Analysis of Shearlet Coorbit Spaces

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joint work with Hartmut Führ

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Motivation

- Signal $f \in L^2(\mathbb{R}^d), W^{k,p}(\mathbb{R}^d), \dots$
- Decompose f with respect to elementary building blocks.
- This decomposition is based on prior transform of f.
- Behind this transform is (in some cases) the action of a group.

Outline

1. Introduction: shearlet groups and associated coorbit spaces

2. Coorbit spaces as decomposition spaces

3. Rigidity of decomposition spaces and coarse geometry

4. Comparison of shearlet coorbit spaces

Overview

Shearlet groups in dimension 2

The original shearlet group in dimension 2 is given by

$$H = \left\{ \pm \begin{pmatrix} a & ab \\ 0 & a^{1/2} \end{pmatrix} : \begin{array}{c} a > 0, \\ b \in \mathbb{R} \end{array} \right\}$$

$$= \left\{ \pm \begin{pmatrix} a & 0 \\ 0 & a^{1/2} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : \begin{array}{c} a > 0, \\ b \in \mathbb{R} \end{array} \right\}.$$

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Generalized Shearlet Groups

Let $H \leq \mathrm{GL}(d, \mathbb{R})$ be an admissible group. H is called *generalized shearlet dilation group* if there exist two closed subgroups $S, D \leq H$ such that

- S is a connected closed abelian subgroup of $T(d, \mathbb{R})$, where $T(d, \mathbb{R})$ is the set of upper triangular matrices with 1 on their diagonal,
- $D = \{\exp(rY) \mid r \in \mathbb{R}\}$ for some diagonal matrix Y,
- every $h \in H$ can be uniquely written as $h = \pm ds$ for some $d \in D$ and $s \in S$.

Shearlet groups in dimension 3

Standard shearlet group

$$H_\lambda := \left\{ \pm \begin{pmatrix} a & 0 & 0 \\ 0 & a^{\lambda_1} & 0 \\ 0 & 0 & a^{\lambda_2} \end{pmatrix} \begin{pmatrix} 1 & b_1 & b_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{l} a > 0, \\ b_1, b_2 \in \mathbb{R} \end{array} \right\} < \mathrm{GL}(3, \mathbb{R})$$

for $\lambda := (\lambda_1, \lambda_2) \in \mathbb{R}^2$.

Toeplitz shearlet group

$$H_\delta := \left\{ \pm \begin{pmatrix} a & 0 & 0 \\ 0 & a^{1-\delta} & 0 \\ 0 & 0 & a^{1-2\delta} \end{pmatrix} \begin{pmatrix} 1 & b_1 & b_2 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{l} a > 0, \\ b_1, b_2 \in \mathbb{R} \end{array} \right\} < \mathrm{GL}(3, \mathbb{R})$$

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for $\delta \in \mathbb{R}$.

Shearlet transform

- For a shearlet group H define $G := \mathbb{R}^d \rtimes H$ with group law

$$(x, h) \circ (y, g) = (x + hy, hg)$$

- Unitary representation π of G on $L^2(\mathbb{R}^d)$

$$[\pi(x, h)\psi](y) = |\det(h)|^{-1/2}\psi(h^{-1}(y - x))$$

- For $\psi, f \in L^2(\mathbb{R}^d)$, we define the *continuous shearlet transform*

$$\mathcal{S}_\psi f : (x, h) \mapsto \langle f, \pi(x, h)\psi \rangle$$

and the associated *continuous shearlet system*

$$\mathcal{S}(\psi) := \{\pi(x, h)\psi : (x, h) \in G\}$$

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Shearlet coorbit spaces

A function $0 \neq \psi \in L^2(\mathbb{R}^d)$ is called *admissible shearlet* if $\mathcal{S}_\psi \psi \in L^2(G)$, i.e.

$$\int_G |\langle \psi, \pi(x,h)\psi \rangle|^2 \, \mathrm{d}\mu_G(x,h) < \infty.$$

For an admissible shearlet ψ the map

$$S_\psi : L^2(\mathbb{R}^d) \to L^2(G), \; f \mapsto S_\psi f$$

is a multiple of an isometry, which implies the inversion formula

$$f = \frac{1}{C_\psi} \int_G \mathcal{S}_\psi f(x,h) \, \pi(x,h)\psi \, \mathrm{d}\mu_G(x,h).$$

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Coorbit theory

For measurable, locally bounded, submultiplicative weight $v : H \to (0, \infty)$ and $p, q \in (1, \infty)$ define the weighted mixed $L^{p,q}$-norm

$$\|f\|_{L_v^{p,q}} := \left(\int_H \left(\int_{\mathbb{R}^3} v(h)^p \, |f(x,h)|^p \, \mathrm{d}x \right)^{q/p} \frac{\mathrm{d}h}{|\det(h)|} \right)^{1/q}$$

