Generalized orthogonal matching pursuit for multiple measurements

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single measurement

Task: Find a sparse solution x of

$$Ax = b,$$

i.e., $\|x\|_0 := |\{j \mid x_j \neq 0\}|$ small.

Problem: Minimizing $\|x\|_0$ is NP-hard.

Solution: Greedy methods, ℓ_1 relaxation.

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Algorithm: Orthogonal **M**atching **P**ursuit

$x = 0,\ r = b,\ J = \emptyset$;

for $k = 1, \ldots, L$ **do**

- $J = J \cup \{\arg\max_i |(A^* r)_i|\}$; `/* Greedy choice */`
- $x = \arg\min_y \|b - Ay\|_2$ where $supp(y) \subseteq J$; `/* Optimization */`
- $r = b - Ax$; `/* Update */`

multiple measurements

Task: Find a solution X of

$$AX = B.$$

(Each column of B is a single measurement.)

Assumption: X column-wise sparse, neighboring columns correlate.

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OMP for multiple measurements

	Idea
V-OMP	$AX(:) = B(:)$
P-OMP	$AX(:,i) = B(:,i)$
S-OMP*	$supp\, X(:,i) = supp\, X(:,j)$

*Tropp, Gilbert, Strauss: Algorithms for simultaneous sparse approximation Part I: Greedy pursuit, Signal Processing 86, 2006.

GM-OMP

Algorithm: GM-OMP

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V-OMP: $\mathbb{J}_V = \{supp(M) \mid M$ is 1 sparse $\}$.

P-OMP: $\mathbb{J}_P = \{supp(M) \mid$ each column of M is 1 sparse $\}$.

S-OMP: $\mathbb{J}_S = \{supp(M) \mid M$ is 1 row-sparse $\}$.

Parametrization of $\mathbb{J}$

Problem: $|\mathbb{J}|$ can have exponential size.

Idea: Describe $\mathbb{J}$ using two parameters $\mathbb{J} = \mathbb{J}(\alpha, \gamma)$.

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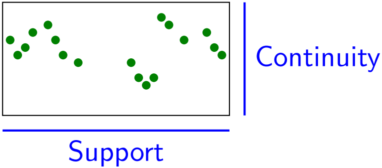
$supp(M) =$ 

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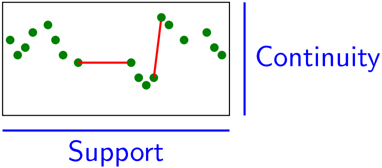

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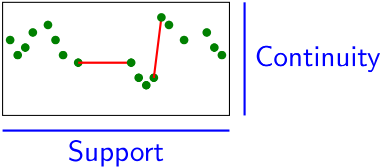

$$supp(M) = \{(i_k, j_k)\}_{k=1}^{K} \in \mathbb{J}(\alpha, \gamma) \Leftrightarrow \begin{array}{l} \min_{k' \neq k} \|j_k - j_{k'}\|_2 \leq \alpha \ \forall k \\ \|i_k - i_{k'}\|_2 \leq \gamma \|j_k - j_{k'}\| \ \forall k, k' \end{array}$$

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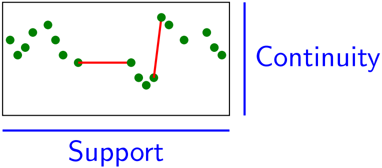

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We obtain: $\mathbb{J}_V = \mathbb{J}(0, \gamma)$, $\mathbb{J}_P = \mathbb{J}(\infty, \infty)$, $\mathbb{J}_S = \mathbb{J}(\infty, 0)$.

Greedy choice problem

Theorem

The problem $\arg\max_{i \in \mathbb{J}(\alpha,\gamma)} \|(A^* R)_i\|_p$ *is NP-hard for* $p < \infty$.

Proof: Reduction of vertex coloring.

