Generalized Taylor operators and Hermite subdivision schemes

Jean-Louis Merrien with Tomas Sauer

Bernried, February 19-23, 2018

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- *Contractive:* if $\varphi = 0$ for any initial data sequence f_0.

4-points scheme

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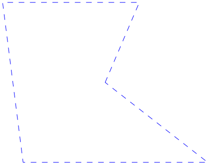

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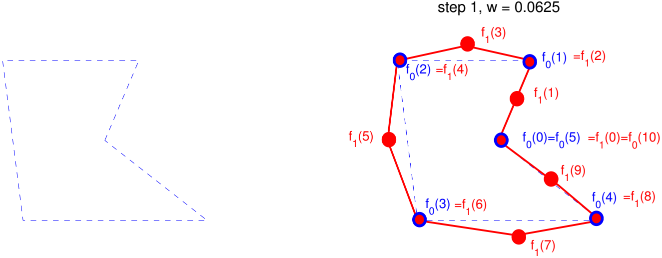

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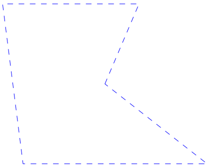
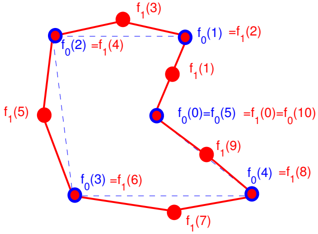
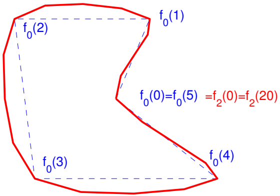


step 1, w = 0.0625



step 2, w = 0.0625



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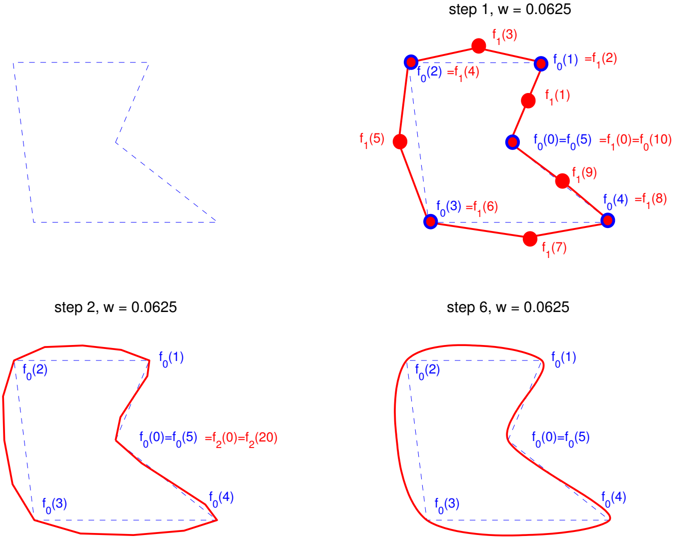

Convergence of scalar schemes

Let S_b be a subdivision scheme.

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$$\sum_{\alpha\in\mathbb{Z}} b(2\alpha) = \sum_{\alpha\in\mathbb{Z}} b(2\alpha+1) = 1,$$

  hence
  $b^*(z) = (1+z)c^*(z) \Leftrightarrow (1-z^{-1})b^*(z) = c^*(z)(1-z^{-2})$, or
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Example with the four points scheme: $b^*(z) = (1+z)c^*(z)$ where
$c^*(z) = wz^{-3} + wz^{-2} + 1/2z^{-1} + 1/2 + wz - wz^2$
and $\sum |c(2\alpha)| = 2|w| + 1/2 = \sum |c(2\alpha+1)|$. so that S_b
converges for $|w| < 1/4$

General Subdivision Schemes

Scalar Subd. Scheme, Mask $\{b\} \in \ell(\mathbb{Z})$.

$$g_0 \in \ell(\mathbb{Z}),\ g_{n+1}(\alpha) = \sum_{\beta \in \mathbb{Z}} b(\alpha - 2\beta) g_n(\beta),\ \alpha \in \mathbb{Z},\ n = 0, 1, \ldots$$

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Hermite Subd. Scheme, Mask $\{\mathbf{A}\} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$

$$\mathbf{f}_0 \in \ell^{d+1}(\mathbb{Z}),\ \mathbf{f}_{n+1}(\alpha) = \sum_{\beta \in \mathbb{Z}} \mathbf{D}^{-n-1} \mathbf{A}(\alpha - 2\beta) \mathbf{D}^n \mathbf{f}_n(\beta),\ \alpha \in \mathbb{Z},\ n \ldots$$

Vector Subdivision Scheme

Stationnary vector subdivision operator $S_{\mathbf{B}} : \ell^r(\mathbb{Z}) \to \ell^r(\mathbb{Z})$,
$S_{\mathbf{B}}\mathbf{c}(\alpha) := \sum_{\beta\in\mathbb{Z}} \mathbf{B}(\alpha - 2\beta)\, \mathbf{c}(\beta)$, $\alpha \in \mathbb{Z}$, $\mathbf{c} \in \ell^r(\mathbb{Z})$.

