TU Graz, Institute of Geometry

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A smoothing procedure for Hermite subdivision schemes

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Joint work with Nira Dyn

March 2, 2017

Overview

- Point subdivision schemes and smoothing

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- Hermite subdivision schemes

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- Examples

Point subdivision schemes

Successive refinement of initial data to create smooth curve.

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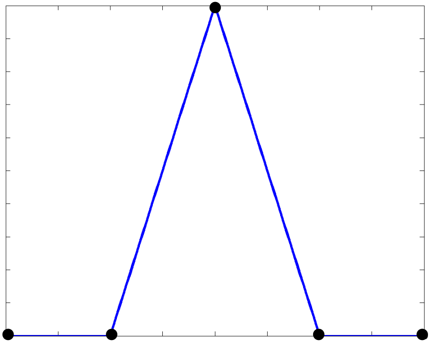
Example: Chaikin's algorithm applied to initial data $\boldsymbol{\delta} = (i, \delta_{i,0})_{i \in \mathbb{Z}}$



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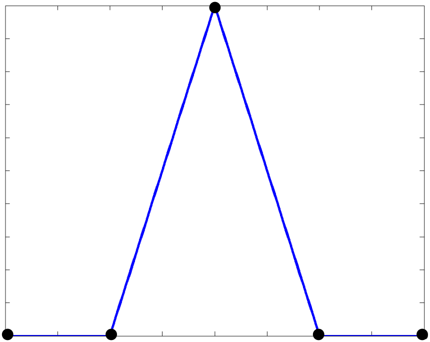

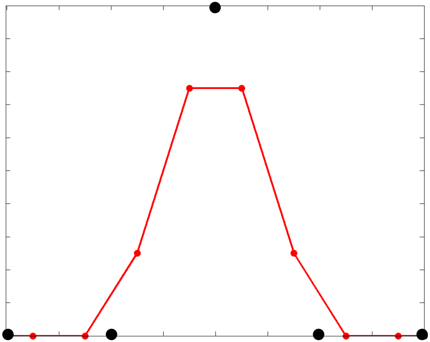

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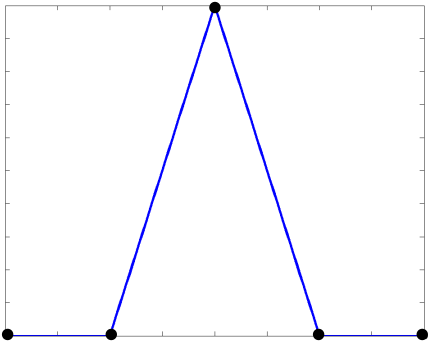

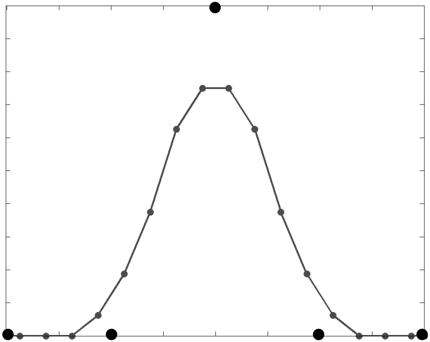

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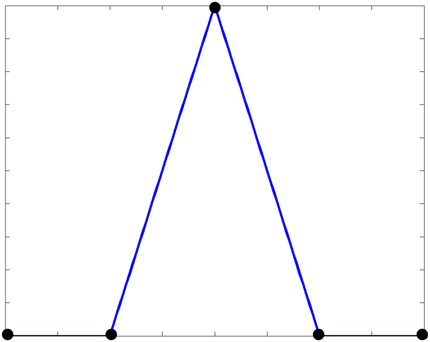