For a shearlet ψ define the coorbit space norm $\|f\|_{\mathrm{Co}(L_v^{p,q})} := \|\mathcal{S}_\psi f\|_{L_v^{p,q}}$.
The *coorbit space* $\mathrm{Co}\left(L_v^{p,q}(G)\right)$ is given as completion of

$$\left\{ f \in L^2(\mathbb{R}^d) \, : \, \mathcal{S}_\psi f \in L_v^{p,q}(G) \right\}.$$

Features of coorbit theory

- Consistency: independence of ψ
- Discretization: $f = \sum_{i \in I} \lambda_i(f) \pi(g_i) \psi$

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Shearlet coorbit spaces

Remark (Führ & Voigtlaender)

Different dilation groups can induce the same coorbit space.

Goal

Different shearlet groups induce different shearlet coorbit spaces!

In order to improve our understanding of the associated coorbit spaces

- $Co\left(L_v^{p,q}(\mathbb{R}^d \rtimes H_\lambda)\right)$
- $Co\left(L_v^{p,q}(\mathbb{R}^d \rtimes H_\delta)\right)$

the next aim is to identify them with certain decomposition spaces.

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Decomposition spaces

Definition (decomposition space)

Let $p, q \in (1, \infty)$, $\mathcal{Q} = (Q_i)_{i \in I}$ a covering of $\mathcal{O}$ and $u : I \to \mathbb{R}^{>0}$ a discrete weight. Then define for a suitable partition of unity $(\varphi_i)_{i \in I}$ subordinate to $\mathcal{Q}$ the norm

$$\|f\|_{\mathcal{D}(\mathcal{Q}, L^p, \ell_u^q)} = \left\| \left(u_i \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p} \right)_{i \in I} \right\|_{\ell^q}$$

and the space

$$\mathcal{D}(\mathcal{Q}, L^p, \ell_u^q) = \left\{ f \in \mathcal{D}'(\mathcal{O}) : \|f\|_{\mathcal{D}(\mathcal{Q}, L^p, \ell_u^q)} < \infty \right\}$$

Dual action and dual orbit

The dual action is given by $H \times \mathbb{R}^d \to \mathbb{R}^d, (h, \xi) \mapsto h^{-t}\xi$ and for a shearlet group this action has a unique open dual orbit $H^{-t}\xi_0 = \mathcal{O} = \mathbb{R}^* \times \mathbb{R}^{d-1}$.

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Decomposition spaces

Definition (induced covering)

The family $\mathcal{Q} = (h_i^{-T} Q)_{i \in I}$ is a covering of $\mathcal{O}$ induced by H if

- $Q \subset \mathcal{O}$ is open with $\overline{Q} \subset \mathcal{O}$ compact
- The set of elements $(h_i)_{i \in I}$ is well-spread in H, i.e.
  - $(h_i V)_{i \in I}$ is pairwise disjoint for a suitable unit neighborhood $V \subset H$
  - $(h_i U)_{i \in I}$ covers H for some relatively compact unit neighborhood $U \subset H$
- $\mathcal{Q}$ covers $\mathcal{O}$

Intuition: This covering *determines* the coorbit/decomposition space associated to it.

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Let $\mathcal{Q}$ be a covering induced by H and define the discrete weight $u = (u_i)_{i \in I}$ by

$$u_i := |\det(h_i)|^{\frac{1}{2} - \frac{1}{q}} v(h_i) \text{ for } i \in I.$$

Theorem (Führ & Voigtlaender 2014)

The Fourier transform

$$\mathcal{F} : \mathrm{Co}(L_v^{p,q}(G)) \to \mathcal{D}(\mathcal{Q}, L^p, \ell_u^q)$$

is an isomorphism of Banach spaces.

Decomposition spaces

Let $\mathcal{Q}$ be a covering induced by H and define the discrete weight $u = (u_i)_{i \in I}$ by

$$u_i := |\det(h_i)|^{\frac{1}{2} - \frac{1}{q}} v(h_i) \text{ for } i \in I.$$

Theorem (Führ & Voigtlaender 2014)

The Fourier transform

$$\mathcal{F} : \mathrm{Co}(L_v^{p,q}(G)) \to \mathcal{D}(\mathcal{Q}, L^p, \ell_u^q)$$

is an isomorphism of Banach spaces.

Overview

Let $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ be two induced coverings (**potentially induced by different groups!**).

