Derivative-Orthogonal Multiwavelets and Multiwavelets on the Interval

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What Is a Wavelet?

- Let $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ and $\psi = (\psi_1, \ldots, \psi_s)^\mathsf{T}$ in $L_2(\mathbb{R})$.
- A system is derived from ϕ, ψ via dilates and integer shifts:

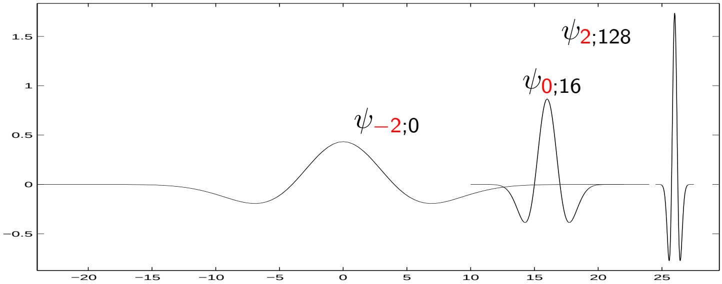
$$\begin{aligned} \mathrm{AS}_0(\phi;\psi) := & \{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\} \cup \\ & \{\psi_{j;k} := 2^{j/2}\psi(2^j \cdot - k) \ : \ j \in \mathbb{N} \cup \{0\}, k \in \mathbb{Z}\}. \end{aligned}$$

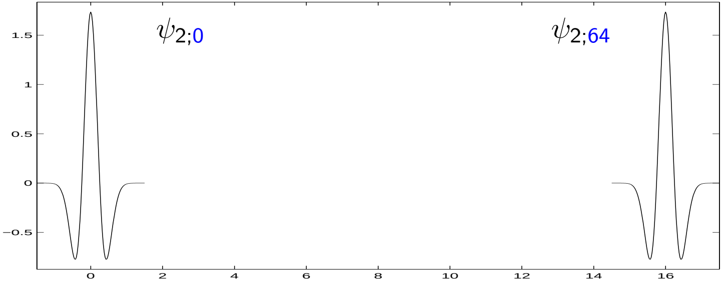
- $\{\phi; \psi\}$ is called an orthogonal wavelet in $L_2(\mathbb{R})$ if $\mathrm{AS}_0(\phi;\psi)$ is an orthonormal basis of $L_2(\mathbb{R})$.
- Wavelet representation:

$$f = \sum_{k\in\mathbb{Z}} \langle f, \phi(\cdot - k)\rangle \phi(\cdot - k) + \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j;k}\rangle \psi_{j;k}, \quad f \in L_2(\mathbb{R}),$$

where $\langle f, g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}^\mathsf{T} dx$ is the inner product.

Dilates and Shifts of a Wavelet





Riesz Wavelets in Sobolev Spaces

- For $\tau \in \mathbb{R}$, the Sobolev space $H^\tau(\mathbb{R})$ consists of all tempered distributions f satisfying

$$\|f\|^2_{H^\tau(\mathbb{R})} := \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f}(\xi)|^2(1+|\xi|^2)^\tau d\xi < \infty,$$

  For $m \in \mathbb{N}\cup\{0\}$, $f \in H^m(\mathbb{R})$ if $f, f', \ldots, f^{(m)} \in L_2(\mathbb{R})$.
- For $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ and $\psi = (\psi_1, \ldots, \psi_r)^\mathsf{T}$, we define

$$\mathrm{AS}_0^\tau(\phi;\psi) = \{\phi(\cdot - k) : k \in \mathbb{Z}\} \cup \{2^{j(\frac{1}{2}-\tau)}\psi(2^j \cdot - k) : k \in \mathbb{Z}\}_{j=0}^\infty.$$

- $\{\phi;\psi\}$ is a Riesz wavelet in the Sobolev space $H^\tau(\mathbb{R})$ if $\mathrm{AS}_0^\tau(\phi;\psi)$ is Riesz basis for $H^\tau(\mathbb{R})$: the linear span of $\mathrm{AS}_0^\tau(\phi;\psi)$ is dense in $H^\tau(\mathbb{R})$ and there exist $C_1, C_2 > 0$ such that

$$C_1 \sum_{h\in \mathrm{AS}_0^\tau(\phi;\psi)} |c_h|^2 \leqslant \Big\| \sum_{h\in \mathrm{AS}_0^\tau(\phi;\psi)} c_h h \Big\|_{H^\tau(\mathbb{R})} \leqslant C_2 \sum_{h\in \mathrm{AS}_0^\tau(\phi;\psi)} |c_h|^2$$

  for all finitely supported sequences $\{c_h\}_{h\in \mathrm{AS}_0^\tau(\phi;\psi)}$.

Derivative-Orthogonal Riesz Wavelets

- Let $m \in \mathbb{N} \cup \{0\}$ be a nonnegative integer.
- Let $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ and $\psi = (\psi_1, \ldots, \psi_r)^\mathsf{T}$ in $H^m(\mathbb{R})$.
- We say that $\{\phi; \psi\}$ is an mth-order derivative-orthogonal Riesz wavelet in the Sobolev space $H^m(\mathbb{R})$ if
  1. $\mathrm{AS}_0^m(\phi; \psi)$ is a Riesz wavelet in $H^m(\mathbb{R})$;
  2. The mth-order derivatives are orthogonal between levels:
     $$\langle \psi^{(m)}, \phi^{(m)}(\cdot - k) \rangle = 0, \qquad \forall\, k \in \mathbb{Z},$$
     and
     $$\langle \psi^{(m)}(2^j \cdot - k), \psi^{(m)}(2^{j'} \cdot - k') \rangle = 0,$$
     for all $k, k' \in \mathbb{Z}, j, j' \in \mathbb{N}_0$ with $j \neq j'$.

For $m = 0$, they are called semi-orthogonal (or pre-) wavelets and well studied, e.g., [Chui-Wang,Jia-Micchelli,Shen-Riemanschneider] through a simple orthogonalization procedure.

Why Derivative-Orthogonal Riesz Wavelets?

- Differential equation with homogeneous boundary condition:

$$u^{(2m)}(x) + \alpha u(x) = f(x), \quad x \in I = [0, 1].$$

- The Galerkin formulation of a weak solution $u \in H^m(I)$ is

$$(-1)^m \langle u^{(m)}, v^{(m)} \rangle + \alpha \langle u, v \rangle = \langle f, v \rangle, \quad v \in H^m(I).$$

- Let S be a Riesz wavelet basis of $H^m(I)$ derived from mth-order derivative-orthogonal wavelet $\{\phi; \psi\}$. Then $u = \sum_{h \in S} c_h h$ and

$$\sum_{h \in S} \Big((-1)^m A_{h,g} + \alpha B_{h,g} \Big) c_h = \langle f, g \rangle, \qquad g \in S$$

with $A = (\langle h^{(m)}, g^{(m)} \rangle)_{h,g \in S}$ and $B = (\langle h, g \rangle)_{h,g \in S}$.

(1) The matrix A is sparse and is almost diagonal.

(2) The condition number of A dominates that of $(-1)^m A + \alpha B$ and is often very small (can be the optimal condition number 1).

