Maximum Principle and Elliptic PDEs

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Contents

Model Problem

- given: bounded domain $\Omega \subset \mathbb{R}^d$.
- sought: approximation $u_h : \Omega \to \mathbb{R}$ of Poisson's equation:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

with $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$.

Galerkin-Approaches

C^0-finite elements:

- triangulation
- hat functions

Isogeometric analysis (IGA)

- NURBS parameterization
- domain-splitting in patches

WEB-Splines

- domains with implicit description
- weight functions for boundary conditions

Alternative: Hierarchical Least-Squares-Collocation

Wanted Properties:

- adaptive refinement with hierarchical B-Splines
- meshingless
- no numerical integration

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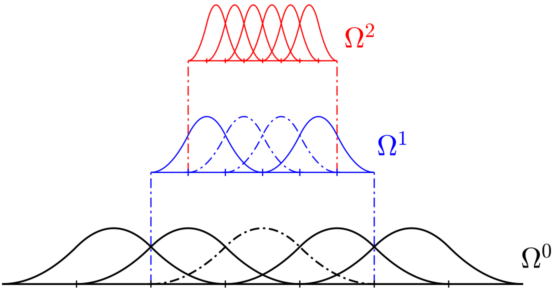
- guaranteed error bound: $\|u - u_h\|_{L^\infty} \leq \epsilon$

Hierarchical B-Splines

- *Forsey, Bartels, 1988:*
  surface constructions
- *Kraft, 1997:*
  linear independent in $\mathbb{R}^2$
- *Giannelli, Jüttler, Speleers, 2012:*
  truncated hierarchical B-Splines

Construction of Hierarchical B-Splines



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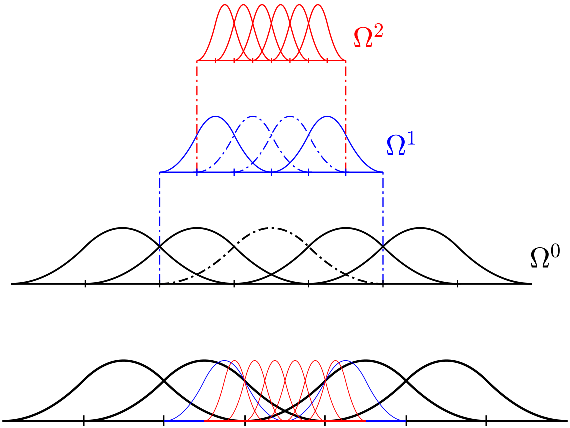
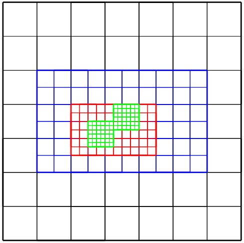
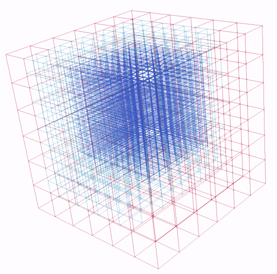


Hierarchical Domains 2D, 3D





- local refinement
- linear independence

Implementation

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- open-source library
- NURBS, hierarchical B-Splines, truncated hierarchical B-Splines any dimension
- isogeometric analysis
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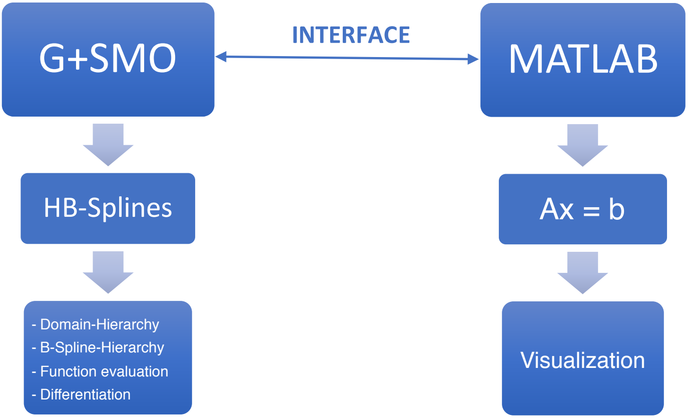
MATLAB:

- versatile programming language
- robust numerical calculations
- simple visualization

G+SMO - MATLAB Interface



Least-Squares Collocation

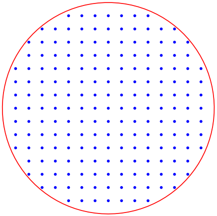
Least-Squares Collocation

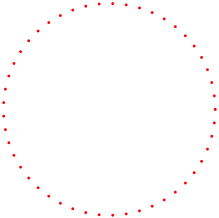
Given:

- Poisson's equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

- $\mathbf{u}_i \in \Omega \setminus \partial\Omega, \quad 1 \leq i \leq N_1$ $\qquad$ $\mathbf{v}_j \in \partial\Omega, \quad 1 \leq j \leq N_2$





Least-Squares Collocation

- hierarchical spline u_h of order $n \geq 4$:

$$u_h : \Omega \to \mathbb{R}, \quad u_h(\mathbf{p}, x) = \sum_{\ell \leq L} \sum_{j \in \mathcal{B}^\ell} p_j^\ell b_j^\ell(x), \quad p_j^\ell \in \mathbb{R}.$$

- we define:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned} \quad \Rightarrow \quad \begin{aligned} R_L &:= -\Delta u - f \quad \text{in } \Omega \\ R_B &:= u - g \quad \text{on } \partial\Omega, \end{aligned}$$

- residuals at collocation points :

$$\begin{aligned} R_L(\mathbf{p}, \mathbf{u}_i) &= -\Delta u_h(\mathbf{p}, \mathbf{u}_i) - f(\mathbf{u}_i), \quad 1 \leq i \leq N_1 \\ R_B(\mathbf{p}, \mathbf{v}_j) &= u_h(\mathbf{p}, \mathbf{v}_j) - g(\mathbf{v}_j), \quad 1 \leq j \leq N_2 \end{aligned}$$

Least-Squares Collocation

- quadratic optimization problem:

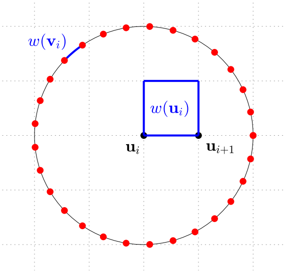
$$\min_{\mathbf{p}} J(\mathbf{p}) \to \min$$

$$J(\mathbf{p}) := \sum_{i=1}^{N_1} [R_L(\mathbf{p}, \mathbf{u}_i)]^2 + \sum_{j=1}^{N_2} [R_B(\mathbf{p}, \mathbf{v}_j)]^2$$

- linear system of equations:

$$A\boldsymbol{p} = G$$

Weighting



$$J(\mathbf{p}) = \sum_{i=1}^{N_1} \omega(\mathbf{u}_i) \left[R_L(\mathbf{p}, \mathbf{u}_i)\right]^2 + \sum_{j=1}^{N_2} \omega(\mathbf{v}_j) \left[R_B(\mathbf{p}, \mathbf{v}_j)\right]^2$$

$$\approx \int_\Omega \left[R_L(\mathbf{p}, x)\right]^2 dx + \int_{\partial\Omega} \left[R_B(\mathbf{p}, x)\right]^2 dx$$

Refinement Strategy

Heuristic:

- determine collocation points with residuals greater than tol
- refine corresponding B-Splines

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Why?

$$\|u - u_h\|_{L^\infty(\Omega)} \leq C_1 \|-\Delta u_h - f\|_{L^\infty(\Omega)} + C_2 \|u_h - g\|_{L^\infty(\partial\Omega)}$$

Wanted Error Estimation

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- $\|u - u_h\|_{L^2} \leq C h^k \ldots$

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Goal:

- given a maximum error ϵ
- guaranteed error bound:

$$\|u - u_h\|_{L^\infty} \leq \epsilon$$

Boundary-Maximum Theorem

Definition

Let Ω be a bounded domain in $\mathbb{R}^d$. A function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is said to be *sub-harmonic* resp. *super-harmonic* if

$$-\Delta u \leq 0 \quad \text{resp.} \quad -\Delta u \geq 0 \quad \text{in } \Omega$$

Theorem (Boundary-Maximum Theorem)

If u is sub-harmonic resp. super-harmonic in Ω, then

$$\max_{x \in \bar{\Omega}} u = \max_{x \in \partial\Omega} u \quad \textit{resp.} \quad \min_{x \in \bar{\Omega}} u = \min_{x \in \partial\Omega} u$$

Monotonicity Theorem

Theorem (Monotonicity Theorem)

Let Ω *be a bounded domain in* $\mathbb{R}^d$. *If* u *satisfies the inequalities*

$$\begin{aligned} -\Delta u &\leq 0 \quad \text{in } \Omega \\ u &\leq 0 \quad \text{on } \partial\Omega, \end{aligned} \qquad \text{resp.} \qquad \begin{aligned} -\Delta u &\geq 0 \quad \text{in } \Omega \\ u &\geq 0 \quad \text{on } \partial\Omega, \end{aligned}$$

then

$$u \leq 0 \quad \text{in } \bar{\Omega} \qquad \text{resp.} \qquad u \geq 0 \quad \text{in } \bar{\Omega}$$

Application to Poisson's equation

- let u be the exact solution of

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

- suppose $v_h \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$\begin{aligned} -\Delta(v_h - u) &\geq 0 \quad \text{in } \Omega \\ (v_h - u) &\geq 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{UB}$$

  then

$$(v_h - u) \geq 0 \quad \Leftrightarrow \quad v_h \geq u$$

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$\Rightarrow$ **v_h is an upper bound of u**

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$\hookrightarrow$ **Question: Construction of v_h?**

Error Bounds Construction

- using any given approximation $v_h := u_h$ with

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- ensure (UB) with $v_h := u_h + w_h$, so that

$$\begin{aligned} -\Delta(u_h + w_h - u) &= (f_h - \Delta w_h) - f \geq 0 \quad \text{in } \Omega \\ (u_h + w_h - u) &= (g_h + w_h) - g \geq 0 \quad \text{on } \partial\Omega \end{aligned}$$

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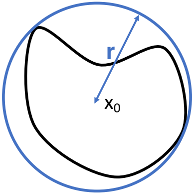
- analog construction for lower bound

General Construction of w_h

- $\Omega \subseteq B(x_0, r) \subset \mathbb{R}^d$
- $F := \|-\Delta(u_h - u)\|_{L^\infty(\Omega)} = \|f_h - f\|_{L^\infty(\Omega)}$
- $G := \|u_h - u\|_{L^\infty(\partial\Omega)} = \|g_h - g\|_{L^\infty(\partial\Omega)}$



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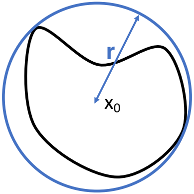

- with

$$\begin{aligned} w_h(x) &:= \frac{1}{2d} r^2 F + G - \frac{1}{2d} F \|x - x_0\|_2^2 \\ \Rightarrow -\Delta w_h(x) &= F \end{aligned}$$

- we have

$$\begin{aligned} -\Delta(u_h + w_h - u) &\geq 0 \quad \text{in } \Omega \\ (u_h + w_h - u) &\geq 0 \quad \text{on } \partial\Omega \end{aligned}$$

General Construction of w_h

- and

$$(u - u_h) \leq w_h \leq \max |w_h| = \frac{1}{2d} F r^2 + G$$

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- final error bound:

$$|u - u_h| \leq \frac{1}{2d} r^2 \|f - f_h\|_{L^\infty(\Omega)} + \|g - g_h\|_{L^\infty(\partial\Omega)}$$

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large overestimation!

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large overestimation!
can we do it better?