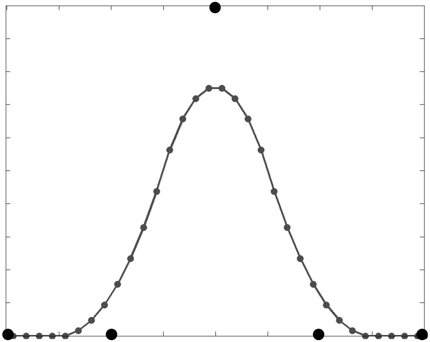

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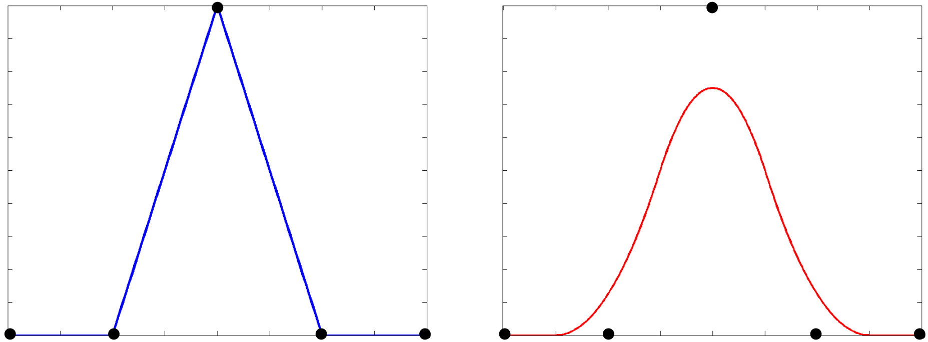

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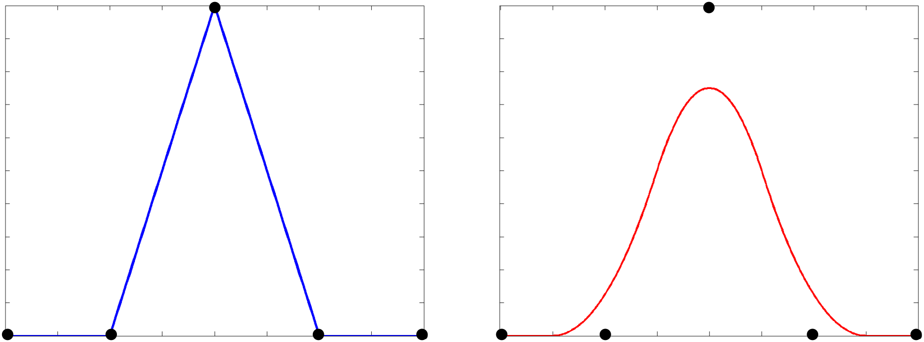

The limit of this subdivision process is a degree 2 spline.

Point subdivision schemes

Chaikin's algorithm in more detail:

$$(S\delta)_{2i} = \frac{3}{4}\delta_i + \frac{1}{4}\delta_{i+1}$$

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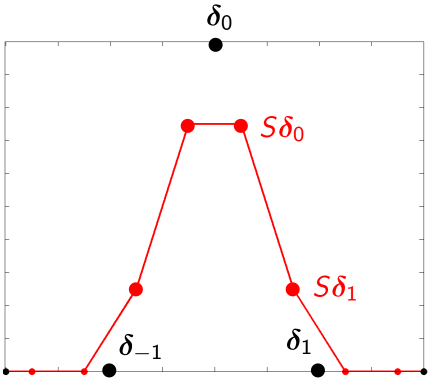

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First step: $S\delta$

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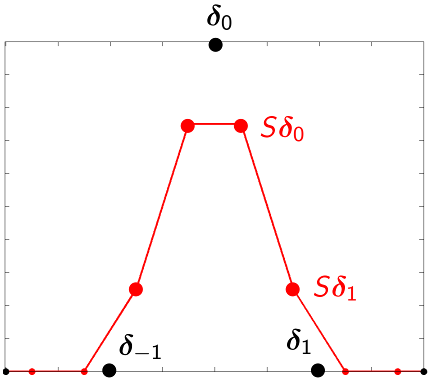

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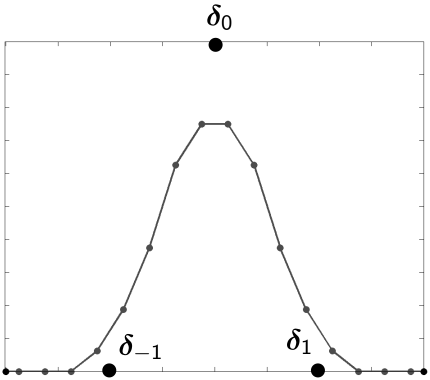