Stable Integer Shifts of Vector Functions

- Let $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ in $H^m(\mathbb{R})$ be compactly supported.
- We say that the integer shifts of ϕ are stable if

$$\mathrm{span}\{\widehat{\phi}(\xi + 2\pi k) \ : \ k \in \mathbb{Z}\} = \mathbb{C}^r, \qquad \forall \xi \in \mathbb{R}.$$

- For $f = (f_1, \ldots, f_r)^\mathsf{T}$ and $g = (g_1, \ldots, g_r)^\mathsf{T}$, bracket product is

$$[\widehat{f}, \widehat{g}](\xi) := \sum_{k \in \mathbb{Z}} \widehat{f}(\xi + 2\pi k)\overline{\widehat{g}(\xi + 2\pi k)}^\mathsf{T}, \quad \xi \in \mathbb{R}.$$

- The integer shifts of ϕ in $H^m(\mathbb{R})$ are stable $\Longleftrightarrow$ $[\widehat{\phi}, \widehat{\phi}](\xi) > 0$ for all $\xi \in \mathbb{R}$ $\Longleftrightarrow$ $\{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\}$ is a Riesz sequence in $H^m(\mathbb{R})$.
- Smoothness of a function is measured by

$$\mathrm{sm}(\phi) := \sup\{\tau \in \mathbb{R} \ : \ \phi \in H^\tau(\mathbb{R})\}.$$

Semi-orthogonal (or Pre-) Wavelets

For $m = 0$, mth order derivative-orthogonal wavelets are called semi-orthogonal (or pre-) wavelets and well studied, e.g., [Chui-Wang,Jia-Micchelli,Shen-Riemanschneider] through

- $\phi \in L_2(\mathbb{R})$ has compact support and satisfies
$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$$
where $\widehat{a}(\xi) := \sum_{k\in\mathbb{Z}} a(k)e^{-ik\xi}$ is a 2π-periodic trigonometric polynomial and $\widehat{\phi}(\xi) := \int_{\mathbb{R}} \phi(x)e^{-ix\xi}dx$.
- Assume the integer shifts of ϕ are stable: $[\widehat{\phi}, \widehat{\phi}](\xi) > 0$.
- Define $\widehat{\psi}(2\xi) := e^{-i\xi}\overline{\widehat{a}(\xi+\pi)}[\widehat{\phi}, \widehat{\phi}](\xi+\pi)\widehat{\phi}(\xi)$.
- Then $\{\phi; \psi\}$ is a semi-orthogonal wavelet in $L_2(\mathbb{R})$, that is,
$$\mathrm{AS}_0^0(\phi;\psi) = \{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j \cdot -k) \ : \ k \in \mathbb{Z}\}_{j=0}^{\infty}$$
is almost an orthonormal basis for $L_2(\mathbb{R})$, except orthogonality among the same scale level j.

Sum Rules of a Filter

- A filter $a \in (l_0(\mathbb{Z}))^{r\times r}$ has order m sum rules if there exists a matching filter $\upsilon \in (l_0(\mathbb{Z}))^{1\times r}$ such that $\widehat{\upsilon}(0) \neq 0$ and

$$\widehat{\upsilon}(2\xi)\widehat{a}(\xi) = \widehat{\upsilon}(\xi) + \mathscr{O}(|\xi|^m), \quad \widehat{\upsilon}(2\xi)\widehat{a}(\xi+\pi) = \mathscr{O}(|\xi|^m), \ \ \xi \to 0.$$

- $f(\xi) = g(\xi) + \mathscr{O}(|\xi|^m), \xi \to 0 \Leftrightarrow f^{(j)}(0) = g^{(j)}(0), 0 \leqslant j < m.$
- $\operatorname{sr}(a)$ denotes the highest order of sum rules satisfied by a.
- For $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ with stable integer shifts, TFAE
  - The filter a has order m sum rules (i.e., $\operatorname{sr}(a) \geqslant m$).
  - All the polynomials of degree $< m$ are contained inside the shift-invariant space

$$S_j(\phi) := \Big\{ \sum_{k\in\mathbb{Z}} v(k)\phi(2^j\cdot - k) \ : \ \text{all sequences } v \text{ on } \mathbb{Z} \Big\}$$

  - The vector function ϕ has approximation order m:

$$\inf_{g\in S_j(\phi)\cap L_2(\mathbb{R})} \|f-g\|_{L_2(\mathbb{R})} \leqslant C2^{-jm}\|f\|_{H^m(\mathbb{R})}, \quad f \in H^m(\mathbb{R}).$$

Main Result: Existence

Theorem

Let $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ be a compactly supported refinable vector function in $H^m(\mathbb{R})$ with $m \in \mathbb{N}_0$ such that $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ for some $a \in (l_0(\mathbb{Z}))^{r\times r}$. Then there exists a finitely supported high-pass filter $b \in (l_0(\mathbb{Z}))^{r\times r}$ such that $\{\phi; \psi\}$ with

$$\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$$

is an mth-order derivative-orthogonal Riesz wavelet in the Sobolev space $H^m(\mathbb{R})$ $\Longleftrightarrow$

1. *the integer shifts of ϕ are stable;*
2. *the filter a has at least order $2m$ sum rules (i.e., $\mathrm{sr}(a) \geqslant 2m$).*

Remark: The proof is quite complicated for $r > 1$ and $m > 0$.

Main Result: Construction

Theorem

Let $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ be a compactly supported refinable vector function in $H^m(\mathbb{R})$ with $m \in \mathbb{N}_0$ such that $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ for some $a \in (l_0(\mathbb{Z}))^{r\times r}$. Suppose that the integer shifts of ϕ are stable. For any $b \in (l_0(\mathbb{Z}))^{r\times r}$, $\{\phi; \psi\}$ with $\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$ is an mth-order derivative-orthogonal Riesz wavelet in $H^m(\mathbb{R})$ $\Longleftrightarrow$

$$\widehat{b}(\xi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi)\overline{\widehat{a}(\xi)}^\mathsf{T} + \widehat{b}(\xi+\pi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi+\pi)\overline{\widehat{a}(\xi+\pi)}^\mathsf{T} = 0,$$

$$\det(\{\widehat{a}; \widehat{b}\})(\xi) := \det\left(\begin{bmatrix} \widehat{a}(\xi) & \widehat{a}(\xi+\pi) \\ \widehat{b}(\xi) & \widehat{b}(\xi+\pi) \end{bmatrix}\right) \neq 0, \qquad \forall\, \xi \in \mathbb{R}.$$

Moreover, $\mathrm{AS}_0^\tau(\phi; \psi)$ is a Riesz basis in the Sobolev space $H^\tau(\mathbb{R})$ for all τ in the nonempty open interval $(2m - \mathrm{sm}(\phi), \mathrm{sm}(\phi))$.

Construction for the Scalar Case $r = 1$

Theorem

Let $m \in \mathbb{N}_0$ and $a \in l_0(\mathbb{Z})$ such that $\widehat{a}(\xi) = 2^{-2m}(1 + e^{-i\xi})^{2m}\widehat{\breve{a}}(\xi)$ with $\breve{a} \in l_0(\mathbb{Z})$ and $\widehat{\breve{a}}(0) = 1$. Define $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$ and $\widehat{\mathring{\phi}}(\xi) := \prod_{j=1}^{\infty} \widehat{\mathring{a}}(2^{-j}\xi)$ with $\widehat{\mathring{a}}(\xi) := 2^{-m}(1 + e^{-i\xi})^m\widehat{\breve{a}}(\xi)$. Suppose that $\phi \in H^m(\mathbb{R})$ and the integer shifts of ϕ are stable. Then for $b \in l_0(\mathbb{Z})$, $\{\phi; \psi\}$ with $\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$ is an mth-order derivative-orthogonal Riesz wavelet in $H^m(\mathbb{R})$ $\iff$

$$\widehat{b}(\xi) = e^{i(m-1)\xi}\overline{\widehat{\breve{a}}(\xi + \pi)}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi + \pi)\widehat{\theta}(2\xi)/\widehat{c}(2\xi),$$

where $\theta \in l_0(\mathbb{Z})$ satisfying $\widehat{\theta}(\xi) \neq 0\ \forall \xi \in \mathbb{R}$ and $c \in l_0(\mathbb{Z})$ is

$$\widehat{c}(2\xi) := \gcd\left(\overline{\widehat{\breve{a}}(\xi)}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi), \overline{\widehat{\breve{a}}(\xi + \pi)}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi + \pi)\right).$$