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Property: $(S_{\mathbf{B}})\mathbf{c}(\cdot + 2) = S_{\mathbf{B}}\big(\mathbf{c}(\cdot + 1)\big)$.

Hermite Subd. Scheme

$\mathbf{f}_0 \in \ell^{d+1}(\mathbb{Z})$, $\mathbf{D}^{n+1}\mathbf{f}_{n+1}(\alpha) = \sum_{\beta\in\mathbb{Z}} \mathbf{A}(\alpha - 2\beta)\mathbf{D}^n\mathbf{f}_n(\beta)$.

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If $\varphi \in C^d(\mathbb{R})$ and $\varphi_n(x) := \varphi(x/2^n)$, then

$$\begin{bmatrix} \varphi_n(x) \\ \vdots \\ \varphi_n^{(d)}(x) \end{bmatrix} = \mathbf{D}^n \begin{bmatrix} \varphi(x/2^n) \\ \vdots \\ \varphi^{(d)}(x/2^n) \end{bmatrix}.$$

If $\mathbf{f}_n(\beta) = [f_n^{(0)}(\beta), f_n^{(1)}(\beta), \ldots, f_n^{(d)}(\beta)]^T$, then

$f_n^{(0)}(\beta) \simeq \varphi(\beta/2^n)$,$f_n^{(1)}(\beta) \simeq \varphi'(\beta/2^n)$,... $f_n^{(d)}(\beta) \simeq \varphi^{(d)}(\beta/2^n)$.

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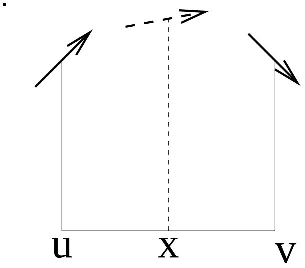
The scheme is C^k–convergent, $k \geq d$, if for any data $\mathbf{f}_0$, there exists $\Phi = [\phi_i]_{i=0,\ldots,d} \in C\left(\mathbb{R}, \mathbb{R}^{d+1}\right)$ with $\phi_0 \in C^k(\mathbb{R})$ and $\dfrac{d^i\phi_0}{dx^i} = \phi_i$ such that for any compact $K \subset \mathbb{R}$

$$\max_{\alpha\in\mathbb{Z}\cap 2^n K} \left\| \mathbf{f}_n^{(i)}(\alpha) - \Phi\left(2^{-n}\alpha\right) \right\| \to 0 \text{ when } n \to \infty.$$

Example: Hermite $\mathbb{P}_3$ interpolant at midpoint



Cubic Interpolation

$$x = \frac{u+v}{2} \quad , \quad h = v - u$$

$$\begin{aligned} f(x) &= \frac{1}{2}\left[f(v) + f(u)\right] \\ &\quad - \frac{1}{8} h \left[f'(v) - f'(u)\right] \\ f'(x) &= \frac{3}{2}\frac{f(v) - f(u)}{h} \\ &\quad + \frac{1}{4}\left[f'(v) + f'(u)\right] \end{aligned}$$

HC^1, a Hermite Subdivision Scheme:

Step 0, f and $p = f'$ are known at points of $\mathbb{Z}$.
Step n, $h_n = 2^{-n}$, if f, f' are known at $u = jh_n$ and $v = (j+1)h_n$, 2 successive points of $\mathcal{D}_n = \{ih_n\}_{i=0,\cdots,2^n}$, then at midpoint $x = h_{n+1}(2j+1)$:

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f(x) &= \lambda_1 \big[f(v) + f(u)\big] + \lambda_2 h_n \big[p(v) - p(u)\big] \\
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Hence, f and p are defined on $\bigcup \mathcal{D}_n$.

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For $\lambda_1 = 1/2$, $\lambda_2 = -1/8$, $\mu_1 = 3/2$, $\mu_2 = -1/4$, f is the cubic spline, for $\lambda_1 = 1/2$, $\lambda_2 = -1/8$, $\mu_1 = 3/2$, $\mu_2 = -1/4$, f is the quadratic spline.

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If the scheme is convergent, then $\lambda_1 = 1/2$ and $\mu_1 + 2\mu_2 = 1$.

Rewriting HC^1:

With $f_n^{(0)}(\alpha) = f(\alpha/2^n)$ and $f_n^{(1)}(\alpha) = p(\alpha/2^n)$, $\alpha \in \mathbb{Z}$,
Step 0, $\mathbf{f}_0 = [f_0^{(0)}, f_0^{(1)}]^T : \mathbb{Z} \to \mathbb{R}^2$.
Step n, for $\alpha \in \mathbb{Z}$,

$$\begin{aligned} f_{n+1}^{(0)}(2\alpha) &= f_n(\alpha),\ f_{n+1}^{(1)}(2\alpha) = f_n^{(1)}(\alpha), \\ f_{n+1}^{(0)}(2\alpha+1) &= \frac{f_n^{(0)}(\alpha+1) + f_n^{(0)}(\alpha)}{2} + \frac{\lambda}{2^n}[f_n^{(1)}(\alpha+1) - f_n^{(1)}(\alpha)], \\ f_{n+1}^{(1)}(2\alpha+1) &= (1-\mu)\frac{f_n^{(0)}(\alpha+1) - f_n^{(0)}(\alpha)}{1/2^n} + \mu\frac{f_n^{(1)}(\alpha+1) + f_n^{(1)}(\alpha)}{2}. \end{aligned}$$

