_i|^{-1} \int_{\omega_i} \nabla f dx$ on ω_i, and which make one dimensional dyadic subdivision of cube. The final partition we denote as Δ. The resulting piecewise constant approximation $s_\Delta(f)$ of f is $s_\Delta(f) := \sum_{\delta \in \Delta} f_\delta \chi_\delta$.

Illustration of the algorithm

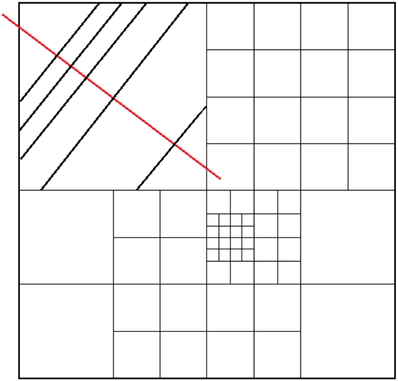

Fig. 3: Demonstration of the algorithm. The cells are shown only for one cube.

Theorem 2

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$ be such that $\frac{2}{d+1} + \frac{1}{p} - \frac{1}{q} < 0$, $\frac{2}{d} - \frac{1}{d^2} + \frac{1}{p} - \frac{1}{q} > 0$ and let $f \in W_q^2(\Omega)$. Then

$$E_N(f)_p \leqslant C(d,p,q) N^{-d\left(\frac{2}{d}+\frac{1}{p}-\frac{1}{q}\right)} \left(|f|^q_{W_q^1(\Omega)} + |f|^q_{W_q^2(\Omega)} \right)^{\frac{1}{q}},$$

with the constant $C(d,p,q)$ independent on f.

Lemma 2

$\Phi(\omega)$ – a nonnegative function of sets $\omega \subset \Omega$, subadditive in sence: if set $\{\omega_i\}$ is such, that $\bigcup_i \omega_i = \omega$ and $\omega_i \cap \omega_j = \varnothing$ then $\sum_i \Phi^{\frac{\gamma}{\alpha}}(\omega_i) \leqslant \Phi^{\frac{\gamma}{\alpha}}(\omega)$. Set

$$g_\alpha(\omega) := |\omega|^\alpha \Phi(\omega), \quad \omega \subset \Omega, \quad \alpha < \gamma.$$

$\{\Delta_k\}_{k=0}^\infty$ – sequence of partitions of bounded domain $\Omega \subset \mathbb{R}^d$ into dyadic cubes. $\Delta_0 = \{\Omega\}$. Obtain Δ_{k+1} from Δ_k by one subdivision into 2^d cubes of those cubes $\omega \in \Delta_k$ for which

$$g_\alpha(\omega) \geqslant 2^{-d\alpha} \max_{\omega \in \Delta} g_\alpha(\omega).$$

For each $k \geqslant 1$ split each $\omega_i \in \Delta_k$ into $N_{i,k} \leqslant |\omega_i|^{-\gamma}$ pieces. $N_k := \sum_i N_{i,k}$, then

$$\max_{\omega \in \Delta_k} g_\alpha(\omega) \leqslant C(d,\alpha) N_k^{-\frac{\alpha}{\gamma}} \Phi(\Omega), \quad k = 0, 1, \ldots$$