Example of Refinable Functions: B-Splines

- For $n \in \mathbb{N}$, the B-spline function B_n of order n is defined to be

$$B_1 := \chi_{(0,1]}, \quad B_n := B_{n-1} * B_1 = \int_0^1 B_{n-1}(\cdot - t)dt.$$

- $B_n = B_n(n - \cdot)$ (symmetry), $\mathrm{sm}(B_n) = n - 1/2$ and B_n is a piecewise polynomial: $B_n|_{(k,k+1)} \in \mathbb{P}_{n-1}$ for all $k \in \mathbb{Z}$.
- $B_n = 2\sum_{k\in\mathbb{Z}} a(k)B_n(2\cdot - k)$, i.e., $\widehat{B_n}(2\xi) = \widehat{a}(\xi)\widehat{B_n}(\xi)$ with

$$\widehat{a}(\xi) := 2^{-n}(1 + e^{-i\xi})^n \quad \text{with} \quad \mathrm{sr}(a) = n.$$

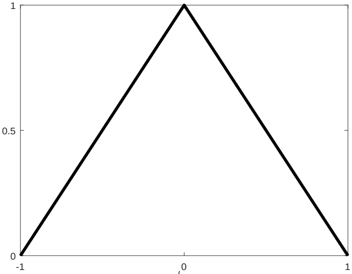
- A compactly supported mth-order derivative-orthogonal Riesz wavelet $\{B_n; \psi\}$ in $H^m(\mathbb{R})$ can be derived from B_n $\Longleftrightarrow$

$$n \geqslant \max(m+1, 2m).$$

Example from B-Spline B_2: $m = 1$

$$a = \left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\}_{[-1,1]}, \qquad b = \left\{\frac{1}{2}\right\}_{[1,1]}.$$

$\operatorname{sr}(a) = 2$ and $\phi = B_2(\cdot - 1)$ is the centered piecewise linear spline B_2 of order 2 satisfying $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$. The wavelet function $\psi = \phi(2x)$. Then $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$, which is closely linked to finite element methods.



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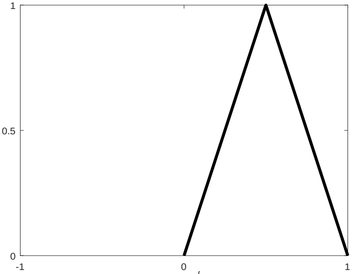

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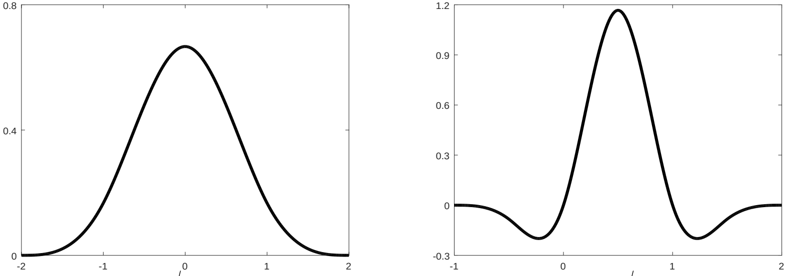
Example from B-Spline B_4: $m = 2$

$$a = \left\{ \frac{1}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{16} \right\}_{[-2,2]}, \quad b = \left\{ -\frac{1}{4}, 1, -\frac{1}{4} \right\}_{[0,2]}$$

$\mathrm{sr}(a) = 4$ and $\phi = B_4(\cdot - 2)$ is the centered B-spline B_4 of order 4 satisfying $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$. The wavelet function $\psi = \phi(2x) - \frac{1}{2}\phi(2x+1) - \frac{1}{2}\phi(2x-1)$. Then $\{\phi; \psi\}$ is a second-order derivative-orthogonal Riesz wavelet in $H^2(\mathbb{R})$, which is used in [Jia-Zhao, Math. Comp., (2011)] for 2D biharmonic equations.



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Hermite Cubic Splines

The Hermite cubic splines are given by

$$\phi_1 = \begin{cases} (1-x)^2(1+2x), & x \in [0,1], \\ (1+x)^2(1-2x), & x \in [-1,0), \\ 0, & \text{otherwise}, \end{cases} \qquad \phi_2 = \begin{cases} (1-x)^2 x, & x \in [0,1], \\ (1+x)^2 x, & x \in [-1,0), \\ 0, & \text{otherwise}. \end{cases}$$

Then $\phi = (\phi_1, \phi_2)^\mathsf{T}$ in $H^2(\mathbb{R})$ satisfies the refinement equation $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ with the filter $a \in (l_0(\mathbb{Z}))^{2\times 2}$:

$$a = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ -\frac{1}{16} & -\frac{1}{16} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{3}{8} \\ \frac{1}{16} & -\frac{1}{16} \end{bmatrix} \right\}_{[-1,1]}.$$

The filter a has order 4 sum rules with $\operatorname{sr}(a) = 4$ and ϕ has the Hermite interpolation property with $\operatorname{sm}(\phi) = 5/2$:

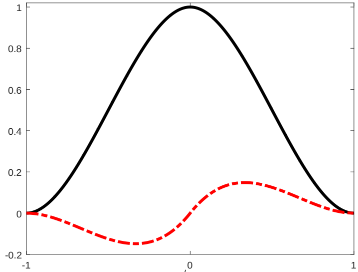
$$\phi_1(k) = \boldsymbol{\delta}(k), \quad \phi_1'(k) = 0, \quad \phi_2(k) = 0, \quad \phi_2'(k) = \boldsymbol{\delta}(k), \quad \forall\, k \in \mathbb{Z},$$

where $\boldsymbol{\delta}(0) = 1$ and $\boldsymbol{\delta}(k) = 0$ for $k \neq 0$.

Example from Hermite Cubic Splines: $m = 1$

$$b = \left\{ \begin{bmatrix} \frac{2}{21} & 1 \\ \frac{1}{9} & 1 \end{bmatrix}, \begin{bmatrix} -\frac{4}{21} & 0 \\ 0 & \frac{4}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{21} & -1 \\ -\frac{1}{9} & 1 \end{bmatrix} \right\}_{[-1,1]}.$$

Then $\{\phi; \psi\}$ with $\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$, which is used in [Jia-Liu, Adv. Comput. Math., (2006)] for Sturm-Liouville equations.



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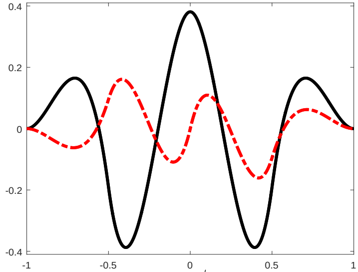

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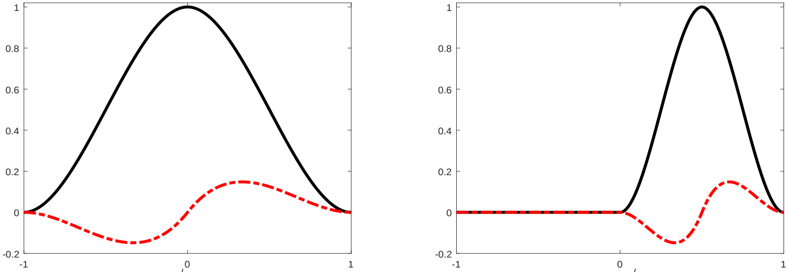
Example from Hermite Cubic Splines: $m = 2$

$$b = \left\{ \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right\}_{[1,1]}.$$

Then $\{\phi; \psi\}$ with $\psi = \phi(2\cdot)$ is a second-order derivative-orthogonal Riesz wavelet in $H^2(\mathbb{R})$.



