Sparse interpolation, exponential analysis, Padé approximation, tensor decomposition

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Signals, images and approximation
Bernried, 18–23/02/2018

Universiteit Antwerpen
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Motivation

Motivation

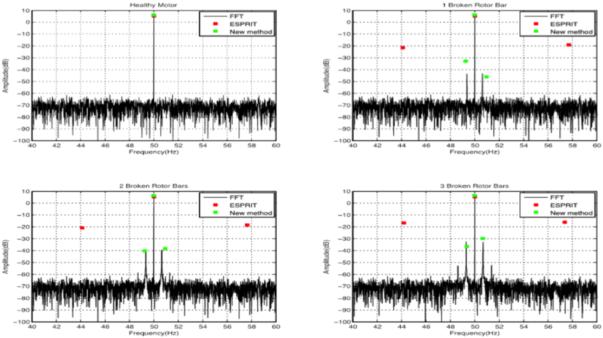
- liquid explosives identification
- induction motor current signature analysis (MCSA)
- magnetic resonance spectroscopy/imaging (MRS/MRI)
- fluorescence lifetime imaging (FLIM)
- direction/angle of arrival problems (DOA/AOA)
- radar imaging (ISAR/SAR)
- automotive radar
- transient detection
- echolocation
- speech/music signal processing
- ...

Motivation

liquid explosives identification

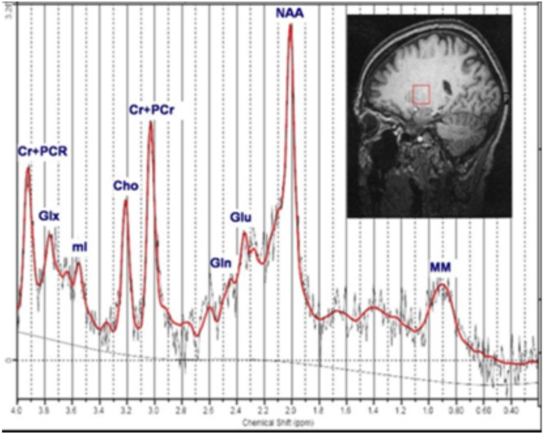
Motivation

induction motor current signature analysis (MCSA)



Motivation

magnetic resonance spectroscopy/imaging (MRS/MRI)



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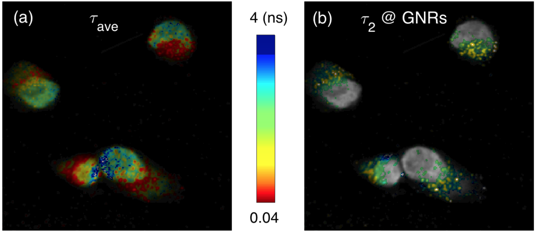
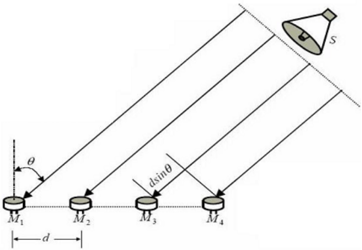
Motivation

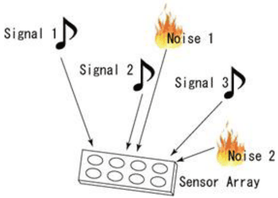
fluorescence lifetime imaging (FLIM)

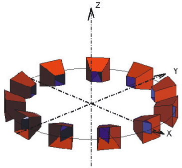


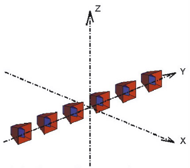
Motivation

direction/angle of arrival problems (DOA/AOA)



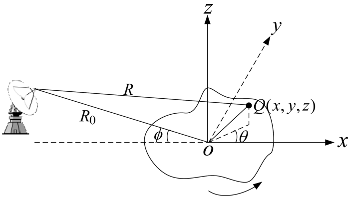


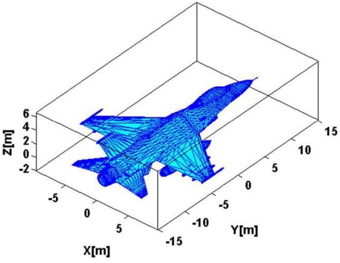




Motivation

radar imaging (ISAR/SAR)





