New non-linear stationary subdivision scheme with trigonometric functions reproduction

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2nd March 2017. Bernried.

Subdivion schemes

$$f^k \longmapsto f^{k+1}$$

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In this talk: $f^k \in l_\infty(\mathbb{Z})$

Univariate, uniform, binary, local, interpolatory:

$$f_{2i}^{k+1} = f_i^k, \qquad f_{2i+1}^{k+1} = \Psi^k(f_{i-q}^k, \ldots, f_i^k, f_{i+1}^k, \ldots, f_{i+q}^k), \qquad i \in \mathbb{Z}.$$

Linear: Ψ^k linear

Reproduction

A subdivision scheme reproduces set of function $\mathcal{F}$ if for any $F \in \mathcal{F}$

$$f^k = (F(i2^{-k}))_{i\in\mathbb{Z}} \quad \Longrightarrow \quad f^{k+1} = (F(i2^{-(k+1)}))_{i\in\mathbb{Z}}.$$

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Nira Dyn, David Levin and Ariel Luzzatto.
Exponentials reproducing subdivision schemes.
Foundations of Computational Mathematics, 3(2):187–206, 2003.

Costanza Conti and Lucia Romani.
Algebraic conditions on non-stationary subdivision symbols for exponential polynomial reproduction.
J. Comput. Appl. Math., 236(4):543–556, September 2011.

Exponential polynomials

$$\mathcal{F} = \{\sum_{i=0}^{g}\sum_{n=0}^{\nu_i} c_{i,n} t^n \exp(\gamma_i t) : c_{i,n} \in \mathbb{R}\}.$$

Applications: Computer Aided Geometric Design

$$\mathcal{F}(\gamma) = \{\tilde{c}_{0,0}\cos(\gamma t) + \tilde{c}_{1,0}\sin(\gamma t) + \tilde{c}_{2,0} : \tilde{c}_{i,n} \in \mathbb{R}\}$$

$$\gamma_0 = \iota\gamma, \gamma_1 = -\iota\gamma,\ \gamma_2 = 0,\ \nu_0 = \nu_1 = \nu_2 = 0, \quad 0 \neq \gamma \in (-\pi, \pi) \cup \iota\mathbb{R}.$$

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Linear non-stationary scheme:

$$f_{2i+1}^{k+1} = \frac{1}{2}f_i^k + \frac{1}{2}f_{i+1}^k - \Gamma_\gamma^k\left(f_{i+2}^k - f_{i+1}^k - f_i^k + f_{i-1}^k\right)$$

$$\Gamma_\gamma^k = \frac{1}{2}\frac{1}{\left(2\sqrt{\frac{1+\cos(\gamma 2^{-k})}{2}} + 1\right)^2 - 1}$$

$$f^{k+1} = T_\gamma^k f^k$$

Problem:

T_γ^k depends on $\cos(\gamma 2^{-k})$

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$cos(\gamma 2^{-k})$ computation

$$f_{i-1}^k - (2\cos(\gamma 2^{-k})+1)f_i^k + (2\cos(\gamma 2^{-k})+1)f_{i+1}^k - f_{i+2}^k = 0, \qquad \forall i \in \mathbb{Z},$$

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$$\cos(\gamma 2^{-k}) = \frac{1}{2}\left(\frac{f_{i+2}^k - f_{i-1}^k}{f_{i+1}^k - f_i^k} - 1\right), \qquad \forall i \in \mathbb{Z}.$$

$$(T_\gamma^k f^k)_{2i+1} = \frac{1}{2} f_i^k + \frac{1}{2} f_{i+1}^k - \Gamma_\gamma^k \left(f_{i+2}^k - f_{i+1}^k - f_i^k + f_{i-1}^k \right)$$

Γ_γ^k computation

$$\Gamma_\gamma^k = \frac{1}{2} \frac{1}{\left(2\sqrt{\frac{1+\cos(\gamma 2^{-k})}{2}} + 1 \right)^2 - 1} = \frac{1}{2} \frac{1}{\left(1 + \sqrt{1 + \frac{f_{i+2}^k - f_{i-1}^k}{f_{i+1}^k - f_i^k}} \right)^2 - 1}$$