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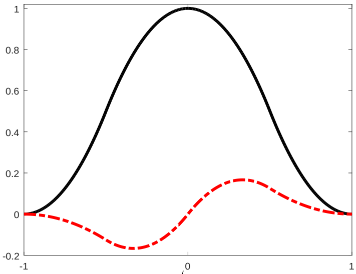
Example from Hermite Quadratic Splines: $m = 1$

$$a = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{16} & -\frac{1}{8} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{16} & -\frac{1}{8} \end{bmatrix} \right\}_{[-1,1]},$$

$$b = \left\{ \begin{bmatrix} \frac{1}{6} & 1 \\ \frac{1}{6} & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} \frac{1}{6} & -1 \\ -\frac{1}{6} & 1 \end{bmatrix} \right\}_{[-1,1]}.$$

Then $\mathrm{sr}(a) = 3$, $\mathrm{sm}(\phi) = 2.5$ and $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$.



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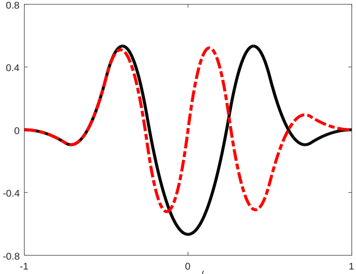

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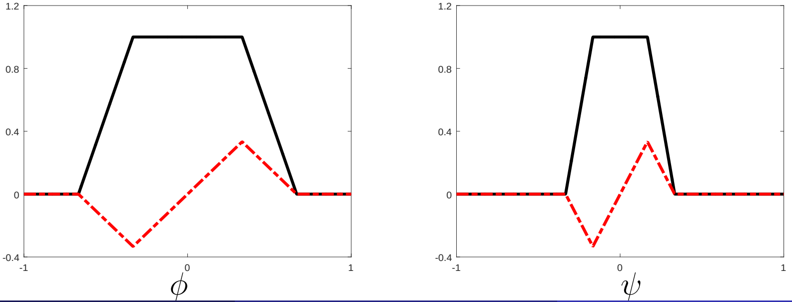
Example from Hermite Linear Splines: $m = 1$

$$a = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{12} & -\frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{12} & -\frac{1}{4} \end{bmatrix} \right\}_{[-1,1]},$$

$$b = \left\{ \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{\sqrt{2}}{4} \end{bmatrix} \right\}_{[0,0]}.$$

Then $\mathrm{sr}(a) = 2$, $\mathrm{sm}(\phi) = 1.5$ and $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$.



Wavelets on $[0, 1]$ with Very Simple Structure

- Let $n > 1$ be the scale/resolution level.
- Let $g_1 = \phi^L(2\cdot), g_2 = \phi_1(2\cdot -1), g_3 = \phi_2(2\cdot -1), g_4 = \phi^R(2\cdot)$.
- For $1 < j \leqslant n$, the boundary wavelets at level j are given by

$$g_{2^j+1} := 2^{j(1/2-m)}\psi^L(2^j\cdot), \quad g_{2^j+2^j} := 2^{j(1/2-m)}\psi^R(2^j\cdot -2^j)$$

with only one boundary wavelet at each endpoint:

$$\phi^L \in \{\phi_1|_{[0,1]}, \phi_2|_{[0,1]}\}, \qquad \phi^R \in \{\phi_1(\cdot - 1)|_{[0,1]}, \phi_2(\cdot - 1)|_{[0,1]}\},$$
$$\psi^L \in \{\psi_1|_{[0,1]}, \psi_2|_{[0,1]}\}, \qquad \psi^R \in \{\psi_1(\cdot - 1)|_{[0,1]}, \psi_2(\cdot - 1)|_{[0,1]}\}$$

- For $k = 1, \ldots, 2^j - 1$, the inner wavelets at level j are

$$g_{2^j+(2k-1)} := 2^{j(1/2-m)}\psi_1(2^j\cdot -k), \quad g_{2^j+2k} := 2^{j(1/2-m)}\psi_2(2^j\cdot -k).$$

- Normalize $\{g_k\}_{k=1}^{2^{n+1}}$ so that $\|g_k^{(m)}\|_{L_2(\mathbb{R})} = 1$ for $k = 1, \ldots, 2^{n+1}$.

Example: 1D Sturm-Liouville Equations

- Consider the following differential equation:

$$\begin{cases} -u'' + \alpha u = f \quad \text{on} \;\; (0,1), \\ u'(0) = 100(1-e^{-1}), \quad u(1) = 200e^{-1} - 100, \end{cases}$$

  where $\alpha = 5$ and $f(x) = -100e^{-x} - 500(1-e^{-x}) - 500e^{-1}x$. The exact solution is $u(x) = 100(1-e^{-x}) - 100e^{-1}x$.
- Its corresponding Galerkin formulation is

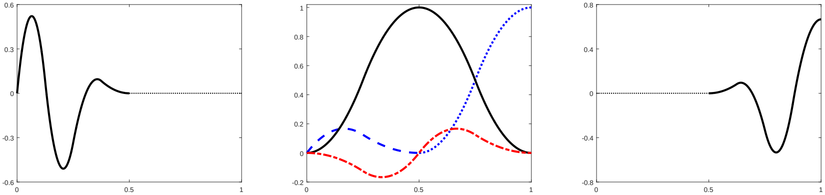
$$\sum_{k=1}^{2^{n+1}} A_{j,k} c_k = \langle g_j, f \rangle, \qquad j = 1, \ldots, 2^{n+1}$$

  with the coefficient matrix $A_{j,k} := \langle g_j', g_k' \rangle + \alpha \langle g_j, g_k \rangle$.
- According to boundary conditions, the boundary elements are

$$\phi^L := \phi_2|_{[0,1]}, \qquad \phi^R := \phi_1(\cdot - 1)|_{[0,1]},$$
$$\psi^L := \psi_2|_{[0,1]}, \qquad \psi^R := \psi_1(\cdot - 1)|_{[0,1]}.$$

Boundary Elements and Numerical Performance

Use the first-order derivative-orthogonal Riesz wavelet derived from Hermite quadratic splines with $m = 1$.



Level	Size	Iteration	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$
5	128	15	3.2106	4.2213e-07	1.8141e-07	—
6	256	15	3.2106	5.2757e-08	2.2607e-08	3.0044
7	512	15	3.2106	6.5944e-09	2.8215e-09	3.0022
8	1024	15	3.2106	8.2365e-10	3.5242e-10	3.0011
9	2048	16	3.2106	1.1561e-10	4.6097e-11	2.9346

Table : Size of linear system, conjugate gradient iterations, condition number κ of the coefficient matrices A, errors, and convergence rates.

Example: 1D Sturm-Liouville Equations

- Consider the following differential equation:

$$\begin{cases} -u'' = f & \text{on } (0,1), \\ u'(0) = 0, & u(1) = 0, \end{cases}$$

where $f(x) = x\ln(x+1)$. The exact solution is $u(x) = (5x^3 - 12x^2 - 12x + 19 - 24\ln(2) - 6(x-2)(x+1)^2\ln(x+1))/36$.