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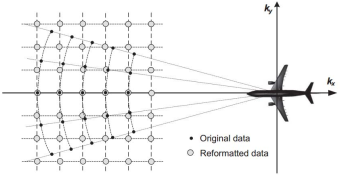
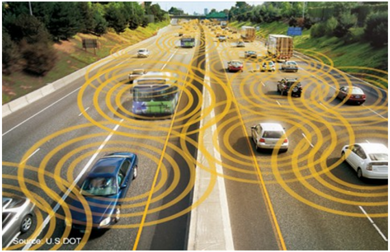
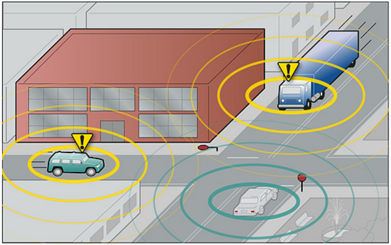


Motivation

automotive radar

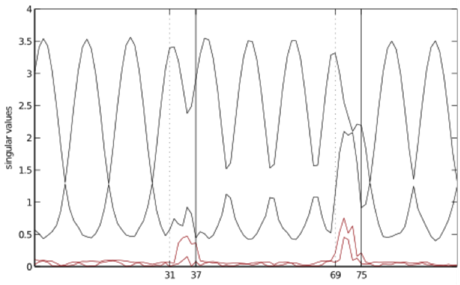




Source: GAO.

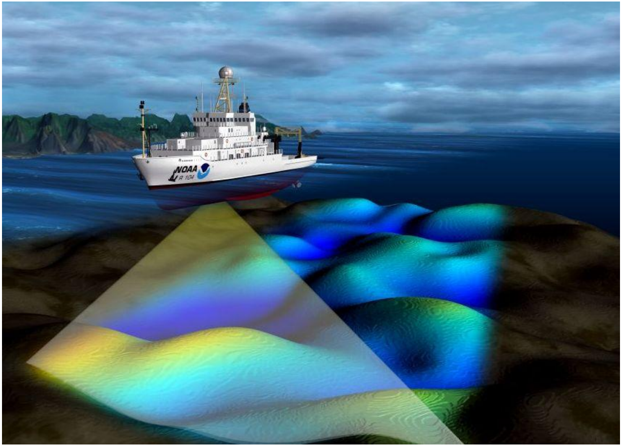
Motivation

transient detection



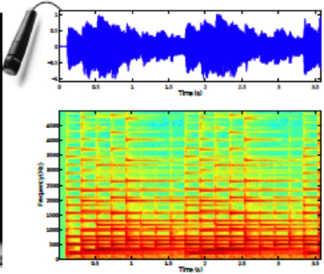
Motivation

echolocation



Motivation

speech/music signal processing



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Sparse interpolation

Sparse interpolation

interpolate

$$f(x) = \alpha_1 + \alpha_2 x^{100}$$

- Newton/Lagrange interpolation: 101 samples
- only 4 unknowns: α_1, α_2, x^0, x^{100}!
- how to solve it from 4 samples?

Sparse interpolation

$$\sum_{i=1}^{n} \alpha_i x_s^{k_i} = f_s, \quad n \ll \max(k_i), \quad k_i \in \mathbb{N}$$

Sparse interpolation

$$\sum_{i=1}^{n} \alpha_i x_s^{k_i} = f_s, \quad n \ll \max(k_i), \quad k_i \in \mathbb{N}$$

$$\sum_{i=1}^{n_1} \alpha_{i,1} \cos(\phi_{i,1} x_s) + \sum_{i=1}^{n_2} \alpha_{i,2} \sin(\phi_{i,2} x_s) = f_s, \quad \phi_{i,\{1,2\}} \in \mathbb{R}$$

Sparse interpolation

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$$\sum_{i=1}^{n} \alpha_i \exp(\phi_i x_s) = f_s, \quad \phi_i \in \mathbb{C}$$

Exponential analysis

$$\phi(x) = \sum_{i=1}^{n} \alpha_i \exp(\phi_i x) + \varepsilon(x)$$

$$\alpha_i \in \mathbb{C}$$

$$\phi_i = \beta_i + \mathtt{i}\gamma_i, \qquad |\Im(\phi_i)| < M/2, \qquad \Delta = 2\pi/M$$