or $\mathbf{D}^{n+1}\mathbf{f}_{n+1}(\alpha) = \sum_{\beta\in\mathbb{Z}} \mathbf{A}(\alpha - 2\beta)\mathbf{D}^n\mathbf{f}_n(\beta)$ where $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$,
$\mathbf{A}(0) = \mathbf{D}$, $\mathbf{A}(\epsilon) = \mathbf{D} \times \begin{bmatrix} 1/2 & -\epsilon\lambda \\ -\epsilon(1-\mu) & \mu/2 \end{bmatrix}$, for $\epsilon = \pm 1$ and $\mathbf{A}(\alpha) = 0$ for $\alpha \notin \{-1, 0, 1\}$

Hermite Scheme $\leftrightarrow$ Vector Scheme

Vector Subd. Scheme, $\mathbf{g}_{n+1}(\alpha) = \sum_{\beta\in\mathbb{Z}} \mathbf{B}(\alpha - 2\beta)\mathbf{g}_n(\beta)$ or $\mathbf{g}_{n+1} = S_{\mathbf{B}}\mathbf{g}_n$

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- For Vector Schemes, it is not so straightforward to prove more than the Hölder regularity of the limit function,
- But the contractivity is usually quite easy to prove with a finite number of matrices products,
- For Hermite subdivision schemes, the notion of convergence automatically includes regularity.
- But prove the convergence is not so easy with the powers of $\mathbf{D}$ and $\mathbf{D}^{-1}$.

Hermite Scheme $\leftrightarrow$ Vector Scheme

Vector Subd. Scheme, $\mathbf{g}_{n+1}(\alpha) = \sum_{\beta\in\mathbb{Z}} \mathbf{B}(\alpha - 2\beta)\mathbf{g}_n(\beta)$ or $\mathbf{g}_{n+1} = S_{\mathbf{B}}\mathbf{g}_n$
Hermite Subd. Scheme, $\mathbf{D}^{n+1}\mathbf{f}_{n+1}(\alpha) = \sum_{\beta\in\mathbb{Z}} \mathbf{A}(\alpha - 2\beta)\mathbf{D}^n\mathbf{f}_n(\beta)$

- For Vector Schemes, it is not so straightforward to prove more than the Hölder regularity of the limit function,
- But the contractivity is usually quite easy to prove with a finite number of matrices products,
- For Hermite subdivision schemes, the notion of convergence automatically includes regularity.
- But prove the convergence is not so easy with the powers of $\mathbf{D}$ and $\mathbf{D}^{-1}$.

Remark:
$\mathbf{g}_n(\cdot) = \mathbf{D}^n\mathbf{f}_n(\cdot)$ is not interesting since the convergence of $\mathbf{g}_n$ does not imply anything for $\mathbf{f}_n$.

From Taylor expansions...

Let $\Phi = [\phi_i]_{i=0,\ldots,d} \in C\left(\mathbb{R}, \mathbb{R}^{d+1}\right)$, $\phi_0 \in C^d(\mathbb{R})$, $\dfrac{d^i \phi_0}{dx^i} = \phi_i$ and

let $\mathbf{f}_n$ be such that $\lim_{n \to +\infty} \left\| \mathbf{f}_n^{(i)}(\cdot) - \Phi\left(2^{-n} \cdot\right) \right\| = 0$

For $r = 0, \ldots, d-1$, for $i = r, \ldots, d$

$$\phi_r\left(\tfrac{\alpha+1}{2^n}\right) \quad = \quad \phi_r\left(\tfrac{\alpha}{2^n}\right) \ldots + \quad \tfrac{2^{n(r-i)}}{(i-r)!}\phi_i\left(\tfrac{\alpha}{2^n}\right) \ldots + \quad \tfrac{2^{n(r-d)}}{(d-r)!}\phi_d\left(\tfrac{\alpha}{2^n}\right) \quad + o(2^{-(d-r)n})$$

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$$\begin{array}{ccccccc} \phi_r\left(\frac{\alpha+1}{2^n}\right) & = & \phi_r\left(\frac{\alpha}{2^n}\right)\ldots + & \frac{2^{n(r-i)}}{(i-r)!}\phi_i\left(\frac{\alpha}{2^n}\right)\ldots + & \frac{2^{n(r-d)}}{(d-r)!}\phi_d\left(\frac{\alpha}{2^n}\right) & +o(2^{-(d-r)n}) \\ \uparrow & & \uparrow & \uparrow & \uparrow & \\ f_n^{(r)}(\alpha+1) & & f_n^{(r)}(\alpha) & f_n^{(i)}(\alpha) & f_n^{(d)}(\alpha) & \end{array}$$