- Its corresponding Galerkin formulation is

$$\sum_{k=1}^{2^{n+1}} A_{j,k} c_k = \langle g_j, f \rangle, \qquad j = 1, \dots, 2^{n+1}$$

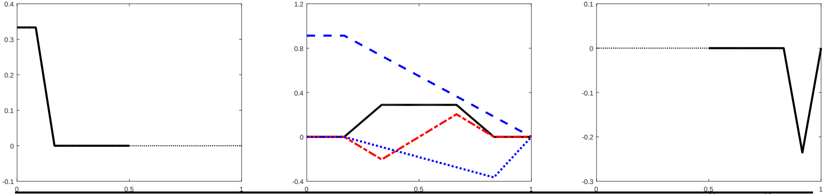
with the coefficient matrix $A_{j,k} := \langle g_j', g_k' \rangle$.

- According to boundary conditions, the boundary elements are

$$\phi^L := \phi_1|_{[0,1]}, \qquad \phi^R := \phi_2(\cdot - 1)|_{[0,1]},$$
$$\psi^L := \psi_1|_{[0,1]}, \qquad \psi^R := \psi_2(\cdot - 1)|_{[0,1]}.$$

Boundary Elements and Numerical Performance

Use the first-order derivative-orthogonal Riesz wavelet derived from Hermite linear splines with $m = 1$.



Level	Size	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$
8	1024	1	1.4640e-07	4.1807e-08	1.9971
9	2048	1	3.6662e-08	1.0462e-08	1.9986
10	4096	1	9.1733e-09	2.6168e-09	1.9995
11	8192	1	2.2943e-09	6.5284e-10	2.0029
12	16384	1	5.7369e-10	1.5988e-10	2.0298

Table : Size of linear system, no iteration is needed, condition number κ of the matrices A (A is the identity matrix), errors, and convergence rates.

Example: 1D Biharmonic Equations

- Consider the following differential equation:

$$\begin{cases} u^{(4)} - \alpha u = f \quad \text{on} \quad (0,1), \\ u(0) = 0, \quad u'(0) = 0, \quad u(1) = 0, \quad u'(1) = 0, \end{cases}$$

  where $\alpha = 11$ and $f(x) = -4\pi^4 \cos(2\pi x) - \frac{11}{4}(1 - \cos(2x))$. The exact solution is $u(x) = \frac{1}{4}(1 - \cos(2\pi x))$.
- Its corresponding Galerkin formulation is

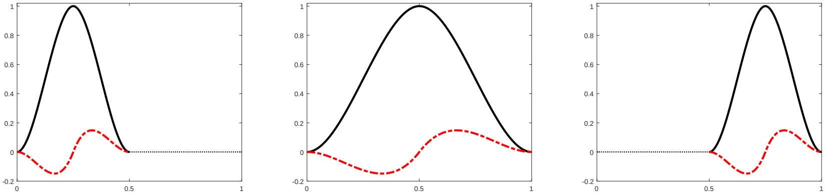
$$\sum_{k=1}^{2^{n+2}} B_{j,k} c_k = \langle g_j, f \rangle, \qquad j = 1, \ldots, 2^{n+2} - 2$$

  with the coefficient matrix $B_{j,k} := \langle g_j'', g_k'' \rangle + \alpha \langle g_j, g_k \rangle$.
- According to boundary conditions, the boundary elements are

$$\phi^L := \emptyset, \qquad \phi^R := \emptyset,$$
$$\psi^L := \emptyset, \qquad \psi^R := \emptyset.$$

Boundary Elements and Numerical Performance

Use the second-order derivative-orthogonal Riesz wavelet derived from Hermite cubic splines with $m = 2$.



Level	Size	Iteration	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$	$\|e_n\|_{H^1}$	$\log_2 \frac{\|e_{n-1}\|_{H_1}}{\|e_n\|_{H_1}}$
2	14	4	1.0225	2.2655e-04	1.1050e-04	—	3.0679e-03	—
3	30	4	1.0225	1.5147e-05	6.9610e-06	3.9886	3.8598e-04	2.9907
4	62	4	1.0225	9.6244e-07	4.3628e-07	3.9971	4.8325e-05	2.9977
5	126	4	1.0225	6.0411e-08	2.7653e-08	3.9993	6.0431e-06	2.9994
6	254	4	1.0225	3.9699e-09	2.1493e-09	4.0000	7.5554e-07	2.9999
7	510	4	1.0225	1.1049e-09	6.8832e-10	4.0650	9.4808e-08	3.0000

Table : Size of linear system, conjugate gradient iteration, condition number κ of the matrices B, errors, and convergence rates.

Example: 1D Biharmonic Equations

- Consider the following differential equation:

$$\begin{cases} u^{(4)} = f \quad \text{on} \;\; (0,1), \\ u(0) = 16, \quad u'(0) = -64, \quad u(1) = 0, \quad u'(1) = 0, \end{cases}$$

where $f(x) = 24(15x^2 - 50x + 41)$. The exact solution is $u(x) = (x-2)^4(x-1)^2$.

- Its corresponding Galerkin formulation is

$$\sum_{k=1}^{2^{n+2}} B_{j,k} c_k = \langle g_j, f\rangle - u(0)\chi_{\{2,3\}}(j)\langle g_j'', g_1''\rangle - u'(0)\chi_{\{2,3\}}(j)\langle g_j'', g_4''\rangle,$$

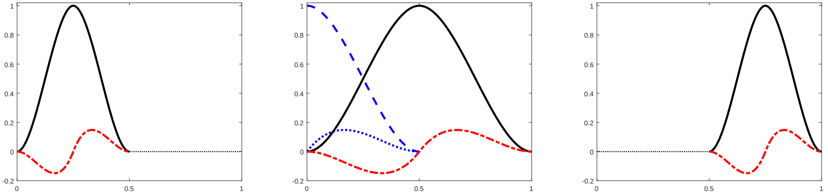
with the coefficient matrix $B_{j,k} := \langle g_j'', g_k''\rangle$.

- According to boundary conditions, the boundary elements are

$$\phi^L := \phi_1|_{[0,1]}, \qquad \phi^R := \phi_2|_{[0,1]},$$
$$\psi^L := \emptyset, \qquad \psi^R := \emptyset.$$

Boundary Elements and Numerical Performance

Use the second-order derivative-orthogonal Riesz wavelet derived from Hermite cubic splines with $m = 2$.



Level	Size	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$	$\|e_n\|_{H_1}$	$\log_2 \frac{\|e_{n-1}\|_{H^1}}{\|e_n\|_{H_1}}$
5	126	1	1.5129e-07	5.5402e-08	—	1.2283e-05	—
6	254	1	9.5006e-09	3.4413e-09	4.0089	1.5354e-06	3.0000
7	510	1	5.9521e-10	2.1642e-10	3.9910	1.9193e-07	3.0000
8	1022	1	3.7247e-11	1.3528e-11	3.9998	2.4004e-08	2.9992
9	2046	1	2.3306e-12	8.4696e-13	3.9975	3.1070e-09	2.9497

Table : Size of linear system, no iteration is needed, condition number κ of the matrices B (B is the identity matrix), errors, and convergence rates.