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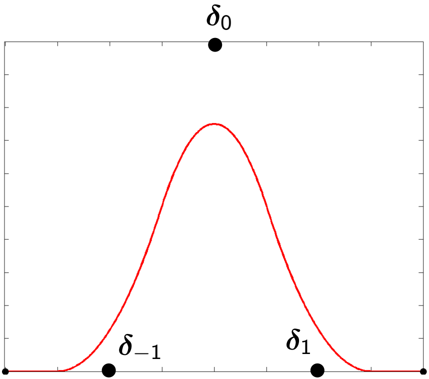

Input: $\boldsymbol{\delta}$

Limit: $S^\infty\boldsymbol{\delta} = B_2$

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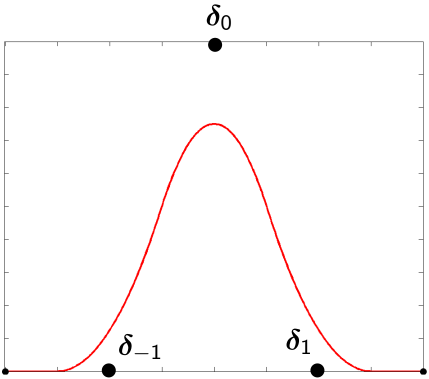
- In this example the limit is C^1



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Point subdivision schemes

Start from input data p, a subdivision operator can be defined by two rules:

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For example, Chaikin's algorithm:

$$a(z) = \frac{1}{4}z^{-2} + \frac{3}{4}z^{-1} + \frac{3}{4} + \frac{1}{4}z.$$

Smoothing of point subdivision schemes

Necessary condition for convergence:

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$$S_*,\ C^\ell \quad \xrightarrow{\times\frac{z+1}{2z}} \quad S,\ C^{\ell+1}$$

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The mask of the Lane-Riesenfeld algorithm for degree k B-Splines:

$$a_k(z) = \frac{(z+1)^{k+1}}{(2z)^k}$$

Apply subdivision operator S_k to 2D input data $\boldsymbol{\delta} = (i, \delta_{i,0})_{i\in\mathbb{Z}}$:

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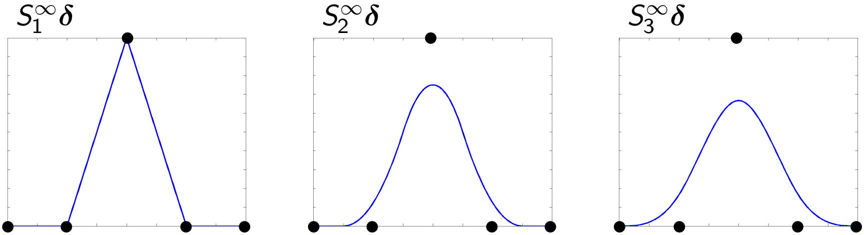

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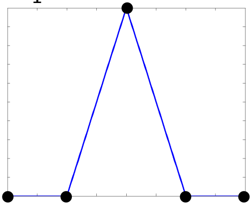

$a_1(z) = \frac{(z+1)^2}{2z}$

C^0

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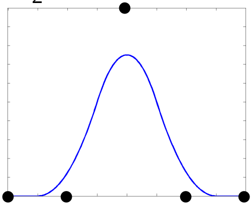
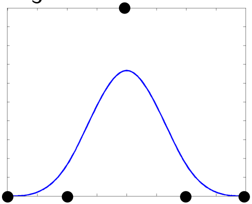


$a_2(z) = \frac{z+1}{2z} a_1(z)$

C^1

$S_3^\infty \boldsymbol{\delta}$



$a_3(z) = \frac{z+1}{2z} a_2(z)$

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Successive refinement of point-vector data for generating a function and its derivative

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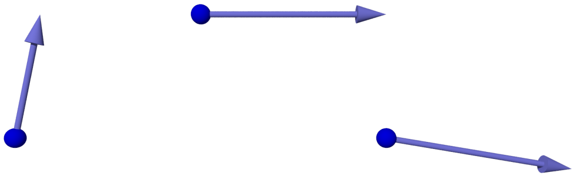

