Optimal spline spaces of higher degree for L^2 n-widths

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Kolmogorov n-widths

Consider the set of functions

$$A^r := \{u \in H^r(0,1) : \|u^{(r)}\| \leq 1\},$$

with $\|\cdot\|$ the L^2 norm on $(0,1)$.

Questions:

1. How well can we approximate functions in A^r by functions from an n-dimensional subspace X_n of $L^2(0,1)$?
2. Which spaces X_n are "optimal" for this?

For a given set of functions $A \subset L^2$ and an n-dimensional subspace $X_n \subset L^2$, let

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be the distance to A from X_n.

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Then the Kolmogorov n-width of A is defined by

$$d_n(A) = \inf_{X_n} E(A, X_n).$$

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A subspace X_n is called an optimal subspace for A provided that

$$d_n(A) = E(A, X_n).$$

Application

If $A = A^r$,

$$\|u - P_n u\| \leq C\|u^{(r)}\|,$$

where P_n is the orthogonal projection of $L^2(0,1)$ onto X_n and $C = E(A^r, X_n)$.

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If X_n is an optimal subspace then $C = d_n(A^r)$, the best possible (least) constant.

Kolmogorov

Kolmogorov (1936) showed that

$$d_n(A^1) = \frac{1}{n\pi},$$

and that an optimal subspace is

$$X_n^0 = [1, \cos \pi x, \cos 2\pi x, \dots, \cos(n-1)\pi x].$$

Melkman and Micchelli

Melkman and Micchelli (1978) showed that A^1 has two further optimal subspaces, both of which are spaces of splines.

For degree d and knot vector

$$0 = \tau_0 < \tau_1 < \cdots < \tau_n < \tau_{n+1} = 1$$

let

$$S_{d,\boldsymbol{\tau}} := \{s \in C^{d-1}[0,1] : \; s|_{[\tau_j,\tau_{j+1}]} \in \Pi_d, \quad j = 0, 1, \ldots, n\}.$$

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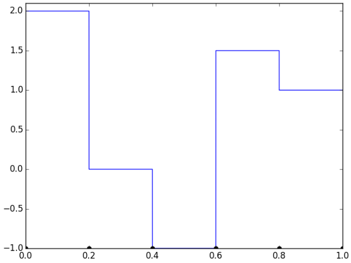
Let $\xi_j = j/n$, $j = 0, 1, \ldots, n$. Then

$$X_n^1 = S_{0,\boldsymbol{\xi}}$$

is an optimal subspace for A^1. Further, let $\eta_j = (2j-1)/(2n)$, $j = 1, \ldots, n$, and let $\eta_0 = 0$ and $\eta_{n+1} = 1$. Then

$$X_n^2 = \{s \in S_{1,\boldsymbol{\eta}} : \quad s'(0) = s'(1) = 0\}$$

is another optimal subspace for A^1.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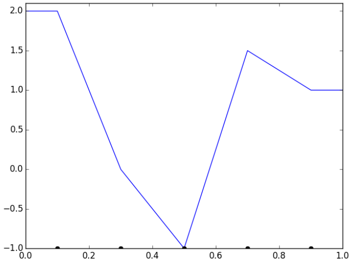
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Conjectures of Evans et al.

In

- J. A. Evans, Y. Bazilevs, I. Babuska, and T. J. R. Hughes (2009), *n-Widths, sup-infs, and optimality ratios for the k-version of the isogeometric finite element method*,

n-widths and optimal subspaces were studied in order to assess the approximation properties of splines for use in finite element methods.
Their numerical tests suggest that for (e.g.) A^1, there may exist optimal spline subspaces of degrees higher than 1.

Our results

For A^1 the following spline spaces are optimal:

$$
\begin{aligned}
X_n^1 &= S_{0,\boldsymbol{\xi}}, \\
X_n^2 &= \{s \in S_{1,\boldsymbol{\eta}} : s'(0) = s'(1) = 0\}, \\
X_n^3 &= \{s \in S_{2,\boldsymbol{\xi}} : s'(0) = s'(1) = 0\}, \\
X_n^4 &= \{s \in S_{3,\boldsymbol{\eta}} : s'(0) = s'(1) = 0,\ s'''(0) = s'''(1) = 0\} \\
X_n^5 &= \{s \in S_{4,\boldsymbol{\xi}} : s'(0) = s'(1) = 0,\ s'''(0) = s'''(1) = 0\}, \\
X_n^6 &= \dots
\end{aligned}
$$

For A^r, there are optimal spline spaces of degrees $r-1, 2r-1, 3r-1, \dots$, but the knots are no longer uniform.