equidistant $x_s = s\Delta, \quad s = 0, 1, 2, \ldots, 2n-1, \ldots$

$$f_s := \phi(x_s), \qquad \varepsilon_s := \varepsilon(x_s)$$

- wide spectrum
- high resolution
- small SNR

Exponential analysis

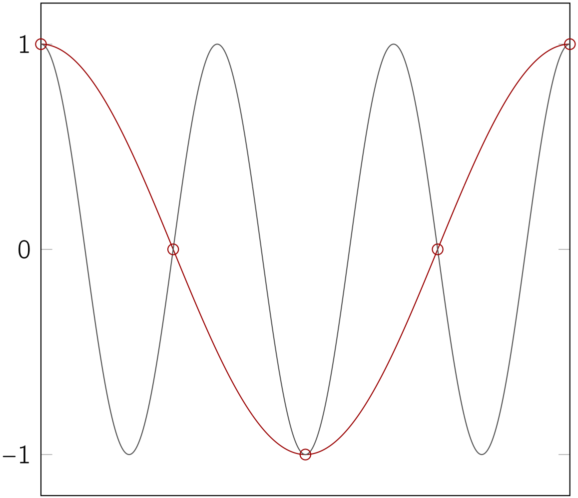


Figure: Sampling too coarsely introduces aliasing

Sparse interpolation

interpolation problem:

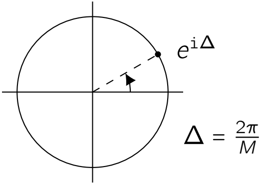
$$\sum_{i=1}^{n} \alpha_i \exp(\phi_i x_s) = f_s, \quad s = 0, \ldots, 2n-1$$

$$x_s = s\Delta, \quad \Delta = 2\pi/M$$

$$|\Im(\phi_i)| < M/2, \quad \Phi_i = \exp(\phi_i \Delta),$$

$$f_s = \sum_{i=1}^{n} \alpha_i \Phi_i^s, \quad s = 0, \ldots, 2n-1$$

$$\begin{cases} \alpha_1 + \cdots + \alpha_n = f_0 \\ \alpha_1 \Phi_1 + \cdots + \alpha_n \Phi_n = f_1 \\ \quad \vdots \\ \alpha_1 \Phi_1^{2n-1} + \cdots + \alpha_n \Phi_n^{2n-1} = f_{2n-1} \end{cases}$$



Sparse interpolation

finding Φ_i:

$$\prod_{i=1}^{n}(z-\Phi_i) = z^n + d_{n-1}z^{n-1} + \cdots + d_1 z + d_0$$

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \alpha_i \Phi_i^s (\Phi_i^n + d_{n-1}\Phi_i^{n-1} + \cdots + d_0) \\ &= \sum_{i=1}^{n} \alpha_i \Phi_i^{n+s} + \sum_{j=0}^{n-1} d_j \left(\sum_{i=1}^{n} \alpha_i \Phi_i^{j+s} \right) \\ &= f_{s+n} + \sum_{j=0}^{n-1} d_j f_{s+j} \end{aligned}$$

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Sparse interpolation

$$\begin{pmatrix} f_0 & \dots & f_{n-1} \\ \vdots & \iddots & \vdots \\ f_{n-1} & \dots & f_{2n-2} \end{pmatrix} \begin{pmatrix} d_0 \\ \vdots \\ d_{n-1} \end{pmatrix} = - \begin{pmatrix} f_n \\ \vdots \\ f_{2n-1} \end{pmatrix}$$

Hankel matrix:

$$H_n^{(r)} = \begin{pmatrix} f_r & \dots & f_{r+n-1} \\ \vdots & \iddots & \vdots \\ f_{r+n-1} & \dots & f_{r+2n-2} \end{pmatrix}$$

Sparse interpolation

Hadamard polynomial:

$$H_n^{(0)}(z) = \begin{vmatrix} f_0 & \dots & f_{n-1} & f_n \\ \vdots & \iddots & \vdots & \vdots \\ f_{n-1} & \dots & f_{2n-2} & f_{2n-1} \\ 1 & \dots & z^{n-1} & z^n \end{vmatrix}$$

$$\begin{aligned}\prod_{i=1}^{n}(z - \Phi_i) &= \frac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|} \\ &= z^n + d_{n-1}z^{n-1} + \dots + d_1 z + d_0\end{aligned}$$

Sparse interpolation

formally orthogonal polynomial:

$$\gamma : z^s \to f_s, \quad s = 0, 1, \ldots$$

$$\frac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|} \perp_\gamma z^i, \quad i = 0, \ldots, n-1$$

$$\gamma\left(z^i \frac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|}\right) = 0, \quad i = 0, \ldots, n-1$$

[Henrici, 1974]