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$$\underset{f_n^{(r)}(\alpha+1)}{\underset{\uparrow}{\phi_r\left(\tfrac{\alpha+1}{2^n}\right)}} = \underset{f_n^{(r)}(\alpha)}{\underset{\uparrow}{\phi_r\left(\tfrac{\alpha}{2^n}\right)}} \dots + \frac{2^{n(r-i)}}{(i-r)!} \underset{f_n^{(i)}(\alpha)}{\underset{\uparrow}{\phi_i\left(\tfrac{\alpha}{2^n}\right)}} \dots + \frac{2^{n(r-d)}}{(d-r)!} \underset{f_n^{(d)}(\alpha)}{\underset{\uparrow}{\phi_d\left(\tfrac{\alpha}{2^n}\right)}} + o(2^{-(d-r)n})$$

Question: Do we have or do we need

$$f_n^{(r)}(\alpha+1) = f_n^{(r)}(\alpha) \dots + \frac{2^{n(r-i)}}{(i-r)!} f_n^{(i)}(\alpha) \dots + \frac{2^{n(r-d)}}{(d-r)!} f_n^{(d)}(\alpha) + o(2^{-(d-r)n})$$

…To a stairway

For $r = 0, \ldots, d-1$

$$\phi_r\left(\frac{\alpha+1}{2^n}\right) = \phi_r\left(\frac{\alpha}{2^n}\right) + \frac{1}{2^n}\phi_{r+1}\left(\frac{\alpha}{2^n}\right) + o(2^{-n})$$

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Taylor operators in $\ell(\mathbb{Z}^{d+1})$

Generalized incomplete Taylor operator:

$$T_d := \begin{bmatrix} \Delta & -1 & * & \dots & * \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & * \\ & & & \Delta & -1 \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} \Delta I & \\ & 1 \end{bmatrix} + [t_{jk}]_{j,k=0,\dots,d} \,,$$

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About polynomials

Π_d: space of polynomials of degree at most d,

$$\mathbb{V}_d = \left\{ \boldsymbol{v} = \begin{bmatrix} v_d \\ \vdots \\ v_0 \end{bmatrix},\ v_j(x) = \frac{1}{j!}x^j + u_j(x) \in \Pi_j,\ u_j \in \Pi_{j-1} \right\}.$$

$(x)_0 := 1$, $(x)_j := \prod_{k=0}^{j-1}(x-k)$, $j \geq 1$ and $[x]_j := \frac{1}{j!}(x)_j$ for $j \geq 0$.

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$$\boldsymbol{v} \in \mathbb{V}_d \Leftrightarrow v_j(x) = [x]_j + \bar{u}_j(x), \quad \bar{u}_j \in \Pi_{j-1} \quad j = 0, \ldots, d.$$

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Prop: For $\boldsymbol{v} \in \mathbb{V}_d$, there exists a unique generalized complete Taylor operator $\widetilde{T}_d$ such that $\widetilde{T}_d \boldsymbol{v} = 0$.

Chains and Taylor operators

A chain of length $d+1$ is a finite sequence $\boldsymbol{V} := [\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d]$ of vectors $\boldsymbol{v}_j = \begin{bmatrix} v_{j,j} \\ \vdots \\ v_{j,0} \end{bmatrix} = \begin{bmatrix} [\cdot]_j \\ \vdots \\ [\cdot]_0 \end{bmatrix} + \boldsymbol{u}_j \in \mathbb{V}_j$, $j = 0, \ldots, d$, that satisfies the condition $\boldsymbol{w}_j := \widetilde{T}(\boldsymbol{v}_j) \begin{bmatrix} v_{j+1,j+1} \\ \vdots \\ v_{j+1,1} \end{bmatrix} \in \mathbb{R}^{j+1}$.

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2/ For any $\boldsymbol{v} \in \mathbb{V}_d$, there exists a chain $\boldsymbol{V}$ of length $d+1$ with $\boldsymbol{v}_d = \boldsymbol{v}$.

Examples of Taylor operators

Let $p_j = [\cdot]_j + q_j$, $q_j \in \Pi_{j-1}$, $j = 0, \ldots, d$

1/ The classical complete Taylor operator:

$$\boldsymbol{v}_j = \begin{bmatrix} p_j \\ p_j' \\ \vdots \\ p_j^{(j)} \end{bmatrix} \text{ for } \widetilde{T}_{C,d} := \begin{bmatrix} \Delta & -1 & -\frac{1}{2!} & -\frac{1}{3!} & \cdots & -\frac{1}{d!} \\ & \Delta & -1 & -\frac{1}{2!} & \cdots & -\frac{1}{(d-1)!} \\ & & \Delta & -1 & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \Delta & -1 \\ & & & & & \Delta \end{bmatrix}$$

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3/ With Cardinal splines for $\widetilde{T}_{S,d} := \begin{bmatrix} \Delta & -1 & \cdots & -1 \\ & \ddots & \ddots & \vdots \\ & & \ddots & -1 \\ & & & \Delta \end{bmatrix}$,

Spectral chains and Factorization

A chain $\boldsymbol{V}$ of length $d+1$ is called spectral chain for a VSS with mask $\boldsymbol{A} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ if $S_{\boldsymbol{A}} \hat{\boldsymbol{v}}_j = 2^{-j} \hat{\boldsymbol{v}}_j$ for $\hat{\boldsymbol{v}}_j := \begin{bmatrix} \boldsymbol{v}_j \\ \boldsymbol{0}_{d-j} \end{bmatrix}$, $j = 0, \ldots, d$. *Generalization of the spectral condition or sum rule:* $\hat{\boldsymbol{v}}_j = [p_j, p_j', \ldots, p_j^{(d)}]^T$.