Improved Construction of w_h

- conditions

$$
\begin{aligned}
-\Delta(u_h + w_h - u) &= (f_h - \Delta w_h) - f &\geq 0 \quad \text{in } \Omega \\
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\end{aligned}
$$

- new approximation problem

$$
\begin{aligned}
-\Delta w_h &\approx f - f_h \quad \text{in } \Omega \\
w_h &\approx g - g_h \quad \text{on } \partial\Omega
\end{aligned}
$$

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$$\begin{aligned} -\Delta w_h &\approx f - f_h \quad \text{in } \Omega \\ w_h &\approx g - g_h \quad \text{on } \partial\Omega \end{aligned}$$

Challenges:

- construction of w_h with low cost
  mesh width $\bar{h} \geq h$, $(\bar{h} = h, 2h, 4h, \dots)$
- valid error bounds

1. Approach

- Example:

$$-u'' = \frac{2x}{(x^2+1)^2} \quad \text{in } (-1,1)$$
$$u(-1) = \gamma_1 = \arctan(-1)$$
$$u(1) = \gamma_2 = \arctan(1)$$

exact solution: $\arctan(x)$

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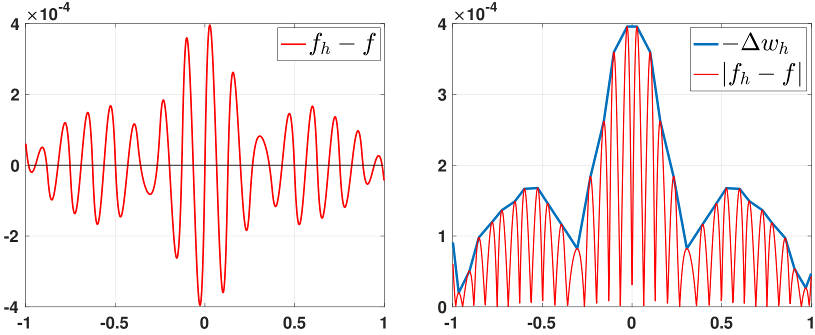
- approximation u_h with uniform B-Splines and least-squares collocation

1. Approach

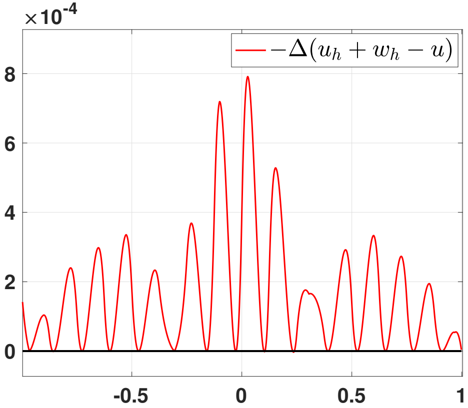


- spline $w_h = \sum_k b_k p_k$, so that

$$-\Delta w_h = |f - f_h|$$

$$w_h = g - g_h$$

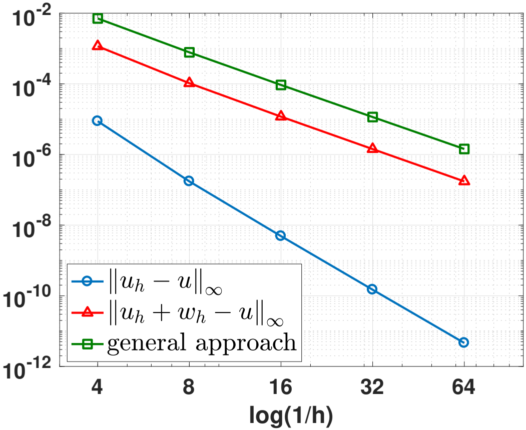
1. Approach



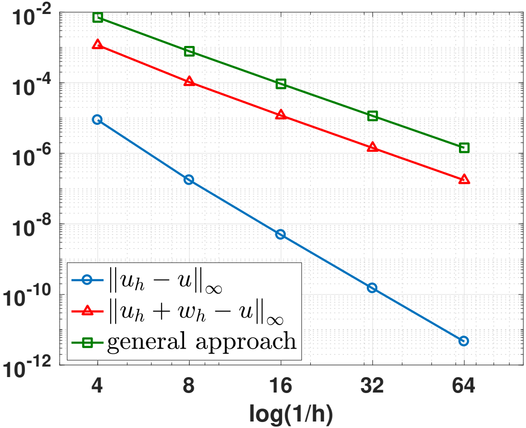
error bounds:

$$-w_h \leq u - u_h \leq w_h$$

1. Approach


10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
4
8
16
32
64
log(1/h)
$\|u_h - u\|_\infty$
$\|u_h + w_h - u\|_\infty$
general approach

