Error analysis for filtered back projection reconstructions in fractional Sobolev spaces

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Outline

1. Filtered Back Projection
   - Basic Reconstruction Problem
   - Approximate Reconstruction Formula
2. Analysis of the Reconstruction Error in the H^σ-Norm
   - Error Estimate
   - Convergence
3. H^σ-Error Analysis for $\mathcal{C}^k$-Window Functions
   - Numerical Observations
   - Error Estimate for $\mathcal{C}^k$-Windows
4. Asymptotic H^σ-Error Analysis

Basic Reconstruction Problem

Problem formulation:
Let $\Omega \subseteq \mathbb{R}^2$ be given. Reconstruct a bivariate function $f \equiv f(x, y)$ with $f \in \mathrm{L}^1(\Omega)$ on its domain Ω from given Radon data

$$\{\mathcal{R}f(t, \theta) \mid t \in \mathbb{R},\, \theta \in [0, \pi)\},$$

where the **Radon transform** $\mathcal{R}f$ of $f \in \mathrm{L}^1(\mathbb{R}^2)$ is defined as

$$\mathcal{R}f(t, \theta) = \int_{\{x \cos(\theta) + y \sin(\theta) = t\}} f(x, y) \, \mathrm{d}x \, \mathrm{d}y \qquad \text{for } (t, \theta) \in \mathbb{R} \times [0, \pi).$$

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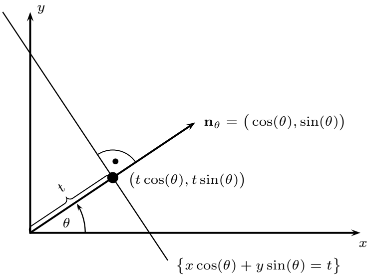

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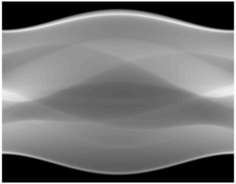
(a) Phantom



(b) Radon transform

Fig.: The Shepp-Logan phantom

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Analytical solution:

The inversion of $\mathcal{R}$ involves the **back projection** $\mathcal{B}h$ of $h \in \mathrm{L}^1(\mathbb{R} \times [0, \pi))$,

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and is given, for $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathcal{C}(\mathbb{R}^2)$, by the **filtered back projection formula**

$$f(x, y) = \frac{1}{2} \mathcal{B}\big(\mathcal{F}^{-1}[|S| \mathcal{F}(\mathcal{R}f)(S, \theta)]\big)(x, y) \quad \forall\, (x, y) \in \mathbb{R}^2.$$

Approximate Reconstruction

Stabilization: Replace the factor $|S|$ by a **low-pass filter** $A_L : \mathbb{R} \longrightarrow \mathbb{R}$,

$$A_L(S) = |S| W(S/L) \quad \text{for } S \in \mathbb{R}$$

with *finite* **bandwidth** $L > 0$ and an *even* **window function** $W : \mathbb{R} \longrightarrow \mathbb{R}$ with *compact support* $\operatorname{supp}(W) \subseteq [-1, 1]$.

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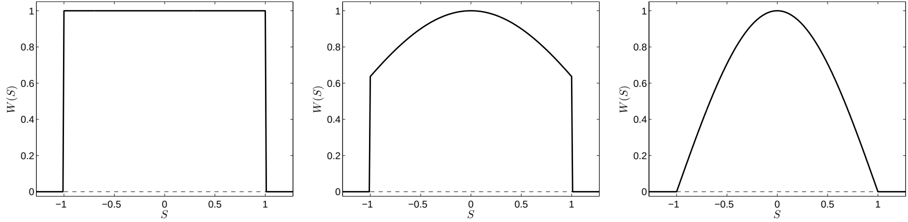

(a) Ram-Lak (b) Shepp-Logan (c) Cosine

Fig.: Window functions of typical low-pass filters

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Applying the low-pass filter A_L yields an *approximate FBP reconstruction* f_L given by