Non-linear stationary scheme

$$(Sf^k)_{2i+1} = \frac{1}{2} f_i^k + \frac{1}{2} f_{i+1}^k - \Gamma(f_{i-1}^k, f_i^k, f_{i+1}^k, f_{i+2}^k) \left(f_{i+2}^k - f_{i+1}^k - f_i^k + f_{i-1}^k \right),$$

$$\Gamma(f_{-1}, f_0, f_1, f_2) = \frac{1}{2} \frac{1}{\left(1 + \sqrt{1 + \frac{f_2 - f_{-1}}{f_1 - f_0}} \right)^2 - 1}.$$

Γ is not well defined

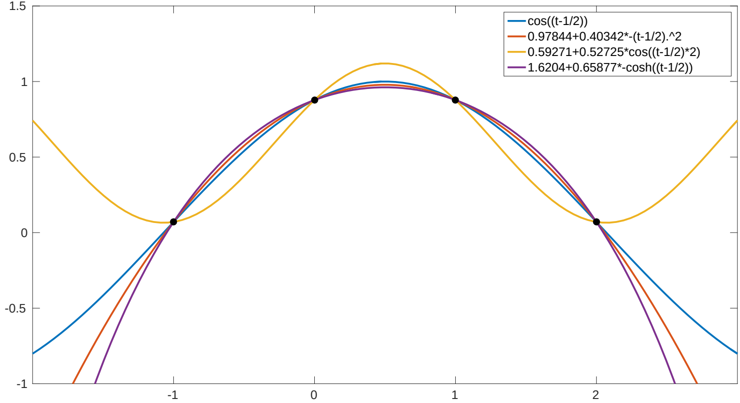
$$\Gamma(f_{i-1}^k, f_i^k, f_{i+1}^k, f_{i+2}^k) = \frac{1}{2}\frac{1}{\left(1+\sqrt{1+\frac{f_{i+2}^k - f_{i-1}^k}{f_{i+1}^k - f_i^k}}\right)^2 - 1}$$

$$\mathcal{F}(\gamma) = \{\tilde{c}_{0,0}\cos(\gamma t) + \tilde{c}_{1,0}\sin(\gamma t) + \tilde{c}_{2,0} : \tilde{c}_{i,n} \in \mathbb{R}\}$$

$$f_{i-1}^k - (2\cos(\gamma 2^{-k}) + 1)f_i^k + (2\cos(\gamma 2^{-k}) + 1)f_{i+1}^k - f_{i+2}^k = 0$$

$$\Downarrow \text{ if } f_i^k = f_{i+1}^k$$

$$f_{i-1}^k \qquad\qquad\qquad - f_{i+2}^k = 0.$$


cos((t-1/2))
0.97844+0.40342*-(t-1/2).^2
0.59271+0.52725*cos((t-1/2)*2)
1.6204+0.65877*-cosh((t-1/2))
1.5
1
0.5
0
-0.5
-1
-1
0
1
2

Truncated version: $\epsilon > 0$

$$(S_\epsilon f)_{2i+1} = \frac{1}{2}f_i + \frac{1}{2}f_{i+1} - \Gamma_\epsilon(f_{i-1}, f_i, f_{i+1}, f_{i+2})\left(f_{i+2} - f_{i+1} - f_i + f_{i-1}\right).$$

$$\Gamma_\epsilon(f_{-1}, f_0, f_1, f_2) = \begin{cases} \frac{1}{2}\frac{1}{\left(1+\sqrt{1+\frac{f_2-f_{-1}}{f_1-f_0}}\right)^2-1} & , f_1 \neq f_0 \text{ and } 2\epsilon \leq 1 + \frac{f_2-f_{-1}}{f_1-f_0} \\ 0 & , f_1 = f_0 \\ \frac{1}{16} & , \text{ otherwise} \end{cases}$$

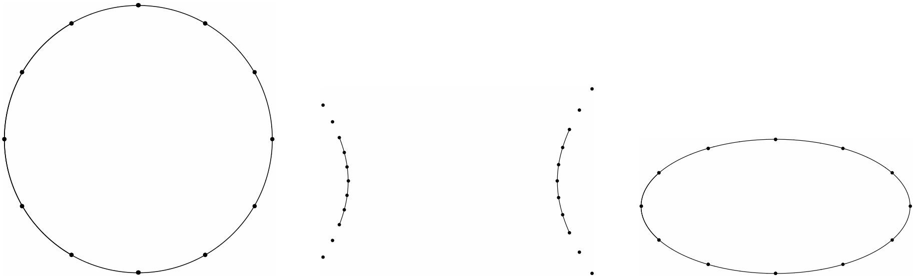
Why this definition? $\gamma = 0 \to \Gamma_0^k = \frac{1}{16}$:

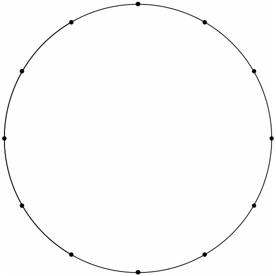
$$(T_0^k f^k)_{2i+1} = -\frac{1}{16}f_{i-1}^k + \frac{9}{16}f_i^k + \frac{9}{16}f_{i+1}^k - \frac{1}{16}f_{i+1}^k.$$

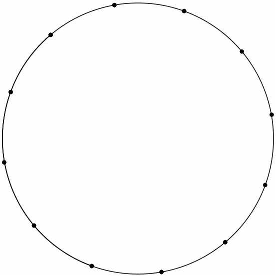
$$\mathcal{F}(\gamma) = \{\tilde{c}_{0,0}\cos(\gamma t) + \tilde{c}_{1,0}\sin(\gamma t) + \tilde{c}_{2,0} : \tilde{c}_{i,n} \in \mathbb{R}\}$$

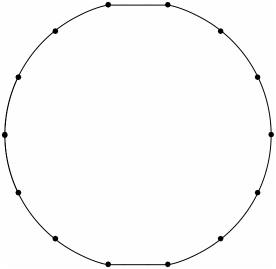
Proposition

S_ϵ reproduces $\{F \in \mathcal{F}(\gamma) : \gamma \in [-\arccos(-1+\epsilon), \arccos(-1+\epsilon)] \cup \imath\mathbb{R}\}$, whenever $f_i^k \neq f_{i+1}^k$ for $i \in \mathbb{Z}$, $f_i^0 = F(i)$.

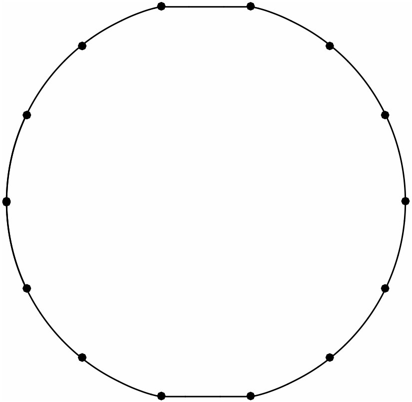








$$\Gamma_\epsilon = \begin{cases} \frac{1}{2}\frac{1}{\left(1+\sqrt{1+\frac{f_2-f_{-1}}{f_1-f_0}}\right)^2-1} & , f_1 \neq f_0 \text{ and } 2\epsilon \leq 1+\frac{f_2-f_{-1}}{f_1-f_0} \\ {\color{red}0} & , f_1 = f_0 \\ \frac{1}{16} & , \text{ otherwise} \end{cases}$$



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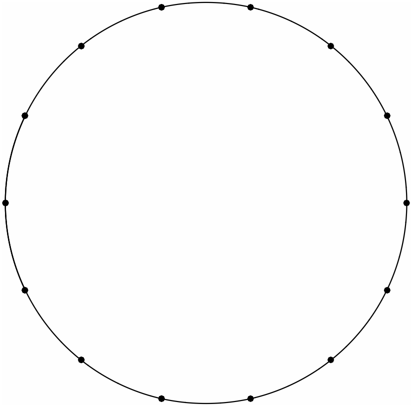


Definition

Monotonicity preserving:

$$\nabla f^k \geq 0 \longrightarrow \nabla f^{k+1} \geq 0.$$

Strictly monotonicity preserving if

$$\nabla f^k > 0 \longrightarrow \nabla f^{k+1} > 0.$$

Theorem

S is strictly monotonicity preserving.