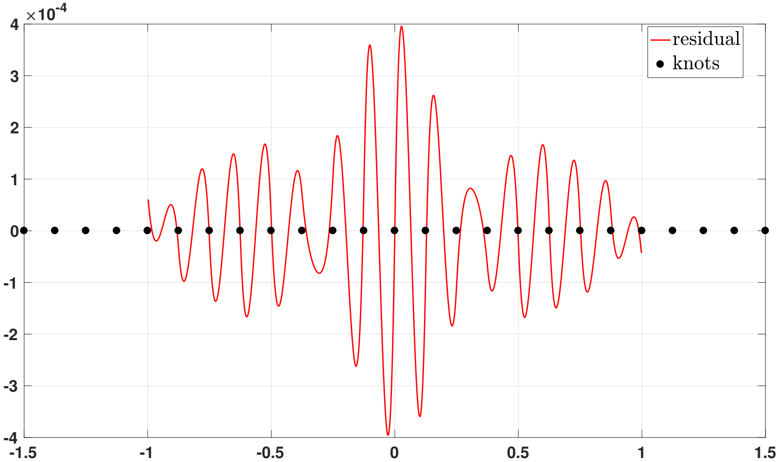
1. Approach



still large overestimation!

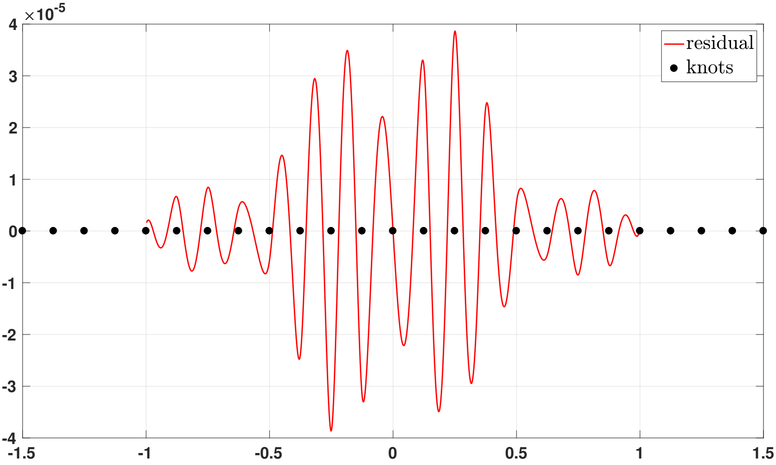
Observations

Uniform B-Spline-Approximation with odd order:



Observations

Uniform B-Spline-Approximation with even order:



Basic Idea

Reproduce the behavior of residuals with suited functions:

- let s_1 and s_2 be two splines of order n and $n+1$
- approximate polynomials t^n and t^{n+1} with knot vector of u_h

$$s_1 \approx t^n, \quad s_2 \approx t^{n+1}$$

- typical error-functions

$$r_1 := s_1 - t^n$$
$$r_2 := s_2 - t^{n+1}$$

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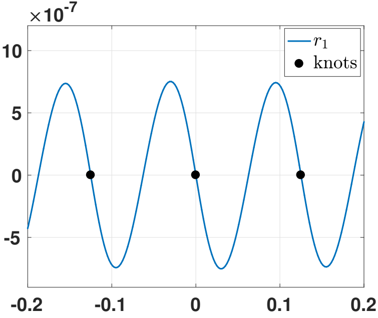
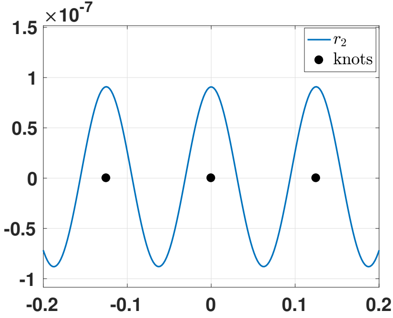
Do they have the same behaviors?