Kernels

The basic idea is to express A^r as

$$A^r = \left\{ \sum_{i=0}^{r-1} a_i x^i + Kf(x) : \quad a_i \in \mathbb{R}, \quad \|f\| \leq 1 \right\}$$

where

$$Kf(x) = \frac{1}{(r-1)!} \int_0^x (x-y)^{r-1} f(y)\, dy = \int_0^1 K(x,y) f(y)\, dy,$$

with

$$K(x,y) = \frac{1}{(r-1)!} (x-y)_+^{r-1}.$$

We then study properties of the kernel $K(x,y)$.

Simplification

The analysis is easier for A of the form

$$A = \{Kf : \|f\| \leq 1\}. \tag{1}$$

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Example

$$A_0^2 = \{u \in H^2(0,1) : \quad \|u''\| \le 1, \quad u(0) = u(1) = 0\}$$

can be expressed as (1) with

$$Kf(x) = \int_0^1 K(x,y) f(y)\, dy,$$

and

$$K(x,y) = \begin{cases} x(1-y) & x \le y, \\ y(1-x) & x \ge y, \end{cases}$$

since $K(x,y)$ is the Green's function for the b.v.p.

$$-u'' = f, \quad u(0) = u(1) = 0.$$

Eigenvalues and eigenfunctions

We denote by K^* the adjoint of the operator K, defined by

$$(f, K^*g) = (Kf, g),$$

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We denote by K^* the adjoint of the operator K, defined by

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where $(\cdot, \cdot)$ is the inner product in $L^2(0, 1)$. The operator K^*K, being symmetric and positive semi-definite, has eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots \geq 0,$$

and corresponding orthogonal eigenfunctions

$$K^*K\phi_n = \lambda_n \phi_n, \qquad n = 1, 2, \ldots.$$

If we further define $\psi_n = K\phi_n$, then

$$KK^*\psi_n = \lambda_n \psi_n, \qquad n = 1, 2, \ldots,$$

and the ψ_n are also orthogonal.

n-width and first optimal subspace

By ‘duality’

$$\inf_{v \in X_n} \|u - v\| = \sup_{v \perp X_n} \frac{(u, v)}{\|v\|},$$

which leads to

$$E(A, X_n) = \sup_{\substack{\|v\| \leq 1 \\ v \perp X_n}} (KK^* v, v)^{1/2}.$$

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Taking the infimum of this over all n-dimensional subspaces X_n one obtains (Pinkus (1985)):

Theorem
$d_n(A) = \lambda_{n+1}^{1/2}$ and $X_n^0 = [\psi_1, \ldots, \psi_n]$ is an optimal subspace for A.

Totally positive kernels

The kernel $K(x, y)$ is totally positive if

$$K\begin{pmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{pmatrix} = \det(K(x_i, y_j))_{i,j=1}^n \geq 0,$$

for all $0 \leq x_1 < x_2 < \cdots < x_n \leq 1$, $0 \leq y_1 < y_2 < \cdots < y_n \leq 1$ and $n = 1, 2, \dots$.

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for all $0 \leq x_1 < x_2 < \cdots < x_n \leq 1$, $0 \leq y_1 < y_2 < \cdots < y_n \leq 1$ and $n = 1, 2, \dots$.
We will call $K(x, y)$ nondegenerate if

$$\dim[K(\cdot, y_1), \dots, K(\cdot, y_n)] = \dim[K(x_1, \cdot), \dots, K(x_n, \cdot)] = n,$$

for all $0 < x_1 < x_2 < \cdots < x_n < 1$, $0 < y_1 < y_2 < \cdots < y_n < 1$ and $n = 1, 2, \dots$.