Sparse interpolation

roots of $\dfrac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|}$ from GEP:

$$H_n^{(0)} = \begin{pmatrix} 1 & \dots & & 1 \\ \Phi_1 & \Phi_2 & \dots & \Phi_n \\ \vdots & & & \vdots \\ \Phi_1^{n-1} & \dots & & \Phi_n^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \begin{pmatrix} 1 & \Phi_1 & \dots & \Phi_1^{n-1} \\ \vdots & \Phi_2 & & \vdots \\ & \vdots & & \vdots \\ 1 & \Phi_n & \dots & \Phi_n^{n-1} \end{pmatrix}$$

$$= V_n^T D_\alpha V_n$$

$$H_n^{(1)} = V_n^T D_\alpha \begin{pmatrix} \Phi_1 & & \\ & \ddots & \\ & & \Phi_n \end{pmatrix} V_n$$

Sparse interpolation

$$\det\left(H_n^{(1)} - \lambda H_n^{(0)}\right) = \det\left(V_n^T D_\alpha \begin{pmatrix} \Phi_1 - \lambda & & \\ & \ddots & \\ & & \Phi_n - \lambda \end{pmatrix} V_n\right)$$
$$= 0 \text{ for } \lambda = \Phi_i, \quad i = 1, \ldots, n$$

[Hua and Sarkar, 1990]

Sparse interpolation

finding ϕ_i:

$$\exp(\phi_i) = \exp(\Re(\phi_i))e^{\mathrm{i}\Im(\phi_i)}$$

$$|\Im(\phi_i)| < \frac{M}{2}:$$

$$\begin{aligned}\arg(\Phi_i) &= \arg(\exp(\phi_i \Delta)) \\ &= \Im(\phi_i)\,\frac{2\pi}{M} \in \left]-\pi, \pi\right[\end{aligned}$$

Sparse interpolation

finding α_i:

$$\sum_{i=1}^{n} \alpha_i \Phi_i^{s+j} = f_{s+j}, \quad s = 0, \ldots, n-1, \quad 0 \le j \le n$$

$$\begin{pmatrix} \Phi_1^j & \ldots & \Phi_n^j \\ \vdots & & \vdots \\ \Phi_1^{j+n-1} & \ldots & \Phi_n^{j+n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} f_j \\ \vdots \\ f_{j+n-1} \end{pmatrix}$$

remaining interpolation conditions are linearly dependent

Sparse interpolation

finding n:

$$N < n : \left|H_N^{(r)}\right| \not\equiv 0, \quad r = 0, 1, \ldots$$

$$N = n : \left|H_N^{(r)}\right| \neq 0 \quad \text{if } \Phi_i \neq \Phi_j \text{ for } i \neq j$$ [Kaltofen and Lee, 2003]

$$N > n : \left|H_N^{(r)}\right| \equiv 0, \quad r = 0, 1, \ldots$$

Exponential analysis

Exponential analysis

$$\phi(x) = \sum_{i=1}^{4} \alpha_i \exp(\phi_i x)$$

$\alpha_1 = 1$	$\phi_1 = 0$
$\alpha_2 = 2.4$	$\phi_2 = -5 + 19.97\mathtt{i}$
$\alpha_3 = -2.1$	$\phi_3 = 3 + 45\mathtt{i}$
$\alpha_4 = 0.2$	$\phi_4 = 5.3\mathtt{i}$

evaluate at $x_s = s\frac{2\pi}{100}$, $\quad M = 100$, $\quad |\Im(\phi_i)| < 50$

sequence $f_0, \ldots, f_7, \ldots$ is linearly generated

Exponential analysis

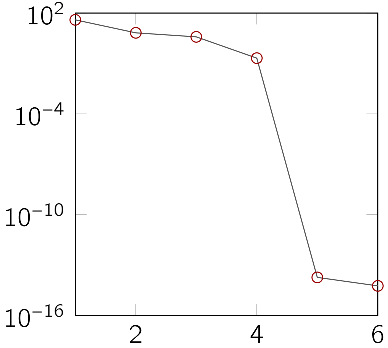


Figure: $H_N^{(0)}$ singular, $N = 6$

Exponential analysis

$$\phi(x) = 174.13 \exp(-x/22) + 19.348 \exp(-x/80) + \varepsilon(x)$$

$$10 \log_{10} \left(\frac{\sum_{s=0}^{255} \phi^2(x_s)}{\sum_{s=0}^{255} \varepsilon^2(x_s)} \right) = 34$$