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Th: If $S_{\boldsymbol{A}}$ has a spectral chain $\boldsymbol{V}$ of length $d+1$ then there exists a finite mask $\boldsymbol{B} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ such that $\widetilde{T}(\boldsymbol{V})\, S_{\boldsymbol{A}} = S_{\boldsymbol{B}}\, \widetilde{T}(\boldsymbol{V})$.

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Th: For the mask $\boldsymbol{A} \in \ell^{(d+1)\times(d+1)}$, if there exists a mask $\boldsymbol{B}$ and a generalized incomplete Taylor operator T_d such that $T_d S_{\boldsymbol{A}} = 2^{-d} S_{\boldsymbol{B}} T_d$ and $S_{\boldsymbol{B}} \boldsymbol{e}_d = \boldsymbol{e}_d$. If a chain $\boldsymbol{V}$ for T_d satisfies $S_{\boldsymbol{A}} \hat{\boldsymbol{v}}_j \in \operatorname{span}\{\hat{\boldsymbol{v}}_0, \ldots, \hat{\boldsymbol{v}}_j\}$, for $j = 0, \ldots, d$, then there exists a spectral chain $\boldsymbol{V}'$ for $S_{\boldsymbol{A}}$.

Complete or incomplete Taylor operator?

Let $\begin{bmatrix} \boldsymbol{I}_d & \\ & \Delta \end{bmatrix} S_{\boldsymbol{B}} = S_{\widetilde{\boldsymbol{B}}} \begin{bmatrix} \boldsymbol{I}_d & \\ & \Delta \end{bmatrix}$, then $S_{\boldsymbol{B}}$ converges to a continuous limit function of the form $\boldsymbol{f}_{\boldsymbol{c}} = f_{\boldsymbol{c}}\, \boldsymbol{e}_d$ if and only if $S_{\widetilde{\boldsymbol{B}}}$ is contractive, $\widetilde{\boldsymbol{B}}_{21}(1) = 0$ and $\widetilde{\boldsymbol{B}}_{22}(1) = 1$.

Convergence

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Th 1: $\boldsymbol{A}, \boldsymbol{B} \in \ell^{d+1}(\mathbb{Z})$, 2 masks $T_d S_{\boldsymbol{A}} = 2^{-d} S_{\boldsymbol{B}} T_d$ for some generalized incomplete Taylor operator T_d. For any initial data $\boldsymbol{f}_0 \in \ell^{d+1}(\mathbb{Z})$ and sequence $\boldsymbol{f}_n$ of the Hermite scheme $H_{\boldsymbol{A}}$ $\boldsymbol{f}_n(0) \to \boldsymbol{y} \in \mathbb{R}^{d+1}$, the VSS $S_{\boldsymbol{B}}$ is convergent, and for any initial data $\boldsymbol{g}_0 = T_d \boldsymbol{f}_0$, the limit function $\Psi = \Psi_{\boldsymbol{g}} \in C\left(\mathbb{R}, \mathbb{R}^{d+1}\right)$ satisfies $\Psi = \begin{bmatrix} \mathbf{0} \\ \psi \end{bmatrix}$, $\psi \in C(\mathbb{R}, \mathbb{R})$, then $H_{\boldsymbol{A}}$ is C^p–convergent.

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Th 2: $\boldsymbol{A}, \tilde{\boldsymbol{B}} \in \ell^{d+1}(\mathbb{Z})$ 2 masks such that $\tilde{T}_d S_{\boldsymbol{A}} = 2^{-d} S_{\tilde{\boldsymbol{B}}} \tilde{T}_d$ for some generalized complete Taylor operator $\tilde{T}_d$. If for any initial data $\boldsymbol{f}_0 \in \ell^{d+1}(\mathbb{Z})$ and refinement sequence $\boldsymbol{f}_n$ of the HSS $H_{\boldsymbol{A}}$, $\boldsymbol{f}_n(0) \to \boldsymbol{y} \in \mathbb{R}^{d+1}$, if $S_{\widetilde{\boldsymbol{B}}}$ is contractive, and if $\widetilde{B}_{21}(1) = 0$ and $\widetilde{B}_{22}(1) = 1$, then $H_{\boldsymbol{A}}$ is C^d–convergent.