Typical Error-Functions



$r_1 = s_1 - t^n$



$r_2 = s_2 - t^{n+1}$

2. Approach

$$w_h(x) := A_1(x) r_1(x) + A_2(x) r_2(x)$$

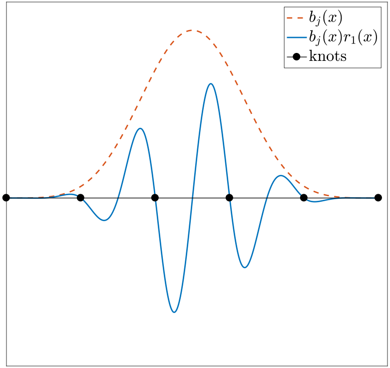
- Amplitudes $A_1(x)$ and $A_2(x)$ are splines with $T = 2h\mathbb{Z}$

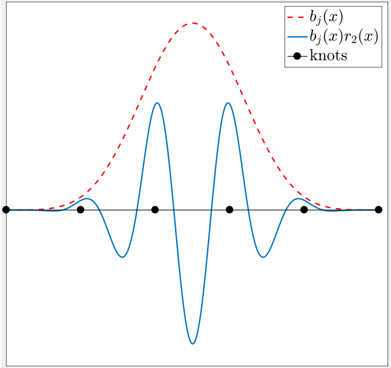
$$A_1(x) = \sum_j b_j p_j, \quad A_2(x) = \sum_k b_k q_k$$

$$w_h(x) = \sum_j [b_j(x) r_1(x)]\, p_j + \sum_k [b_k(x) r_2(x)]\, q_k$$

Oscillating Basis

$$w_h(x) = \sum_j [b_j(x) r_1(x)]\, p_j + \sum_k [b_k(x) r_2(x)]\, q_k$$





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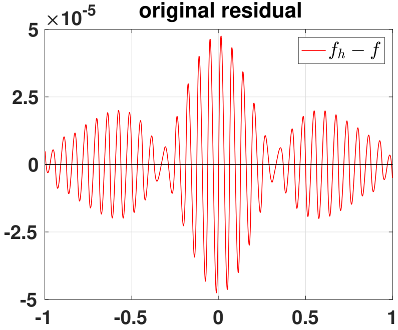
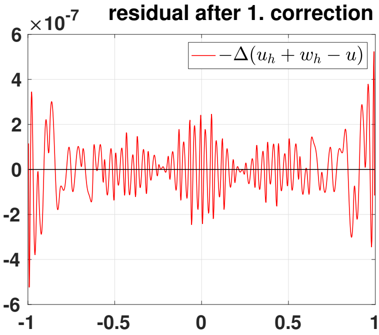
$$\begin{aligned} -\Delta w_h &\approx f - f_h && \text{in } \Omega \\ w_h &\approx g - g_h && \text{on } \partial\Omega \end{aligned}$$