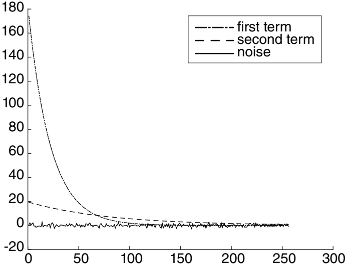


Figure: Individual components including the discrete noise

Exponential analysis

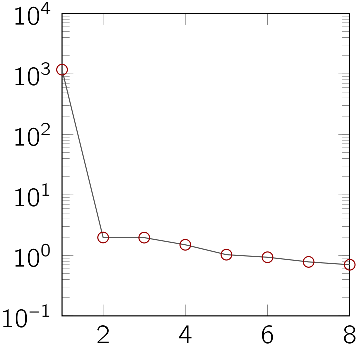


Figure: Log-plot of singular values of $H_8^{(0)}$ with 34 dB white Gaussian noise added

Approximation theory

Approximation theory

$$f_s = \sum_{i=1}^{n} \alpha_i \exp(\phi_i x_s), \qquad s = 0, 1, \ldots, 2n-1$$

$$f(z) = \sum_{j=0}^{\infty} f_j z^j, \qquad f_j = 0, \qquad j < 0$$

$$p(z) = \sum_{i=0}^{m} a_i z^i,$$

$$q(z) = \sum_{i=0}^{n} b_i z^i$$

$$\left(\sum_{j=0}^{\infty} f_j z^j\right) q(z) - p(z) = \sum_{i \geq m+n+1} c_i z^i$$

Approximation theory

$$\begin{cases} f_0 b_0 = a_0 \\ f_1 b_0 + f_0 b_1 = a_1 \\ \quad \vdots \\ f_m b_0 + \cdots + f_{m-n} b_n = a_m \end{cases}$$

$$b_0 = 1$$

$$\begin{cases} f_{m+1} b_0 + \cdots + f_{m-n+1} b_n = 0 \\ \quad \vdots \\ f_{m+n} b_0 + \cdots + f_m b_n = 0 \end{cases}$$

$$H_n^{(m+1-n)} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = - \begin{pmatrix} f_{m+1} \\ \vdots \\ f_{m+n} \end{pmatrix}$$

$$[m/n](z) := p(z)/q(z)$$

Approximation theory

$$\begin{aligned} f_s &= \sum_{i=1}^{n} \alpha_i \exp(\phi_i x_s) \\ &= \sum_{i=1}^{n} \alpha_i \Phi_i^s \end{aligned}$$

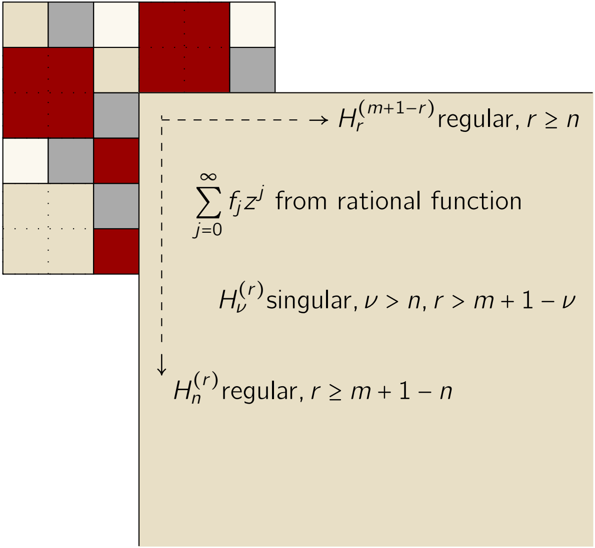
$$\begin{aligned} f(z) &= \sum_{j=0}^{\infty} f_j z^j \\ &= \sum_{i=1}^{n} \frac{\alpha_i}{1 - z\Phi_i} \\ &= \text{Laplace transform of } \sum_{i=1}^{n} \alpha_i \exp(\phi_i x) \end{aligned}$$

Approximation theory

$$f(z) = [n-1/n](z) = p(z)/q(z)$$

$$\begin{aligned} q(z) &= \prod_{i=1}^{n} (1 - z\Phi_i) \\ &= z^n \frac{H_n^{(0)}(1/z)}{\left|H_n^{(0)}\right|} \\ &= d_0 z^n + d_1 z^{n-1} + \cdots + d_{n-1} z + 1 \end{aligned}$$

Approximation theory



$\rightarrow H_r^{(m+1-r)}$ regular, $r \geq n$

$\sum_{j=0}^{\infty} f_j z^j$ from rational function

$H_\nu^{(r)}$ singular, $\nu > n, r > m+1-\nu$

$\downarrow H_n^{(r)}$ regular, $r \geq m+1-n$

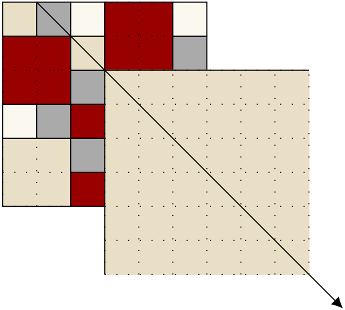
Approximation theory

$f(z) + \varepsilon(z)$ analytic except for a countable number of poles [Nuttall, 1970] and essential singularities [Pommerenke, 1973]

$$\Downarrow$$

$[\nu - 1/\nu](z) \to f(z) + \varepsilon(z)$ in measure on compact sets, i.e.