Definition (intersection sets)

Define the intersection sets of $\mathcal{Q}$ and $\mathcal{P}$ for $i \in I$ and $j \in J$ by

$$I_j := \{i \in I : Q_i \cap P_j \neq \emptyset\} \text{ and } J_i := \{j \in J : Q_i \cap P_j \neq \emptyset\}.$$

Definition (weak equivalence)

We call the coverings $\mathcal{Q}$ and $\mathcal{P}$ weakly equivalent if

$$\sup_{j \in J} |I_j| < \infty \text{ and } \sup_{i \in I} |J_i| < \infty.$$

Rigidity of decomposition spaces

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Rigidity of decomposition spaces

Rigidity Theorem (Voigtlaender 2016)

Let $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ be induced coverings, $p_1, p_2, q_1, q_2 \in (1, \infty)$ and $u = (u_i)_{i \in I}$, $u' = (u'_j)_{j \in J}$ discrete weights. If

$$\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right) = \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell^{q_2}_{u'}\right),$$

then

1. $p_1 = p_2$ and $q_1 = q_2$,
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Coarse Geometry

- Study of metric spaces from a *large scale point of view*.
- Induced coverings can be used to define a metric on their associated orbits.
- For a metric space (X, d) define the *bounded coarse structure* on X as the family of sets

$$\mathcal{C} := \left\{ A \subseteq X \times X : \sup_{(x,y)\in A} d(x,y) < \infty \right\}.$$

- Weak equivalence of coverings is a property of their coarse structure

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Definition (quasi-isometry)

Let $(X, d_X), (Y, d_Y)$ be metric spaces. We call a map $f : X \to Y$ *quasi-isometric (embedding)* if there exist $c, d, C, D > 0$ such that

$$cd_X(x, x') - d \leq d_Y(f(x), f(x')) \leq Cd_X(x, x') + D$$

holds for all $x, x' \in X$.

coarse equivalence

Let $(X, d_X), (Y, d_Y)$ be metric spaces. Then X and Y are *coarsely equivalent* if there exist quasi-isometric maps $f : X \to Y$, $g : Y \to X$ such that

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Definition (induced metric on dual orbit)

For an induced covering $\mathcal{Q} = (Q_i)_{i \in I}$ of an orbit $\mathcal{O} \subseteq \mathbb{R}^n$ consisting of relatively compact, open and connected sets, define a metric $d_{\mathcal{Q}}$ on $\mathcal{O}$ by

$$d_{\mathcal{Q}}(x,y) := \begin{cases} \inf \left\{ n \in \mathbb{N} : \begin{array}{l} \exists Q_{i_1}, \ldots, Q_{i_n} \in \mathcal{Q} \text{ s.t. } x \in Q_{i_1}, \\ y \in Q_{i_n}, Q_{i_j} \cap Q_{i_{j+1}} \neq \emptyset \end{array} \right\}, & \text{for } x \neq y \\ 0, & \text{for } x = y. \end{cases}$$

Definition (induced metric on the group)

For a group $H \leq \mathrm{GL}(d, \mathbb{R})$ and a relatively compact, symmetric, connected neighborhood of the identity $W \subset H$, define a metric d_W on H by

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Theorem (Führ & K. 2017)

Let $\mathcal{Q}, \mathcal{P}$ be induced coverings of an orbit $\mathcal{O}$ consisting of open connected sets. Then the following statements are equivalent

1. $\mathcal{Q}$ and $\mathcal{P}$ are weakly equivalent.
2. $\mathrm{id} : (\mathcal{O}, d_{\mathcal{Q}}) \to (\mathcal{O}, d_{\mathcal{P}})$ is a quasi-isometry.

Theorem (Führ & K. 2017)

If $H \leq \mathrm{GL}(d, \mathbb{R})$ is an admissible group with dual orbit $\mathcal{O} \subset \mathbb{R}^d$ and $\xi \in \mathcal{O}$ such that $H_\xi \subset H_0$, where $H_\xi = \{h \in H : \; h^{-t}\xi = \xi\}$ and H_0 is the connected component of E, then the orbit map

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Overview

Standard shearlet group

$$H_\lambda := \left\{ \pm \begin{pmatrix} a & 0 & 0 \\ 0 & a^{\lambda_1} & 0 \\ 0 & 0 & a^{\lambda_2} \end{pmatrix} \begin{pmatrix} 1 & b_1 & b_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{l} a > 0, \\ b_1, b_2 \in \mathbb{R} \end{array} \right\} < \mathrm{GL}(3, \mathbb{R})$$

for $\lambda := (\lambda_1, \lambda_2) \in \mathbb{R}^2$.