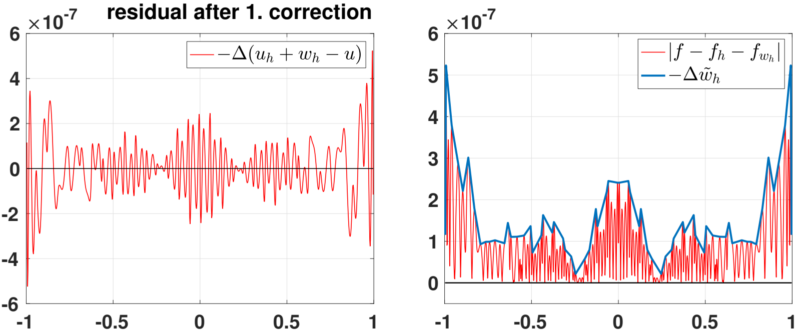




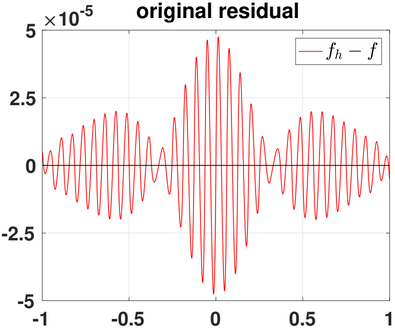
2. Approach

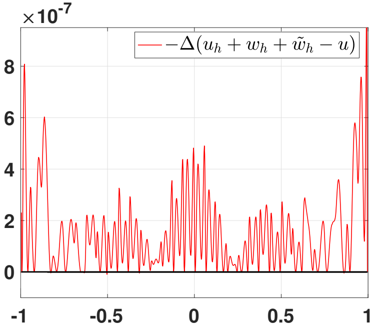
second correction $\tilde{w}_h$:

$$-\Delta \tilde{w}_h = |f - f_h - f_{w_h}| \quad \text{in } \Omega$$
$$\tilde{w}_h = g - g_h - g_{w_h} \quad \text{on } \partial\Omega$$



2. Approach





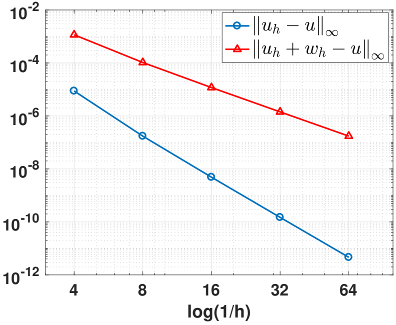
error bound:

$$-\tilde{w}_h + w_h \leq u - u_h \leq w_h + \tilde{w}_h$$

$$|u - u_h| \leq \|w_h\|_\infty + \|\tilde{w}_h\|_\infty$$

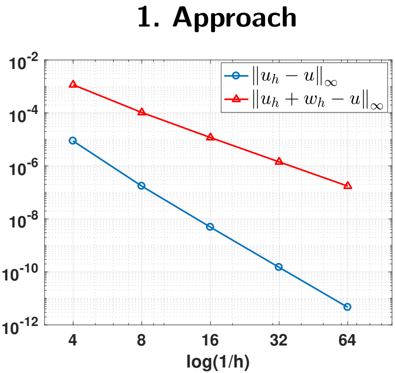
Error Comparison

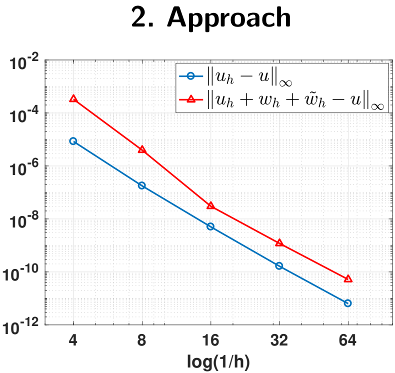
1. Approach



2. Approach

Error Comparison





Algorithm

Input:

- max. error $\epsilon > 0$

Do:

1. calculate approximation u_h
2. calculate first correction w_h with the oscillating basis
3. calculate second correction $\tilde{w}_h$ with the absolute value approach
4. calculate $M := \|w_h\|_\infty + \|\tilde{w}_h\|_\infty$
5. **if** $M > \epsilon$ **then**
   - refine locally
   - go to 1

Conclusion

- no triangulation, parameterization or weight function
- no numerical integration
- adaptive refinement
- easy to implement
- framework (G+SMO $\leftrightarrow$ MATLAB)
- guaranteed error bounds by applying maximum principle

$$\|u - u_h\|_{L^\infty} \leq \epsilon$$

Conclusion

current work:

- hierarchical B-Splines for construction of oscillating basis
- extension to higher dimensions
- ...

Conclusion

current work:

- hierarchical B-Splines for construction of oscillating basis
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Thanks

for your attention!