$$\Lambda_2\left(\{z : |f(z) + \varepsilon(z) - [\nu - 1/\nu](z)| \geq \tau\}\right) \to 0$$



Approximation theory

mathematical (noise free):

1. build $H_\nu^{(0)}, \quad \nu = 0, 1, 2, \ldots$
2. $H_\nu^{(0)} = U\Sigma V^T$ singular value decomposition
3. $\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_\nu \end{pmatrix}, \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > \sigma_{n+1} = \cdots = \sigma_\nu = 0$
4. find $\Phi_i, \phi_i, \alpha_i, i = 1, \ldots, n$ from $H_n^{(1)} v_i = \Phi_i H_n^{(0)} v_i$ and the interpolation conditions

Approximation theory

numerical (with noise):

1. take ν large enough so that in the singular value decomposition noise is clearly separated from n
2. solve $H_\nu^{(1)} v_i = \lambda_i H_\nu^{(0)} v_i, \quad i = 1, \ldots, \nu, \quad \lambda_i = \Phi_i, \quad i = 1, \ldots, n$
3. find ϕ_i
4. solve $\displaystyle\sum_{i=1}^{n} \alpha_i \exp(\phi_i x_j) = f_j, \quad 0 \le j \le 2\nu - 1$

Exponential analysis

Exponential analysis

$$\phi(x) = \sum_{i=1}^{4} \alpha_i \exp(\phi_i x)$$

$$\begin{aligned} \alpha_1 &= 1 & \phi_1 &= 0 \\ \alpha_2 &= 2 & \phi_2 &= -0.2 + 39.5\mathtt{i} \\ \alpha_3 &= 4 & \phi_3 &= -0.5 + 40\mathtt{i} \\ \alpha_4 &= 8 & \phi_4 &= -1 \end{aligned}$$

evaluate at $x_s = s\frac{2\pi}{100}$, $M = 100$, $|\Im(\phi_i)| < 50$

$\|\varepsilon(z)\|_\infty = 10^{-2}$, uniform random noise

Exponential analysis

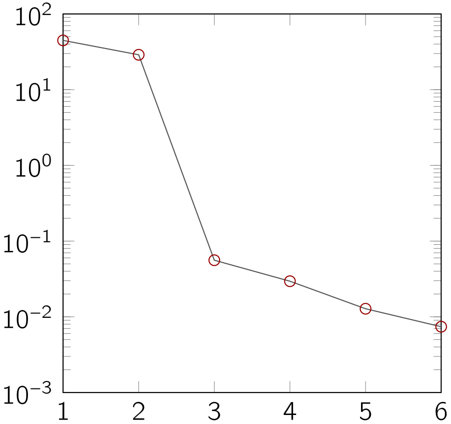


Figure: Singular values $H_\nu^{(0)}$ with $n = 4, \nu = 6$

Exponential analysis

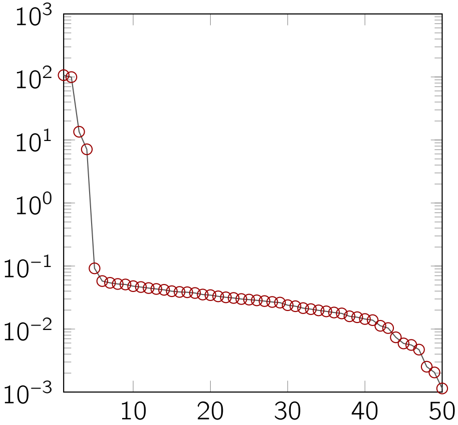


Figure: Singular values $H_\nu^{(0)}$ with $n = 4, \nu = 50$

Exponential analysis

$$\phi(x) = 174.13\,\exp(-x/22) + 19.348\,\exp(-x/80) + \varepsilon(x)$$

$$10\log_{10}\left(\frac{\sum_{s=0}^{255}\phi^2(x_s)}{\sum_{s=0}^{255}\varepsilon^2(x_s)}\right) = 34$$