General Construction of Wavelets on $[0, 1]$

Theorem

Let $\Phi = \{\phi^1, \dots, \phi^r\}$, $\Psi = \{\psi^1, \dots, \psi^s\}$, *and* $\tilde{\Phi} = \{\tilde{\phi}^1, \dots, \tilde{\phi}^r\}$, $\tilde{\Psi} = \{\tilde{\psi}^1, \dots, \tilde{\psi}^s\}$ *be sets of compactly supported functions in* $L_2(\mathbb{R})$*:*

$\phi^\ell(c^\phi_\ell - \cdot) = \epsilon^\phi_\ell \phi^\ell, \quad \tilde{\phi}^\ell(c^\phi_\ell - \cdot) = \epsilon^\phi_\ell \tilde{\phi}^\ell$ *with* $c^\phi_\ell \in \mathbb{Z}, \epsilon^\phi_\ell \in \{-1, 1\}$,

$\psi^\ell(c^\psi_\ell - \cdot) = \epsilon^\psi_\ell \psi^\ell, \; \tilde{\psi}^\ell(c^\psi_\ell - \cdot) = \epsilon^\psi_\ell \tilde{\psi}^\ell$ *with* $c^\psi_\ell \in \mathbb{Z}, \epsilon^\psi_\ell \in \{-1, 1\}$.

For $\epsilon \in \{-1, 1\}$ *and* $c \in \mathbb{Z}$, *define* $\mathcal{I} := [\frac{c}{2}, \frac{c}{2} + 1]$. *For* $j \in \mathbb{N}_0$,

$$d^\phi_{j,\ell} := \lfloor 2^{j-1} c - 2^{-1} c^\phi_\ell \rfloor, \quad d^\psi_{j,\ell} := \lfloor 2^{j-1} c - 2^{-1} c^\psi_\ell \rfloor,$$
$$o^\phi_{j,\ell} := \operatorname{odd}(2^j c - c^\phi_\ell), \qquad o^\psi_{j,\ell} := \operatorname{odd}(2^j c - c^\psi_\ell),$$

where $\operatorname{odd}(m) := 1$ *if* m *is odd and* $\operatorname{odd}(m) := 0$ *if* m *is even. Let* $\chi_{\mathcal{I}}$ *denote the characteristic function of the interval* $\mathcal{I}$.

General Construction of Wavelets on $[0, 1]$...

Theorem

For $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, define Ψ_j^ℓ to be

$$\begin{cases} \{F_{c,\epsilon}(\psi^\ell_{2^j;k})\chi_{\mathcal{I}} \;:\; k = d^\psi_{j,\ell}+1,\ldots,d^\psi_{j,\ell}+2^j\}, & o^\psi_{j,\ell} = 1,\\ \{F_{c,\epsilon}(\psi^\ell_{2^j;k})\chi_{\mathcal{I}} \;:\; k = d^\psi_{j,\ell}+1,\ldots,d^\psi_{j,\ell}+2^j-1\}, & o^\psi_{j,\ell} = 0,\ \epsilon^\psi_\ell = -\epsilon,\\ \{F_{c,\epsilon}(\psi^\ell_{2^j;k})\chi_{\mathcal{I}} \;:\; k = d^\psi_{j,\ell}+1,\ldots,d^\psi_{j,\ell}+2^j-1\} & \\ \qquad \cup\{\frac{1}{\sqrt{2}}F_{c,\epsilon}(\psi^\ell_{2^j;d^\psi_{j,\ell}})\chi_{\mathcal{I}}, \frac{1}{\sqrt{2}}F_{c,\epsilon}(\psi^\ell_{2^j;d^\psi_{j,\ell}+2^j})\chi_{\mathcal{I}}\}, & o^\psi_{j,\ell} = 0,\ \epsilon^\psi_\ell = \epsilon, \end{cases}$$

where $F_{c,\epsilon}(f) := \sum_{k\in\mathbb{Z}}(f(\cdot - 2k) + \epsilon f(c + 2k - \cdot))$, and define $\tilde{\Psi}_j^\ell$, $\Phi_j^\ell, \tilde{\Phi}_j^\ell$ similarly. For $J \in \mathbb{N}_0$, define

$$\mathscr{B}_J := (\cup_{\ell=1}^r \Phi_J^\ell) \cup \cup_{j=J}^\infty(\cup_{\ell=1}^s \Psi_j^\ell), \quad \tilde{\mathscr{B}}_J := (\cup_{\ell=1}^r \tilde{\Phi}_J^\ell) \cup \cup_{j=J}^\infty(\cup_{\ell=1}^s \tilde{\Psi}_j^\ell).$$

If $(\{\tilde{\Phi}; \tilde{\Psi}\}, \{\Phi; \Psi\})$ is a biorthogonal wavelet in $L_2(\mathbb{R})$, then $(\tilde{\mathscr{B}}_J, \mathscr{B}_J)$ is a pair of biorthogonal bases for $L_2(\mathcal{I})$ for all $J \in \mathbb{N}_0$. Moreover, $\mathscr{B}_J$ and $\tilde{\mathscr{B}}_J$ are Riesz bases for $L_2(\mathcal{I})$.

Wavelets on $[0, 1]$ with Simple Structure

Let ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$ be compactly supported vector functions in $L_2(\mathbb{R})$ with symmetry and all the symmetry centers satisfy

$$c_1^\phi = \cdots = c_r^\phi = c_1^\psi = \cdots = c_s^\psi = 0.$$

In addition we assume that

all the elements/entries in ϕ and ψ vanish outside $[-1, 1]$.

Take $c = 0$. Since $d_{j,\ell}^\phi = d_{j,\ell}^\psi = o_{j,\ell}^\phi = o_{j,\ell}^\psi = 0$, Ψ_j^ℓ becomes

$$\begin{cases} \{\psi_{2^j;k}^\ell \ : \ k = 1, \ldots, 2^j - 1\}, & \epsilon_\ell^\psi = -\epsilon, \\ \{\psi_{2^j;k}^\ell \ : \ k = 1, \ldots, 2^j - 1\} \cup \{\sqrt{2}\psi_{2^j;0}^\ell \chi_{[0,1]}, \sqrt{2}\psi_{2^j;2^j}^\ell \chi_{[0,1]}\}, & \epsilon_\ell^\psi = \epsilon. \end{cases}$$

(i) If $\epsilon = -1$ and all entries in ϕ, ψ are continuous, then $h(0) = h(1) = 0$ for all $h \in \mathscr{B}_J$ (Dirichlet boundary condition).

(ii) If $\epsilon = 1$ and all entries in ϕ, ψ are in $\mathscr{C}^1(\mathbb{R})$, then $h'(0) = h'(1) = 0$ for all $h \in \mathscr{B}_J$ (von Neumann boundary).

Tight Framelet from Hermite Linear Splines

A tight framelet $\{\phi; \psi\}$ is given by

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi)$$

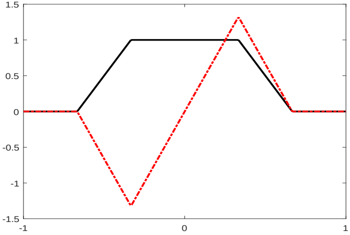
through a tight framelet filter bank $\{a; b\}$:

$$a = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{7}}{14} \\ -\frac{\sqrt{7}}{8} & -\frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{7}}{14} \\ \frac{\sqrt{7}}{8} & -\frac{1}{4} \end{bmatrix} \right\}_{[-1,1]},$$

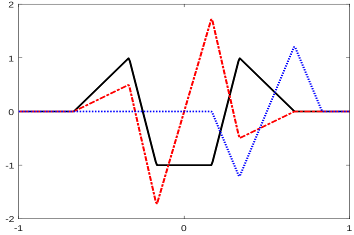
$$b = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{7}}{14} \\ \frac{1}{8} & \frac{\sqrt{7}}{28} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{\sqrt{7}}{4} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{7}}{14} \\ -\frac{1}{8} & \frac{\sqrt{7}}{28} \\ 0 & \frac{\sqrt{42}}{14} \end{bmatrix} \right\}_{[-1,1]}.$$

Then $\{\phi; \psi\}$ generates a tight frame in $L_2(\mathbb{R})$ and has symmetry property.