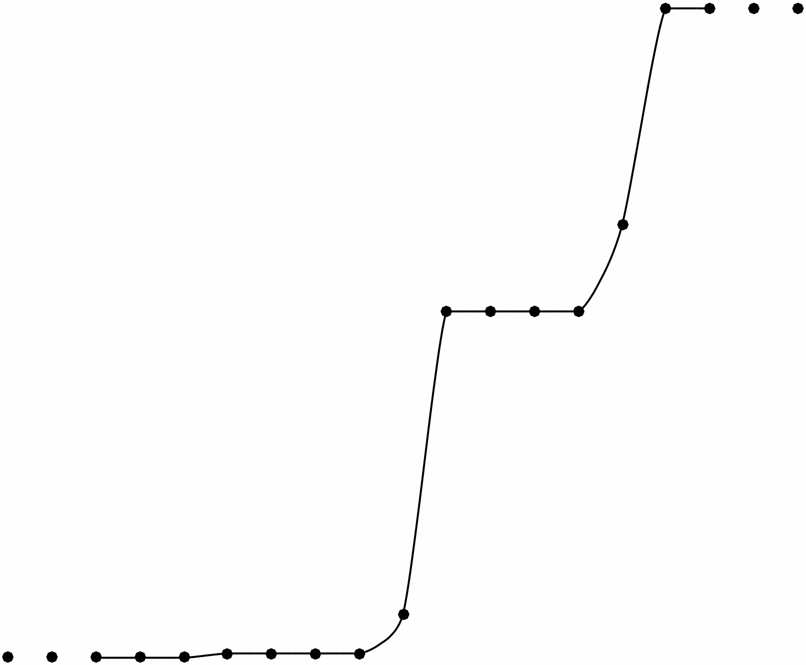
$$(S_\epsilon f)_{2i+1} = \frac{1}{2}f_i + \frac{1}{2}f_{i+1} - \Gamma_\epsilon(f_{i-1}, f_i, f_{i+1}, f_{i+2})\left(f_{i+2} - f_{i+1} - f_i + f_{i-1}\right).$$

$$\Gamma_\epsilon(f_{-1}, f_0, f_1, f_2) = \begin{cases} \frac{1}{2}\frac{1}{\left(1+\sqrt{1+\frac{f_2-f_{-1}}{f_1-f_0}}\right)^2-1} & , f_1 \neq f_0 \text{ and } 2\epsilon \leq 1 + \frac{f_2-f_{-1}}{f_1-f_0} \\ 0 & , f_1 = f_0 \\ \frac{1}{16} & , \text{ otherwise} \end{cases}$$

Corolary

S_ϵ is (strictly) monotonicity preserving if $\epsilon \leq 1$.

Rosa Donat, Sergio López-Ureña, and Maria Santágueda.
A family of non-oscillatory 6-point interpolatory subdivision schemes.
Advances in Computational Mathematics, pages 1–35, 2017.



Theorem

Let be S of the form

$$Sf = Tf + \mathcal{H}(Lf), \tag{1}$$

where T is a convergent linear stationary SS, L is a linear continuous operator and $\mathcal{H}$ could be a non-linear operator. If

$$\exists M > 0 \ : \|\mathcal{H}(f)\| \leq M\|f\|,$$

$$\exists J > 0, 0 < \mu < 1 \ : \|(L \circ S \circ S \circ \overset{J}{\cdots} \circ S)(f)\| \leq \mu\|Lf\|,$$

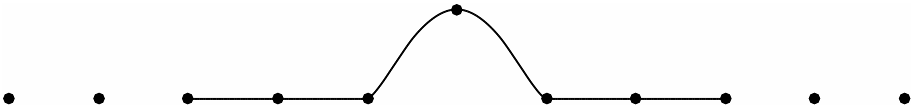
then S is convergent. Moreover, if T converges to $\mathcal{C}^{\alpha-}$ functions, then S converges to $\mathcal{C}^{\beta-}$ with $\beta = \text{mín}\{-\log_2(\mu)/J, \alpha\}$.

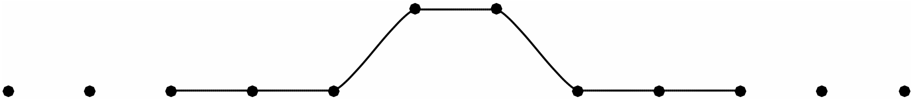
Karine Dadourian.
Schémas de subdivision, analyses multirésolutions non-linéaires.