Interpolating schemes

A necessary condition for the C^d-convergence of an interpolating HSS ($\boldsymbol{A}(2\alpha) = \boldsymbol{D} \times \delta_{\alpha 0}$) is the reproduction of polynomials of Π_d (Dyn and Levin). In that case the spectral condition is satisfied for $p_r(x) = x^r$ and we have Taylor factorization with the classical Taylor operators.
Example of HC^2

$$\mathbf{A}(-1) = \mathbf{D} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}, \mathbf{A}(0) = \mathbf{D}, \mathbf{A}(1) = \mathbf{D} \begin{bmatrix} \alpha_1 & -\alpha_2 & \alpha_3 \\ -\beta_1 & \beta_2 & -\beta_3 \\ \gamma_1 & -\gamma_2 & \gamma_3 \end{bmatrix}.$$

Reproduction of degree 3 polynomials, among the free parameters, (Guglielmi and Manni): $\alpha_1 = \frac{1}{2}$, $\gamma_1 = 0$, $\beta_2 = \frac{1-\beta_1}{2}$, $\gamma_3 = \frac{1-\gamma_2}{2}$, $\alpha_3 = \frac{-1-8\alpha_2}{16}$, $\beta_3 = \frac{2\beta_1-3}{24}$.

de Rham scheme

Given a mask $\{\mathbf{A}\}$ of degree d, $\bar{\mathbf{f}}_0 = \mathbf{f}_0 : \mathbb{Z} \to \mathbb{R}^{d+1}$, $\bar{\mathbf{f}}_n \to \bar{\mathbf{f}}_{n+1}$

$$\begin{aligned}
\boldsymbol{D}^{n+1}\mathbf{g}(\beta) &= \sum_{\gamma\in\mathbb{Z}} \mathbf{A}(\beta - 2\gamma)\mathbf{D}^n\bar{\mathbf{f}}_n(\gamma),\ \beta \in \mathbb{Z} \\
\mathbf{D}^{n+2}\mathbf{v}(\alpha) &= \sum_{\beta\in\mathbb{Z}} \mathbf{A}(\alpha - 2\beta)\mathbf{D}^{n+1}\mathbf{g}(\beta),\ \alpha \in \mathbb{Z} \\
\bar{\mathbf{f}}_{n+1}(\alpha) &= \mathbf{v}(2\alpha + 1),\ \alpha \in \mathbb{Z}.
\end{aligned}$$

Then $\mathbf{D}^{n+1}\bar{\mathbf{f}}_{n+1}(\alpha) = \sum_{\gamma\in\mathbb{Z}} \bar{\mathbf{A}}(\alpha - 2\gamma)\mathbf{D}^n\bar{\mathbf{f}}_n(\gamma)$, then
$\bar{\mathbf{A}}(\alpha) = \mathbf{D}^{-1}\sum_{\beta\in\mathbb{Z}} \mathbf{A}(2\alpha + 1 - 2\beta)\mathbf{A}(\beta)$, de Rham scheme.

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$\text{supp}\{\mathbf{A}\} \subset [\sigma, \sigma'] \Rightarrow \text{supp}\{\bar{\mathbf{A}}\} \subset [(3\sigma - 1)/2, (3\sigma' - 1)/2]$.

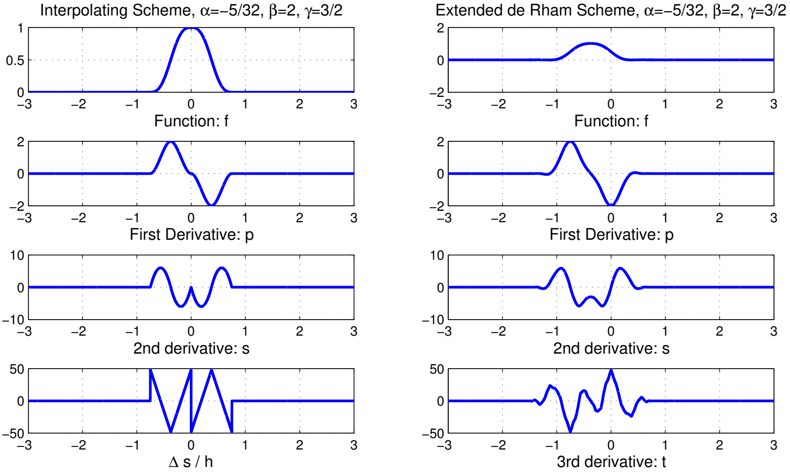
de Rham scheme from HC^2

If the initial HSS satisfies the spectral conditions with $p_r(x) = x^r$, $r = 1\ldots,\ell$ with $\ell \geq d$ then the corresponding de Rham scheme satisfies the spectral conditions with $\bar{p}_r(x) = (x - 1/2)^r$.

$$\begin{bmatrix} z^{-1}-1 & -1 & -1/2 \\ 0 & z^{-1}-1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \overline{\boldsymbol{A}}^*(z) = \frac{1}{4}\overline{\boldsymbol{B}}^*(z) \begin{bmatrix} z^{-2}-1 & -1 & -1/2 \\ 0 & z^{-2}-1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} z^{-1}-1 & -1 & -1/2 \\ 0 & z^{-1}-1 & -1 \\ 0 & 0 & z^{-1}-1 \end{bmatrix} \overline{\boldsymbol{A}}^*(z) = \frac{1}{4}\widetilde{\overline{\boldsymbol{B}}}^*(z) \begin{bmatrix} z^{-2}-1 & -1 & -1/2 \\ 0 & z^{-2}-1 & -1 \\ 0 & 0 & z^{-2}-1 \end{bmatrix}.$$

HC^2 and associated de Rham



Cardinal spline functions and Hermite schemes

Let $\varphi_0(x) = \chi_{[0,1]} = \begin{cases} 1 \text{ if } x \in [0,1], \\ 0 \text{ if } x \notin [0,1]. \end{cases}$ and for $r = 1, 2, \ldots,$