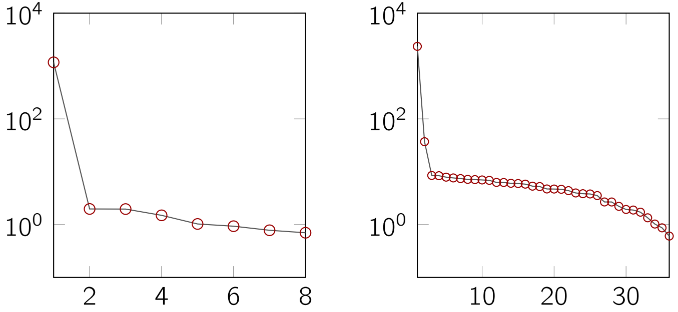


Figure: Log-plot of singular values of $H_8^{(0)}$ and $H_{36}^{(0)}$ with 34 dB white Gaussian noise added

Tensor decomposition

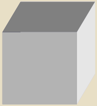
Tensor decomposition

tensor $T = \left(t_{j_1 \dots j_k}\right)_{j_1,\dots,j_k=0}^{d} \in \mathbb{C}^{(d+1)\times\cdots\times(d+1)}$

of order k and dimension $d+1$

e.g. $k = 3,\ d = 1$

$$\left(\begin{array}{cc|cc} t_{000} & t_{010} & t_{001} & t_{011} \\ t_{100} & t_{110} & t_{101} & t_{111} \end{array}\right)$$



associate $(d+1)$-variate homogeneous polynomial of degree k to T:

$$\sum_{j_1 \le \cdots \le j_k = 0}^{d} \left(\sum_{\pi} t_{\pi(j_1)\dots\pi(j_k)} \right) z_{j_1} \dots z_{j_k}$$

e.g. $t_{000} z_0^3 + (t_{100} + t_{010} + t_{001}) z_0^2 z_1 + (t_{110} + t_{101} + t_{011}) z_0 z_1^2 + t_{111} z_1^3$

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Tensor decomposition

homogeneous polynomial compactly written as

$$\sum_{|\kappa|=k} c_\kappa Z^\kappa,$$

$$Z = (z_0, \ldots, z_d), \quad \kappa = (k_0, \ldots, k_d)$$

$$Z^\kappa = z_0^{k_0} \ldots z_d^{k_d}, \quad |\kappa| = k_0 + \cdots + k_d$$

set $z_0 = 1$ and for $d = 1$, $z := z_1$, $k := k_1$:

$$\sum_{|\kappa|=k} c_\kappa Z^\kappa = \sum_{j=0}^{k} c_j z^j$$

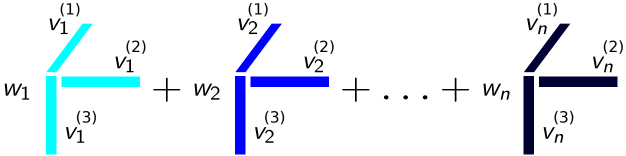
e.g. $k = 3$, $d = 1$

$$t_{000} + (t_{100} + t_{010} + t_{001})z + (t_{110} + t_{101} + t_{011})z^2 + t_{111}z^3$$

Tensor decomposition

tensor decomposition (general) if rank $T = n$:

$$T = \sum_{i=1}^{n} w_i v_i^{(1)} \otimes \cdots \otimes v_i^{(k)}$$



for order $k = 3$: $t_{jhl} = \sum_{i=1}^{n} w_i v_{i,j}^{(1)} v_{i,h}^{(2)} v_{i,l}^{(3)}$

Tensor decomposition

tensor decomposition (order $k = 3$, dimension $n \times n \times 2$)

$$t_{jhl} := f_{j+h+l}, \qquad 0 \le j, h \le n-1, \quad 0 \le l \le 1$$

$$T = \left(\begin{array}{cccc|cccc} f_0 & f_1 & \dots & f_{n-1} & f_1 & f_2 & \dots & f_n \\ f_1 & & & \vdots & f_2 & & & \vdots \\ \vdots & & & & \vdots & & & \\ f_{n-1} & \dots & & f_{2n-2} & f_n & \dots & & f_{2n-1} \end{array}\right)$$

$$= \left(H_n^{(0)} \;\middle|\; H_n^{(1)} \right)$$

$$= \sum_{i=1}^{n} \alpha_i \begin{pmatrix} 1 \\ \Phi_i \\ \vdots \\ \Phi_i^{n-1} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \Phi_i \\ \vdots \\ \Phi_i^{n-1} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \Phi_i \end{pmatrix}, \qquad \Phi_i = \exp(\phi_i \Delta)$$

[Van Barel]