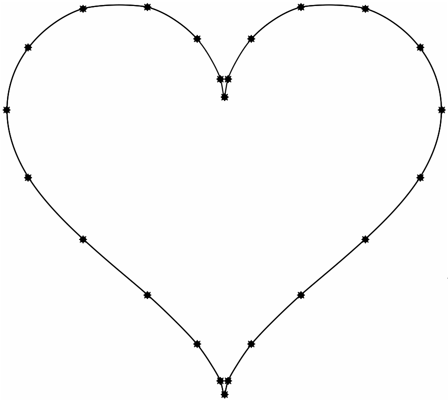
Ingrid Daubechies, Olof Runborg, and Wim Sweldens.
Normal multiresolution approximation of curves.

Theorem

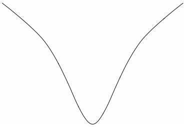
If $\epsilon > \frac{1}{2}(\sqrt{3}-1)^2 \approx 0{,}268$, then S_ϵ converges and its smoothness is at least $\mathcal{C}^{\beta-}$, $\beta = \text{mín}\{-\log_2(5/8), -\log_2(\frac{1}{2} + \frac{1}{(1+\sqrt{2\epsilon})^2-1})\} \in (0,1)$, $-\log_2(5/8) \simeq 0{,}678$.







$\mathcal{C}^{2-}$



Definition

F smooth. $f = (F(t^h_{2i}))_{i\in\mathbb{Z}}$, $t^h_i = ih$, $h > 0$.
A convergent S has approximation order r if

$$\|S^\infty f - F\| \leq Ch^r, \qquad \forall h > 0.$$

Definition

Approximation order r after-one-step

$$\|Sf - (F(t^h_i))_{i\in\mathbb{Z}}\| \leq Ch^r, \qquad \forall h > 0.$$

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Definition

Approximation order r after-one-step

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In strictly monotone regions:

$$(Sf)_{2i+1} = F(t^h_{2i+1}) + \frac{3h^4}{128}\left(\frac{F^{(3)}(t^h_{2i+1})F''(t^h_{2i+1})}{F'(t^h_{2i+1})} - F^{(4)}(t^h_{2i+1})\right) + O(h^5).$$

$$
\begin{aligned}
(S_\epsilon f)_{2i+1} &= \frac{1}{2} f_i + \frac{1}{2} f_{i+1} - \Gamma_\epsilon(f_{i-1}, f_i, f_{i+1}, f_{i+2})\,(f_{i+2} - f_{i+1} - f_i + f_{i-1}) \\
(T_0 f)_{2i+1} &= \frac{1}{2} f_i + \frac{1}{2} f_{i+1} - \frac{1}{16}\,(f_{i+2} - f_{i+1} - f_i + f_{i-1}) \\
&= -\frac{1}{16} f_{i-1} + \frac{9}{16} f_i + \frac{9}{16} f_{i+1} - \frac{1}{16} f_{i+1}.
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\end{aligned}
$$

How close?

$$
\begin{aligned}
\|S_\epsilon f - T_0 f\| &\leq C(\|\nabla f\|^3 + \|\nabla f\| \|\nabla^2 f\|), \qquad \|\nabla f\|, \|\nabla^2 f\| \leq \delta \\
\|S_\epsilon^{(1)} f - T_0^{(1)} f\| &\leq C\|\nabla f\|^2, \qquad \|\nabla f\| \leq \delta
\end{aligned}
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\end{aligned}
$$

Only for monotone regions, and

$$
\sup_{i\in\mathbb{Z}} \left| \frac{\nabla f_{i+j}^k}{\nabla f_i^k} - 1 \right| \overset{k\to\infty}{\to} 0, \quad -1 \leq j \leq 1
$$

If $\left|\frac{\nabla f^0_{i+j}}{\nabla f^0_i} - 1\right| \leq \delta$,

- S_ϵ is C^{2-}
- S_ϵ is stable

For F_ϵ such that $F' > \tilde{\delta} > 0$,

- S_ϵ has approximation order 4

Numerical experiments:

- S_ϵ is at least $C^{1,34}$
- S_ϵ has at least 3 order of approximation

- Linear non-stationary subdivision schemes reproducing $\mathcal{F}(\gamma)$
  $\Longrightarrow$ Non-linear stationary subdivision scheme reproducing $\bigcup_\gamma \mathcal{F}(\gamma)$
- Detect γ from $f^k = (F(i2^{-k}))_{i\in\mathbb{Z}}$, $F \in \mathcal{F}(\gamma)$
- Definition limitations $\Longrightarrow$ Truncated version
- Properties are related with the monotonicity

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