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Then $\varphi_r(x) = \frac{1}{2^r} \sum_{\alpha \in \mathbb{Z}} \binom{r+1}{\alpha} \varphi_r(2x - \alpha)$

Cardinal spline functions and Hermite schemes

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Derivative formula

$$\frac{d^i v}{dx^i}(x) = \sum_{\alpha\in\mathbb{Z}} 2^{ni} \Delta^i f_n^{(0)}(\alpha - i)\, \varphi_{r-i}\left(2^n x - \alpha\right), \qquad i = 0, \ldots, r-1.$$

Properties of the assoicated scalar scheme

Transformation of polynomials

$$p \in \mathcal{P}_r \Rightarrow S_{a_r} p \in \mathcal{P}_r.$$

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A spectral property

$$\ell_r(x) = \frac{1}{r!} \prod_{j=1}^{r} (x + j) \Rightarrow S_{a_r} \ell_r^{(i)} = \frac{1}{2^{r-i}} \ell_r^{(i)},\ i = 0, \ldots, r.$$

Define

$$p_j = \ell_r^{(r-j)}, \quad j = 0, \ldots, r$$

Extension of S_{a_r} to Hermite subdivision schemes

For $d \leq r$, define the mask in $\ell^{(d+1)\times(d+1)}(\mathbb{Z})$

$$\mathbf{A}(\alpha) = \begin{bmatrix} a_r(\alpha) & 0 & \dots & 0 \\ \Delta a_r(\alpha - 1) & 0 & \dots & 0 \\ \Delta^2 a_r(\alpha - 2) & 0 & \dots & 0 \\ \vdots & & & \\ \Delta^d a_r(\alpha - d) & 0 & \dots & 0 \end{bmatrix}$$

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For $p \in \Pi_d$ if $\hat{\boldsymbol{v}}_p = [p, \Delta p(\cdot-1), \dots, \Delta^d p(\cdot-d)]^T$ then
$S_{\boldsymbol{A}} \hat{\boldsymbol{v}}_{p_j} = \frac{1}{2^j} \hat{\boldsymbol{v}}_{p_j}$ for $j = 0, \dots, d$ and the $\{\hat{\boldsymbol{v}}_j\}$ forms a chain. There exists a finite mask $\boldsymbol{B} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ such that
$\widetilde{T}_{S,d}\, S_{\boldsymbol{A}} = S_{\widetilde{\boldsymbol{B}}}\, \widetilde{T}_{S,d}$.

Cardinal Spline with $r = 4$, $d = 3$

$$\boldsymbol{A}^*(z) = \frac{(1+z)^5}{2^4} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ (1-z) & 0 & 0 & 0 & 0 \\ (1-z)^2 & 0 & 0 & 0 & 0 \\ (1-z)^3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\tilde{\boldsymbol{B}}^*(z) = \begin{bmatrix} -\frac{(z-1)^3\, z\,(1+z)^4}{2} & \frac{(z-1)^2\, z^3\,(1+z)^3}{2} & -\frac{(z-1)\, z^3\,(1+z)^2}{2} & \frac{z^3\,(1+z)}{2} \\ -\frac{(z-1)^3\, z\,(1+z)^4}{2} & \frac{(z-1)^2\, z^3\,(1+z)^3}{2} & -\frac{(z-1)\, z^3\,(1+z)^2}{2} & \frac{z^3\,(1+z)}{2} \\ -\frac{(z-1)^3\, z\,(1+z)^4}{2} & \frac{(z-1)^2\, z^3\,(1+z)^3}{2} & -\frac{(z-1)\, z^3\,(1+z)^2}{2} & \frac{z^3\,(1+z)}{2} \\ -\frac{(z-1)^3\, z\,(1+z)^4}{2} & \frac{(z-1)^2\, z^3\,(1+z)^3}{2} & -\frac{(z-1)\, z^3\,(1+z)^2}{2} & \frac{z^3\,(1+z)}{2} \end{bmatrix}.$$

Vector Scheme $\rightarrow$ Hermite Scheme, first construction

If the nonzero elements of the matrix $\widetilde{\boldsymbol{B}}^*$ are of the form

$$\begin{aligned}
\widetilde{b}^*_{jk}(z) &= (z^{-1}-1)^{j+1}\, h^*_{jk}(z), \qquad 0 \le k < j < d, \\
\widetilde{b}^*_{jj}(z) &= \frac{(z^{-1}-1)^{j+1}}{2^{j+1}}, \qquad j = 0, \ldots, d-1, \\
\widetilde{b}^*_{dj}(z) &= \frac{1}{2}(z^{-1}+1)\left(z^{-2}-1\right)^{d-j} \qquad j = 0, \ldots, d,
\end{aligned}$$

then there exists a C^d–convergent Hermite subdivision scheme whose mask $\boldsymbol{A}$ satisfies $\widetilde{T}_\Delta S_{\boldsymbol{A}} = 2^{-d} S_{\widetilde{\boldsymbol{B}}} \widetilde{T}_\Delta$.