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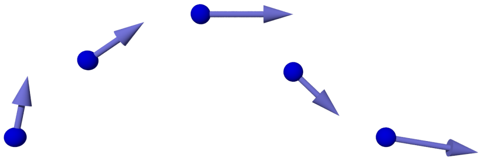

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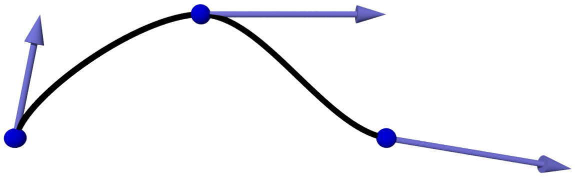

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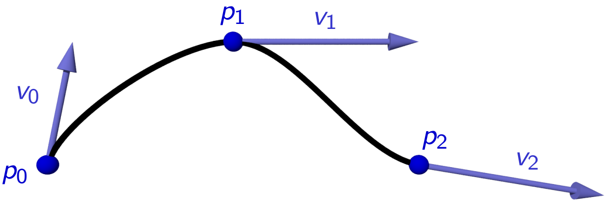

Subdivision operator: $S\binom{p}{v}_i = \sum_{j\in\mathbb{Z}} \begin{pmatrix} a_{i-2j} & b_{i-2j} \\ c_{i-2j} & d_{i-2j} \end{pmatrix} \binom{p_j}{v_j}$.

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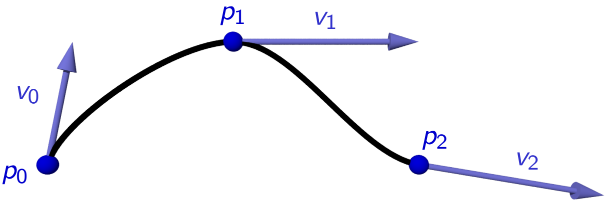

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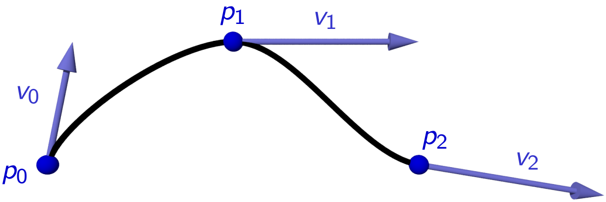

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- The iterates $S^n\binom{p}{v}$ describe the refined point-vector data
- $S^n\binom{p}{v}$ converges to function and its derivative (after appropriate scaling)

Smoothing of Hermite subdivision schemes

The spectral condition implies the existence of the Taylor scheme S_* with respect to the Taylor operator:

$$\begin{pmatrix} \Delta & -1 \\ 0 & 1 \end{pmatrix} S = \frac{1}{2} S_* \begin{pmatrix} \Delta & -1 \\ 0 & 1 \end{pmatrix}$$

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Disadvantage: Makes support larger by a maximum of 5.

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$$\text{Compute} \begin{pmatrix} J_{11}(z)(z^{-1}+1) & J_{12}(z)/(z^{-1}-1) \\ J_{21}(z)(z^{-2}-1) & J_{22}(z) \end{pmatrix}$$

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Change to eigenbasis of $A_*(1)$ transforms $A_*(z)$ to $J(z)$

$\downarrow$

Compute $\begin{pmatrix} J_{11}(z)(z^{-1}+1) & J_{12}(z)/(z^{-1}-1) \\ J_{21}(z)(z^{-2}-1) & J_{22}(z) \end{pmatrix}$

$\downarrow$

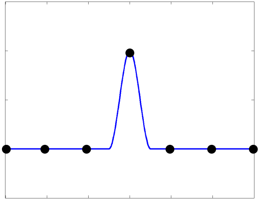
Undo change of basis, invert Taylor factorization

Examples

We smoothen an interpolatory C^1 Hermite scheme by J.-L. Merrien.