Tensor decomposition

symmetric tensor decomposition (order $k = 2n - 1$, dimension $d + 1 = 2$)

$$t_{j_1 \ldots j_k} = f_{j_1 + \cdots + j_{2n-1}}, \qquad 0 \leq j_k \leq 1$$

tensor slice

$$t_{\bullet \bullet j_3 \ldots j_k} = \begin{pmatrix} t_{00\, j_3 \ldots j_k} & t_{01\, j_3 \ldots j_k} \\ t_{10\, j_3 \ldots j_k} & t_{11\, j_3 \ldots j_k} \end{pmatrix} = \begin{pmatrix} f_{j_3 + \cdots + j_k} & f_{j_3 + \cdots + j_k + 1} \\ f_{j_3 + \cdots + j_k + 1} & f_{j_3 + \cdots + j_k + 2} \end{pmatrix}$$

$$T = \sum_{i=1}^{n} \alpha_i \begin{pmatrix} 1 \\ \Phi_i \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ \Phi_i \end{pmatrix}$$

Tensor decomposition

e.g. $k = 3,\ d = 1$

$$\left(\begin{array}{cc|cc} t_{000} & t_{010} & t_{001} & t_{011} \\ t_{100} & t_{110} & t_{101} & t_{111} \end{array}\right) = \left(\begin{array}{cc|cc} 4 & 3 & 3 & 6 \\ 3 & 6 & 6 & 17 \end{array}\right) \qquad \text{symmetric}$$

Tensor decomposition

symmetric tensor $t_{j_1 j_2 j_3} = f_{j_1+j_2+j_3}, \qquad 0 \le j_1, j_2 \le 1, \quad 0 \le j_3 \le 1$

$$c_0 = f_0 = 4$$
$$c_1 = 3f_1 = 9$$
$$c_2 = 3f_2 = 18$$
$$c_3 = f_3 = 17$$

$$\sum_{j=0}^{k} c_j z^j = 4 + 9z + 18z^2 + 17z^3 = \sum_{i=1}^{n} \alpha_i (1 + \Phi_i z)^k$$

$$T = \left(\begin{array}{cc|cc} f_0 & f_1 & f_1 & f_2 \\ f_1 & f_2 & f_2 & f_3 \end{array}\right) \in \mathbb{C}^{n \times n \times 2}, \quad \in \mathbb{C}^{2\times 2\times 2} \quad (n = 2)$$

$$= \frac{5}{8}\begin{pmatrix}1\\3\end{pmatrix} \otimes \begin{pmatrix}1\\3\end{pmatrix} \otimes \begin{pmatrix}1\\3\end{pmatrix} + \frac{27}{8}\begin{pmatrix}1\\\frac{1}{3}\end{pmatrix} \otimes \begin{pmatrix}1\\\frac{1}{3}\end{pmatrix} \otimes \begin{pmatrix}1\\\frac{1}{3}\end{pmatrix}$$

Tensor decomposition

also:

- roots of Hadamard polynomial

$$\begin{vmatrix} 4 & 3 & 6 \\ 3 & 6 & 17 \\ 1 & z & z^2 \end{vmatrix} / \left| H_2^{(0)} \right| = z^2 - \frac{10}{3} z + 1$$

- generalized eigenvalues Φ_1 and Φ_2
- Hadamard polynomial equals reverse of Padé denominator [1/2] for

$$f_0 + f_1 z + f_2 z^2 + f_3 z^3 + \dots$$

References

References

P. Henrici. *Applied and computational complex analysis I*. John Wiley & Sons, New York, 1974.

Y. Hua and T. K. Sarkar. Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise. *IEEE Trans. Acoust. Speech Signal Process.*, 38: 814–824, 1990. http://dx.doi.org/10.1109/29.56027.

E. Kaltofen and W.-s. Lee. Early termination in sparse interpolation algorithms. *J. Symbolic Comput.*, 36(3-4):365–400, 2003. http://dx.doi.org/10.1016/S0747-7171(03)00088-9. International Symposium on Symbolic and Algebraic Computation (ISSAC'2002) (Lille).

J. Nuttall. The convergence of Padé approximants of meromorphic functions. *J. Math. Anal. Appl.*, 31:147–153, 1970. http://dx.doi.org/10.1016/0022-247X(70)90126-5.

C. Pommerenke. Padé approximants and convergence in capacity. *J. Math. Anal. Appl.*, 41: 775–780, 1973. http://dx.doi.org/10.1016/0022-247X(73)90248-5.

M. Van Barel. Decomposing multivariate exponential sums using the canonical polyadic decomposition. Private communication.