Shearlet groups are admissible with $H_\xi = \{E\}$ for $\xi = e_1$. We get the following commutative diagram:

$$\begin{array}{ccc}
(\mathcal{O}, d_{\mathcal{Q}_\lambda}) & \xrightarrow{\quad \mathrm{id} \quad} & (\mathcal{O}, d_{\mathcal{Q}_{\lambda'}}) \\
\Big\uparrow p_\xi^\lambda & & \Big\downarrow \left[p_\xi^{\lambda'}\right]^{-1} \\
(H_\lambda, d_{W_\lambda}) & \xrightarrow{\quad \varphi_\xi^{\lambda,\lambda'} \quad} & (H_{\lambda'}, d_{W_{\lambda'}})
\end{array}$$

with $\varphi_\xi^{\lambda,\lambda'} := \left[p_\xi^{\lambda'}\right]^{-1} \circ \mathrm{id} \circ p_\xi^\lambda$.

Lemma (Führ & K. 2017)

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Application to standard shearlet groups ($\lambda \neq \lambda'$):

- Determine sequences $(h_n)_{n\in\mathbb{N}}$, $(h'_n)_{n\in\mathbb{N}}$ in H_λ such that

$$d_{W_\lambda}(h_n, h'_n) \leq K$$

for some $K > 0$ and all $n \in \mathbb{N}$, but

$$d_{W_{\lambda'}}\left(\varphi_\xi^{\lambda,\lambda'}(h_n), \varphi_\xi^{\lambda,\lambda'}(h'_n)\right) \xrightarrow{n\to\infty} \infty.$$

- This implies that $\varphi_\xi^{\lambda,\lambda'}$ is not a quasi-isometry.
- The coverings $\mathcal{Q}_\lambda$ and $\mathcal{Q}_{\lambda'}$ are not weakly equivalent.
- The associated decomposition spaces don't coincide.

Main point

Different shearlet groups lead to substantially different coverings of the dual orbit through the dual action. We can show this **without** actually computing any covering.

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Main point

Different shearlet groups lead to substantially different coverings of the dual orbit through the dual action. We can show this **without** actually computing any covering.

Assume $\lambda_1 < \lambda_1'$ and let

$$W_\lambda' := \left\{ \begin{pmatrix} a & ab_1 & ab_2 \\ 0 & a^{\lambda_1} & 0 \\ 0 & 0 & a^{\lambda_2} \end{pmatrix} \middle| \begin{array}{l} 2/3 \le a \le 4/3 \\ -1 \le b_i \le 1 \end{array} \right\},$$

then set $W_\lambda := W_\lambda' \cup (W_\lambda')^{-1}$. Furthermore, define $a_n := n$ for $n \in \mathbb{N}$, $b := 1$, $b' := 2$ and

$$h_n := \begin{pmatrix} a_n & a_n b & 0 \\ 0 & a_n^{\lambda_1} & 0 \\ 0 & 0 & a_n^{\lambda_2} \end{pmatrix}, \quad h_n' := \begin{pmatrix} a_n & a_n b' & 0 \\ 0 & a_n^{\lambda_1} & 0 \\ 0 & 0 & a_n^{\lambda_2} \end{pmatrix}.$$

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Then we get

$$h_n^{-1} h_n' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in W_\lambda \Longrightarrow d_{W_\lambda}(h_n, h_n') = 1,$$

and

$$A_n := \left[\varphi_\xi^{\lambda,\lambda'}(h_n)\right]^{-1} \varphi_\xi^{\lambda,\lambda'}(h_n') = \begin{pmatrix} 1 & a_n^{\lambda_1'-\lambda_1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\left|a_n^{\lambda_1'-\lambda_1}\right| \xrightarrow{n\to\infty} \infty$, the sequence $(A_n)_{n\in\mathbb{N}}$ is not contained in any relatively compact set.

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$$h_n^{-1} h_n' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in W_\lambda \Longrightarrow d_{W_\lambda}(h_n, h_n') = 1,$$

and

$$A_n := \left[\varphi_\xi^{\lambda,\lambda'}(h_n)\right]^{-1} \varphi_\xi^{\lambda,\lambda'}(h_n') = \begin{pmatrix} 1 & a_n^{\lambda_1'-\lambda_1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\left|a_n^{\lambda_1'-\lambda_1}\right| \xrightarrow{n\to\infty} \infty$, the sequence $(A_n)_{n\in\mathbb{N}}$ is not contained in any relatively compact set.

Then we get

$$h_n^{-1} h'_n = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in W_\lambda \Longrightarrow d_{W_\lambda}(h_n, h'_n) = 1,$$

and

$$A_n := \left[\varphi_\xi^{\lambda,\lambda'}(h_n)\right]^{-1} \varphi_\xi^{\lambda,\lambda'}(h'_n) = \begin{pmatrix} 1 & a_n^{\lambda'_1 - \lambda_1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\left| a_n^{\lambda'_1 - \lambda_1} \right| \xrightarrow{n \to \infty} \infty$, the sequence $(A_n)_{n \in \mathbb{N}}$ is not contained in any relatively compact set.

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