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while $C \neq 0$ **do**

  $(i,j) = \arg\max C$, such that the support constraint holds;

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Theorem

The greedy choice solves $p = \infty$ *and is optimal for* $\mathbb{J}_V$ *and* $\mathbb{J}_P$*.*

Exact recovery

Theorem

GM-OMP recovers the exact sparsity pattern of the solution and its L structures if:

- *no* (α,γ)*-intersection,*
- $\mu_1(0,L-1)+\mu_1(0,L)<1$,
- λ*-separation with* $\mu_1(\lambda,l)<\beta_l(1-\mu_1(\lambda,l-1))$, $l\leq L$.

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Post processing:

Approximate $supp(M)=\{(i_k,j_k)\}_{k=1}^K$ by a function f:

$$supp(f)\approx\{j_k\}_{k=1}^K,\quad f(j_k)\approx i_k.$$

Example 1: comparison (setting)

Setting: $AX_\xi = B$ with A convolution matrix (Gauss kernel) and

$$X_\xi(i,j) = \begin{cases} 1 & i = -[j \tan \xi] \\ 0 & \text{otherwise} \end{cases},$$

where $-500 \leq i,j \leq 500$ ($X_{45^\circ} = I$, X_{0° has row-sparsity 1).

Compare: P-OMP, S-OMP, SBL (sparse bayesian learning).

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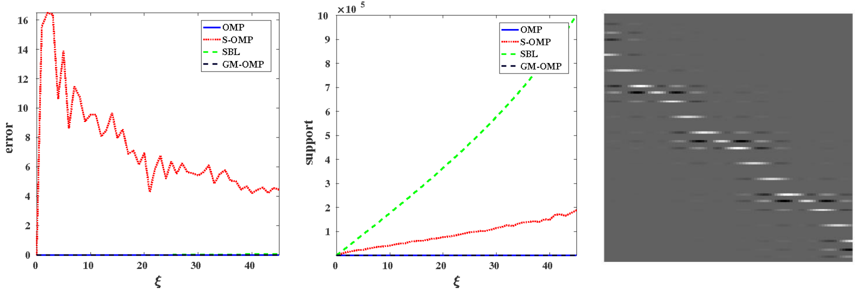
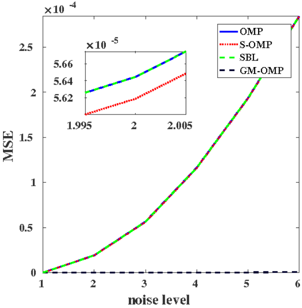


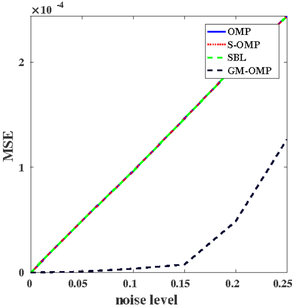
Error and sparsity of the reconstructed solution, stair-casing effect for S-OMP.

Example 1: comparison (noisy data)

$$X_0(i,j) = \begin{cases} 1 & i = \varepsilon_{\text{uniform}} \\ 0 & \text{otherwise} \end{cases},$$

$$X_0(i,j) = \begin{cases} \varepsilon_{\{0,1\}} & i = 0 \\ 0 & \text{otherwise} \end{cases},$$

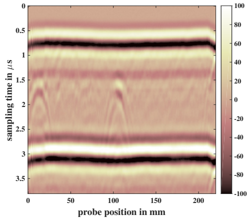


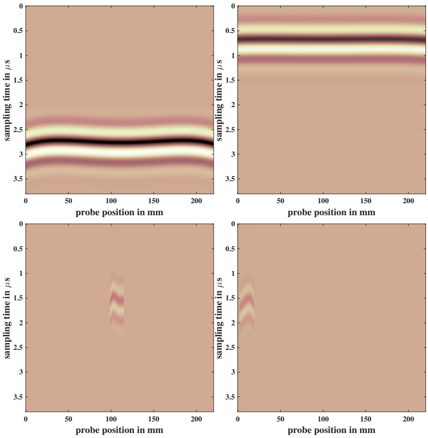


Denoising by polynomial approximation (degree 3).

Example 2: ultrasonic testing

A convolution matrix (Gabor impulse: $g(t) = e^{-pt^2}\cos(\phi t + \psi)$).





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...The End!