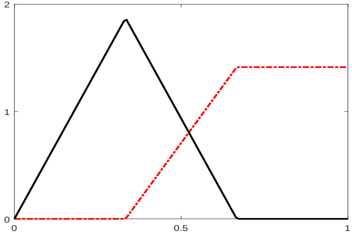
Tight Framelet in $L_2([0,1])$ from Linear Splines



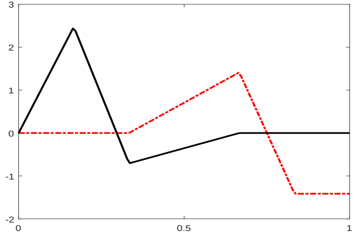
(a) $\phi = (\phi^1, \phi^2)^\mathsf{T}$



(b) $\psi = (\psi^1, \psi^2, \psi^3)^\mathsf{T}$



(c) ϕ^L, ϕ^R



(d) ψ^L, ψ^R

Tight Framelet from Hermite Quadratic Splines

A tight framelet $\{\phi; \psi\}$ is given by

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi)$$

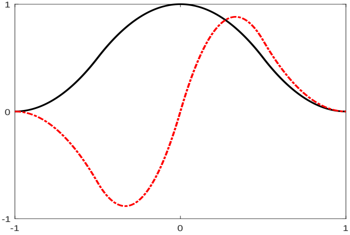
through a tight framelet filter bank $\{a; b\}$:

$$a = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{7}}{28} \\ -\frac{\sqrt{7}}{8} & -\frac{1}{8} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{7}}{28} \\ \frac{\sqrt{7}}{8} & -\frac{1}{8} \end{bmatrix} \right\}_{[-1,1]},$$

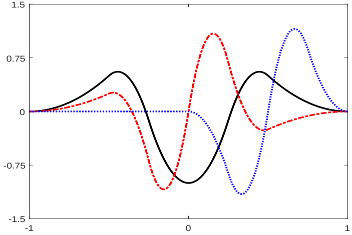
$$b = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{7}}{28} \\ \frac{1}{8} & \frac{\sqrt{7}}{56} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{\sqrt{7}}{4} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{7}}{28} \\ -\frac{1}{8} & \frac{\sqrt{7}}{56} \\ 0 & \frac{\sqrt{21}}{7} \end{bmatrix} \right\}_{[-1,1]}.$$

Then $\{\phi; \psi\}$ generates a tight frame in $L_2(\mathbb{R})$ and has symmetry property.

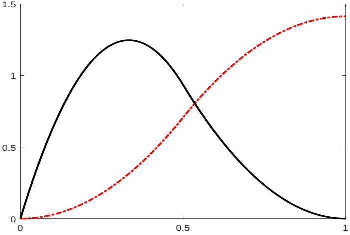
Tight Framelet in $L_2([0,1])$ from Quadratic Splines



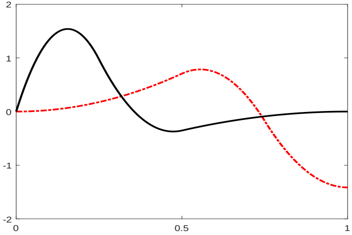
(a) $\phi = (\phi^1, \phi^2)^\mathsf{T}$



(b) $\psi = (\psi^1, \psi^2, \psi^3)^\mathsf{T}$



(c) ϕ^L, ϕ^R



(d) ψ^L, ψ^R

Riesz Wavelet from Hermite Cubic Splines

A Riesz wavelet $\{\phi; \psi\}$ is given by

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi)$$

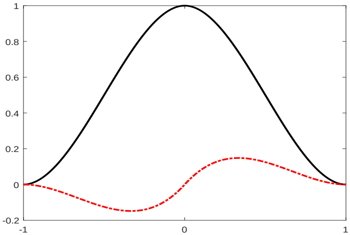
through a filter bank $\{a; b\}$ (part of a biorthogonal wavelet filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})$:

$$a = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ -\frac{1}{16} & -\frac{1}{16} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{3}{8} \\ \frac{1}{16} & -\frac{1}{16} \end{bmatrix} \right\}_{[-1,1]},$$

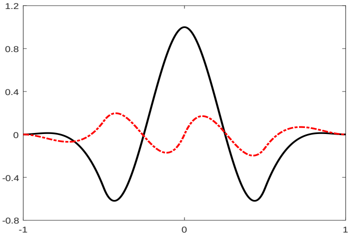
$$b = \left\{ \begin{bmatrix} -\frac{1}{4} & -\frac{23}{24} \\ \frac{1}{16} & \frac{91}{176} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{37}{44} \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} & \frac{23}{24} \\ -\frac{1}{16} & \frac{91}{176} \end{bmatrix} \right\}_{[-1,1]},$$

Then $\{\phi; \psi\}$ generates a Riesz wavelet in $L_2(\mathbb{R})$ and has symmetry property.

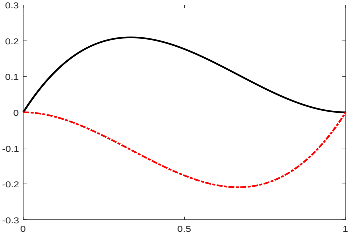
Riesz Wavelet in $L_2([0,1])$ from Cubic Splines



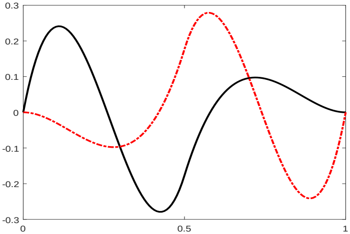
(a) $\phi = (\phi^1, \phi^2)^\mathsf{T}$



(b) $\psi = (\psi^1, \psi^2)^\mathsf{T}$



(c) ϕ^L, ϕ^R



(d) ψ^L, ψ^R

Riesz Wavelet from B_2 Spline

A Riesz wavelet $\{\phi; \psi\}$ is given by

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi)$$

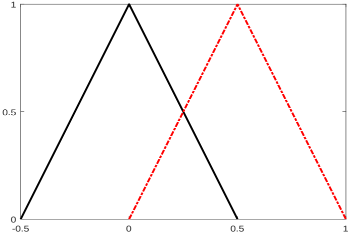
through a filter bank $\{a; b\}$ (part of a biorthogonal wavelet filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})$:

$$a = \left\{ \begin{bmatrix} 0 & \frac{1}{4} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} \right\}_{[-1,1]},$$

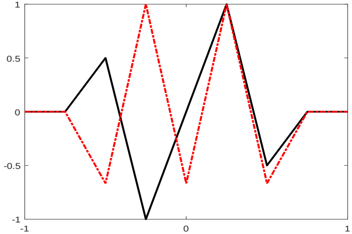
$$b = \left\{ \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} & 0 \\ -\frac{1}{3} & 0 \end{bmatrix} \right\}_{[-1,1]},$$

Then $\{\phi; \psi\}$ generates a Riesz wavelet in $L_2(\mathbb{R})$ and has symmetry property.