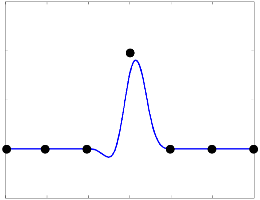
C^1 Hermite limit

$S^\infty\delta$



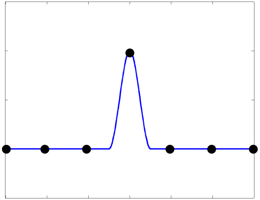
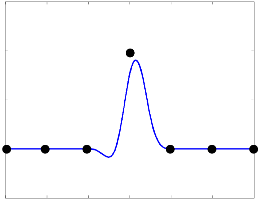
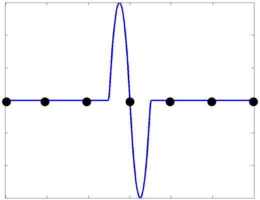
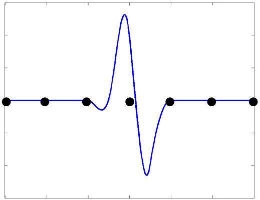
C^2 Hermite limit

$S^\infty\delta$



Examples

We smoothen an interpolatory C^1 Hermite scheme by J.-L. Merrien.

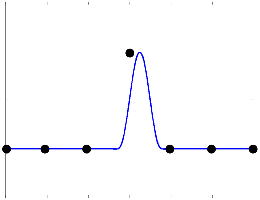
	C^1 Hermite limit	C^2 Hermite limit
$S^\infty\delta$		
$S^{\infty\prime}\delta$		

Examples

We smoothen a C^2 Hermite scheme constructed by a de Rham transform.

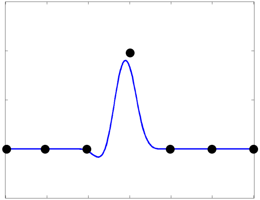
C^2 Hermite limit

$S^\infty \delta$



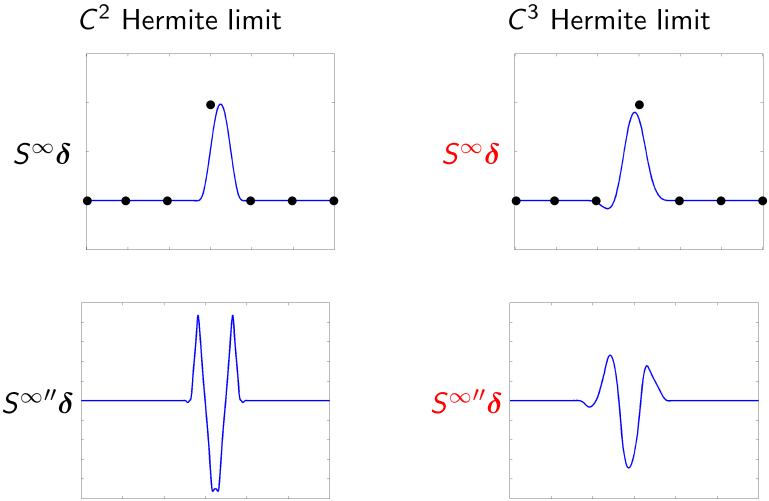
C^3 Hermite limit

$S^\infty \delta$



Examples

We smoothen a C^2 Hermite scheme constructed by a de Rham transform.



Conclusion

- We can smoothen Hermite schemes in a manner similar to point subdivision schemes, by manipulating symbols.

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Thank you!

FWF Der Wissenschaftsfonds.

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Smoothing of Hermite schemes

For example, if S has the mask $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $b(1) = 0$, then the mask of the new Hermite scheme S is given by

$$a(z) = \frac{(z+1)}{2z}\Big((z^{-2} - 2)b(z) + a(z)\Big),$$

$$b(z) = \frac{1}{2}\frac{zb(z)}{(1-z)},$$

$$c(z) = \frac{1}{2}(z^{-2} - 1)\Big(c(z) - a(z)(z^{-1} - 2) + d(z)(z^{-2} - 2) - b(z)(z^{-1} - 2)(z^{-2} - 2)\Big),$$

$$d(z) = \frac{1}{2}(d(z) - (z^{-1} - 2)b(z)).$$