If K is NTP (nondegenerate totally positive) then, by a theorem of Kellogg (1918), $\lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots > 0$, and the eigenfunctions ϕ_{n+1} and ψ_{n+1} have exactly n simple zeros in $(0, 1)$,

$$\phi_{n+1}(\xi_j) = \psi_{n+1}(\eta_j) = 0, \quad j = 1, 2, \ldots, n,$$

$$0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1, \qquad 0 < \eta_1 < \eta_2 < \cdots < \eta_n < 1.$$

Optimal subspaces of Melkman and Micchelli

Melkman and Micchelli (1978) showed

Theorem

If $K(x,y)$ is an NTP kernel, then

$$X_n^1 = [K(\cdot,\xi_1),\ldots,K(\cdot,\xi_n)]$$

and

$$X_n^2 = [(KK^*)(\cdot,\eta_1),\ldots,(KK^*)(\cdot,\eta_n)]$$

are also optimal subspaces for A.

Further optimal subspaces

Let

$$A^* := \{K^* f : \|f\| \leq 1\}.$$

Our idea is:

Lemma

For any n-dimensional subspace X_n,

$$E(A, K(X_n)) \leq E(A^*, X_n).$$

Further optimal subspaces

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Lemma

For any n-dimensional subspace X_n,

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Proof:

$$\begin{aligned} E(A, K(X_n)) &\leq \sup_{\|f\| \leq 1} \|(K - KP_n)f\| \\ &= \sup_{\|f\| \leq 1} \|(K^* - P_n K^*)f\| = E(A^*, X_n). \end{aligned}$$

Since $d_n(A^*) = d_n(A)$, it follows that if X_n is optimal for A^* then $K(X_n)$ is optimal for A.

Example. Since X_n^1 is optimal for A, by reversing the roles of K and K^*,

$$(X_n^1)^* := [K^*(\eta_1, \cdot), \ldots, K^*(\eta_n, \cdot)]$$

is optimal for A^*. So by the lemma, $K((X_n^1)^*) = X_n^2$ is optimal for A.

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Applying the lemma twice gives

$$E(A, KK^*(X_n)) \leq E(A, X_n),$$

and so if X_n is optimal for A, $KK^*(X_n)$ is also optimal for A.

Applying either the lemma recursively, or the double step recursively, we obtain

Theorem

If K is NTP, then for $l = 1, 2, 3, \ldots,$

$$(KK^*)^l(X_n^1) = [(KK^*)^l K(\cdot, \xi_1), \ldots, (KK^*)^l K(\cdot, \xi_n)]$$

and

$$(KK^*)^l(X_n^2) = [(KK^*)^{l+1}(\cdot, \eta_1), \ldots, (KK^*)^{l+1}(\cdot, \eta_n)]$$

are optimal subspaces for the n-width of A.

Back to H^r

Recall that $A_0^2 = \{Kf : \|f\| \le 1\}$, where

$$K(x,y) = \begin{cases} x(1-y) & x \le y; \\ y(1-x) & x \ge y, \end{cases}$$

Then $K^*K = KK^*$ has eigenvalues $1/(k\pi)^4$ and eigenfunctions $\sin k\pi x$, $k = 1, 2, \ldots$, and so $\xi_j = \eta_j = j/(n+1)$, $j = 1, \ldots, n$.

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Then $K^*K = KK^*$ has eigenvalues $1/(k\pi)^4$ and eigenfunctions $\sin k\pi x$, $k = 1, 2, \ldots$, and so $\xi_j = \eta_j = j/(n+1)$, $j = 1, \ldots, n$. So

$$\begin{aligned}
X_n^1 &= \{s \in S_{1,\boldsymbol{\xi}} : \quad s(0) = s(1) = 0\} \\
X_n^2 &= \{s \in S_{3,\boldsymbol{\xi}} : \quad s(0) = s(1) = 0,\ s''(0) = s''(1) = 0\} \\
X_n^3 &= \ldots
\end{aligned}$$

Recall that

$$A^r = \left\{ \sum_{i=0}^{r-1} a_i x^i + Kf(x) : a_i \in \mathbb{R}, \|f\| \leq 1 \right\}$$

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$$K(x,y) = \frac{1}{(r-1)!}(x-y)_+^{r-1}.$$

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where

$$K(x,y) = \frac{1}{(r-1)!}(x-y)_+^{r-1}.$$

To obtain the n-width and optimal subspaces we must first define

$$K_r = (I - Q_r)K,$$

where Q_r is the orthogonal projection onto

$$\Pi_r = [x^i : i = 0, 1, \ldots r-1].$$

Then we work with the eigenvalues and eigenfunctions of $K_r^* K_r$ and $K_r K_r^*$...
Although K_r is not totally positive, $K_r^* K_r$ is.

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