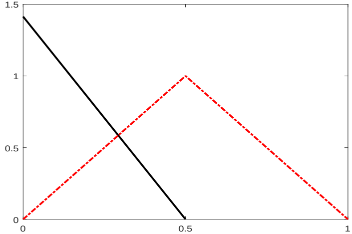
Riesz Wavelet in $L_2([0,1])$ from B_2 Spline



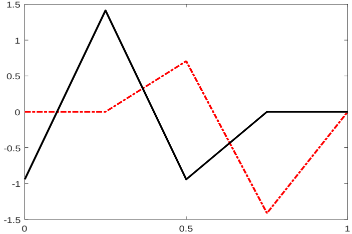
(a) $\phi = (\phi^1, \phi^2)^\mathsf{T}$



(b) $\psi = (\psi^1, \psi^2)^\mathsf{T}$



(c) ϕ^L, ϕ^R



(d) ψ^L, ψ^R

Orthogonal Multiwavelet

An orthogonal wavelet $\{\phi; \psi\}$ (Gernimo-Hardin-Massopust) is given by

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi)$$

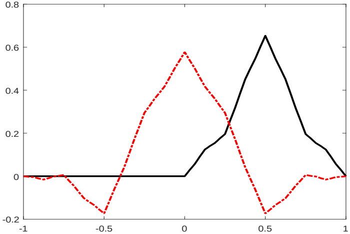
through a filter bank $\{a; b\}$ (part of a biorthogonal wavelet filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})$:

$$a = \left\{ \begin{bmatrix} \frac{3}{5} & \frac{4\sqrt{2}}{5} \\ -\frac{\sqrt{2}}{20} & -\frac{3}{10} \end{bmatrix}, \begin{bmatrix} \frac{3}{5} & 0 \\ \frac{9\sqrt{2}}{20} & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \frac{9\sqrt{2}}{20} & -\frac{3}{10} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -\frac{\sqrt{2}}{20} & 0 \end{bmatrix} \right\}_{[0,3]},$$

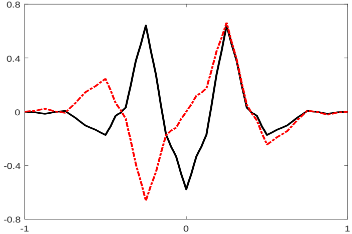
$$b = \left\{ \begin{bmatrix} -\frac{\sqrt{2}}{20} & -\frac{3}{10} \\ \frac{1}{10} & \frac{3\sqrt{2}}{10} \end{bmatrix}, \begin{bmatrix} \frac{9\sqrt{2}}{20} & -1 \\ -\frac{9}{10} & 0 \end{bmatrix}, \begin{bmatrix} \frac{9\sqrt{2}}{20} & -\frac{3}{10} \\ \frac{9}{10} & -\frac{3\sqrt{2}}{10} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{20} & 0 \\ -\frac{1}{10} & 0 \end{bmatrix} \right\}_{[0,3]}.$$

Then $\{\phi; \psi\}$ generates an orthogonal wavelet in $L_2(\mathbb{R})$ and has symmetry property.

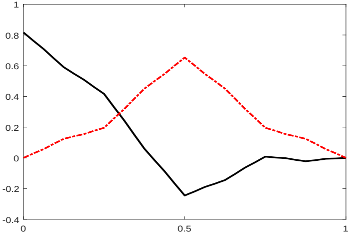
Orthogonal Multiwavelet in $L_2([0,1])$



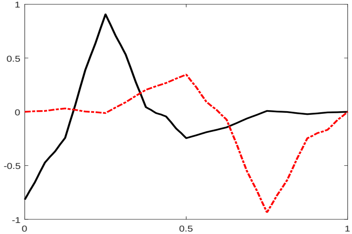
(a) $\phi = (\phi^1, \phi^2)^\mathsf{T}$



(b) $\psi = (\psi^1, \psi^2)^\mathsf{T}$



(c) ϕ^L, ϕ^R



(d) ψ^L, ψ^R

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$$\widehat{\phi}(2\xi)=\widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(2\xi)=\widehat{b}(\xi)\widehat{\phi}(\xi)$$

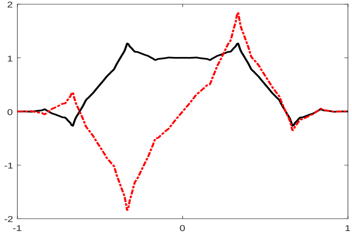
through a filter bank $\{a;b\}$ (part of a biorthogonal wavelet filter bank $(\{a;b\},\{\tilde{a};\tilde{b}\})$:

$$a=\left\{\begin{bmatrix}\frac{1}{4} & \frac{1}{4}\\ \frac{-\sqrt{7}}{8} & \frac{-\sqrt{7}}{8}\end{bmatrix},\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{4}\end{bmatrix},\begin{bmatrix}\frac{1}{4} & -\frac{1}{4}\\ \frac{\sqrt{7}}{8} & -\frac{\sqrt{7}}{8}\end{bmatrix}\right\}_{[-1,1]},$$

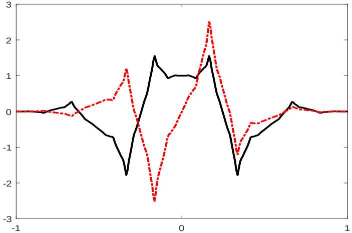
$$b=\left\{\begin{bmatrix}-\frac{1}{4} & -\frac{1}{4}\\ \frac{1}{8} & \frac{1}{8}\end{bmatrix},\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{\sqrt{7}}{4}\end{bmatrix},\begin{bmatrix}-\frac{1}{4} & \frac{1}{4}\\ -\frac{1}{8} & \frac{1}{8}\end{bmatrix}\right\}_{[-1,1]}.$$

Then $\{\phi;\psi\}$ generates an orthogonal wavelet in $L_2(\mathbb{R})$ and has symmetry property.

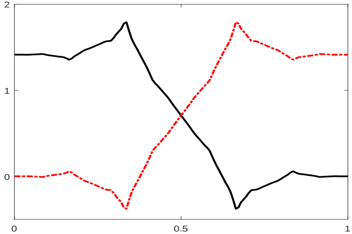
Orthogonal Multiwavelet in $L_2([0,1])$



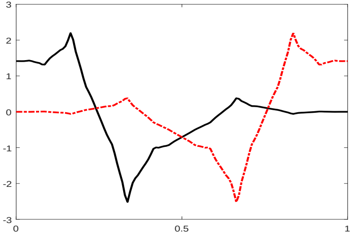
(a) $\phi = (\phi^1, \phi^2)^\mathsf{T}$



(b) $\psi = (\psi^1, \psi^2)^\mathsf{T}$



(c) ϕ^L, ϕ^R



(d) ψ^L, ψ^R

Summary and Conclusion

- Established existence and construction of mth-order derivative-orthogonal Riesz wavelets in $H^m(\mathbb{R})$, $m \in \mathbb{N} \cup \{0\}$.
- The constructed derivative-orthogonal Riesz wavelets in $H^m(\mathbb{R})$ from Hermite cubic splines have very short support making the construction of wavelets on $[0,1]$ very simple.
- The condition numbers of the stiffness matrices are often very small (even the optimal condition number one).
- These derivative-orthogonal Riesz wavelets in $H^m(\mathbb{R})$ are of particular interest for high dimension problems on $[0,1]^d$.
- General method for constructing wavelets on $[0,1]$.
- E-mail: `bhan@ualberta.ca`
URL: http://www.ualberta.ca/∼bhan

Thank You!

Papers

This talk is based on the following papers:

- B. Han and M. Michelle, Derivative-orthogonal Riesz wavelets in Sobolev spaces with applications to differential equations, *Applied and Computational Harmonic Analysis*, published online, (2017).
- B. Han and M. Michelle, Construction of wavelets and framelets on a bounded interval, *Analysis and Applications*, in press, (2018).

and Section 7.5 in the following book:

- B. Han, Framelets and Wavelets: Algorithms, Analysis, and Applications, *Applied and Numerical Harmonic Analysis*, Birkhauser/Springer, Cham, (2017), 724 pages