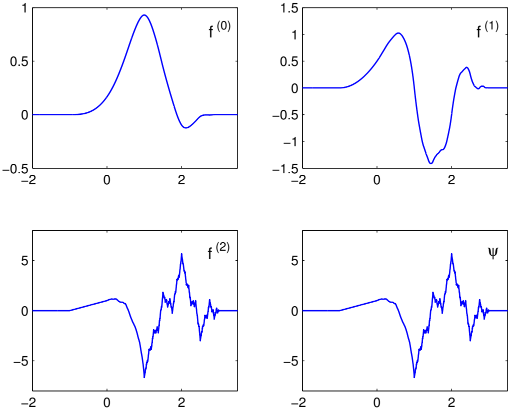
Example with $d = 2$

$$\widetilde{\boldsymbol{B}}^*(z) = \begin{bmatrix} -\frac{z-1}{2z} & 0 & 0 \\ \frac{(z-1)^2}{z^2} & \frac{(z-1)^2}{4z^2} & 0 \\ \frac{(z-1)^2(1+z)^3}{2z^5} & -\frac{(z-1)(1+z)^2}{2z^3} & \frac{1+z}{2z} \end{bmatrix}$$

and get the corresponding

$$\boldsymbol{A}^*(z) = 1/4 \begin{bmatrix} -\frac{(1+z)\left(-1-3z-6z^2+2z^3\right)}{2z^4} & -\frac{7z^2-1}{4z^2} & -\frac{1}{4} \\ \frac{(z-1)(1+z)\left(-1-3z-5z^2+z^3\right)}{2z^5} & \frac{(z-1)\left(5z^2-1\right)}{4z^3} & \frac{z-1}{4z} \\ \frac{(z-1)^2(1+z)^4}{2z^6} & 0 & 0 \end{bmatrix}$$

which give a C^2–convergent Hermite subdivision scheme



Limit functions showing the three entries of the limit function of the Hermite subdivision scheme and the nonzero limit function of the associated convergent difference scheme, $S_{\boldsymbol{B}}$.

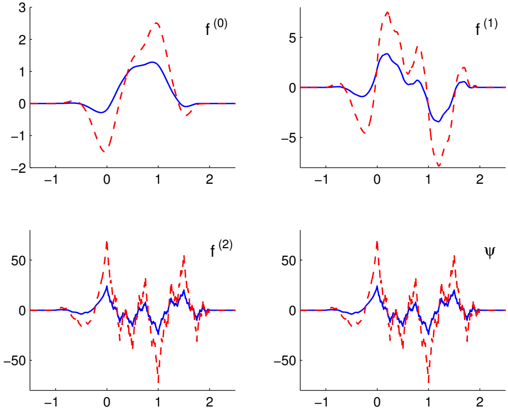
Vector Scheme $\to$ Hermite Scheme, generic construction

For any $d \in \mathbb{N}$ and any generalized Taylor operator $\widetilde{T}$ of order d there exists a convergent Hermite subdivision scheme with mask $\boldsymbol{A}$ such that $\widetilde{T} S_{\boldsymbol{A}} = 2^{-d} S_{\widetilde{\boldsymbol{B}}} \widetilde{T}$ for some appropriate $\widetilde{\boldsymbol{B}}$.

$$\widetilde{\boldsymbol{B}}^*(z) =$$

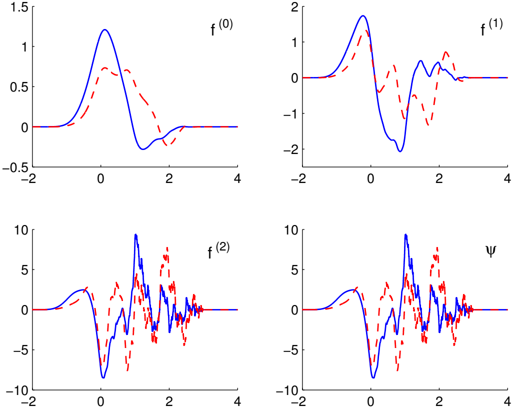
$$\begin{bmatrix} \frac{z^{-1}-1}{2} & & & & \\ (z^{-1}-1)^2\, h^*_{10}(z) & \frac{(z^{-1}-1)^2}{4} & & & \\ \vdots & \ddots & \ddots & & \\ (z^{-1}-1)^d\, h^*_{d-1,0}(z) & \dots & (z^{-1}-1)^d\, h^*_{d-1,d-2}(z) & \frac{(z^{-1}-1)^d}{2^d} & \\ \widetilde{b}^*_{d0}(z) & \dots & \widetilde{b}^*_{d,d-2}(z) & \widetilde{b}^*_{d,d-1}(z) & \widetilde{b}^*_{dd}(z) \end{bmatrix}$$

Example for $d = 2$ with a free parameter w_{21}



Limit functions showing the three entries of the limit function of the Hermite subdivision scheme and the nonzero limit function of the associated convergent difference scheme, $S_{\boldsymbol{B}}$, $w_{21} = \frac{1}{2}$ (blue, solid) and $w_{21} = 1$ (red, dashed).

New example with $d = 2$



Limit functions $w_{21} = \frac{1}{2}$ (blue, solid) and $w_{21} = 1$ (red, dashed).

Next work

Can we obtain C^p–convergence with $p > d$?