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Proposition

Let $f \in \mathrm{L}^1(\mathbb{R}^2)$ and $W \in \mathrm{L}^\infty(\mathbb{R})$ be even with $\operatorname{supp}(W) \subseteq [-1, 1]$. Then, for all $L > 0$, the approximate FBP reconstruction f_L is defined almost everywhere on $\mathbb{R}^2$ and can be rewritten as

$$f_L = \frac{1}{2} \mathcal{B}\big(\mathcal{F}^{-1} A_L * \mathcal{R}f\big). \qquad \square$$

Approximate Reconstruction

Proposition

For all $L > 0$, the approximate FBP reconstruction f_L satisfies $f_L \in \mathrm{L}^2(\mathbb{R}^2)$ and

$$f_L = f * K_L$$

with the convolution kernel $K_L : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *given by*

$$K_L(x, y) = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1} A_L\big)(x, y) \qquad \textit{for } (x, y) \in \mathbb{R}^2.$$

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We define the bivariate window function $W_L : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as

$$W_L(x,y) = W\left(\frac{r(x,y)}{L}\right) \quad \text{for } (x,y) \in \mathbb{R}^2,$$

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For all $L > 0$, the convolution kernel K_L satisfies $K_L \in \mathcal{C}_0(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$ and

$$\mathcal{F}K_L(x,y) = W_L(x,y) \quad \text{for almost all } (x,y) \in \mathbb{R}^2.$$

□

Analysis of the Reconstruction Error

Aim

Analyse the FBP reconstruction error

$$e_L = f - f_L$$

depending on the window function W and the bandwidth $L > 0$.

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In this talk:

- Sobolev error estimates for target functions f from fractional Sobolev spaces, i.e.,

$$f \in \mathrm{H}^\alpha(\mathbb{R}^2) = \left\{ g \in \mathcal{S}'(\mathbb{R}^2) \mid \|g\|_\alpha < \infty \right\} \qquad \text{for } \alpha > 0,$$

where

$$\|g\|_\alpha^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}g(x,y)|^2 \, \mathrm{d}(x,y)$$

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- Convergence rates ($L \longrightarrow \infty$) in terms of bandwidth L and smoothness α

H^σ-Error Analysis

Theorem (H^σ-error estimate)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ *for some* $\alpha > 0$ *and let* $W \in \mathrm{L}^\infty(\mathbb{R})$ *be even with* $\mathrm{supp}(W) \in [-1, 1]$*. Then, for* $0 \leq \sigma \leq \alpha$*, the* H^σ*-norm of the FBP reconstruction error* $e_L = f - f_L$ *is bounded above by*

$$\|e_L\|_\sigma \leq \left(\Phi_{\alpha-\sigma,W}^{1/2}(L) + L^{\sigma-\alpha}\right)\|f\|_\alpha,$$

where

$$\Phi_{\gamma,W}(L) = \sup_{S\in[-1,1]} \frac{(1 - W(S))^2}{(1 + L^2 S^2)^\gamma} \quad \text{for } L > 0.$$

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□

Theorem (Convergence of $\Phi_{\gamma,W}$)

Let the window function W *be continuous on* $[-1,1]$ *and satisfy* $W(0) = 1$*. Then, for all* $\gamma > 0$*,*

$$\Phi_{\gamma,W}(L) = \max_{S\in[0,1]} \frac{(1-W(S))^2}{(1+L^2S^2)^\gamma} \longrightarrow 0 \quad \text{for} \quad L \longrightarrow \infty.$$

□

H^σ-Convergence of the FBP Reconstruction

Corollary (H^σ-convergence of the FBP reconstruction)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ *for some* $\alpha > 0$ *and* $W \in \mathcal{C}([-1,1])$ *with* $W(0) = 1$*. Then, for* $0 \leq \sigma < \alpha$*, the* H^σ*-norm of the FBP reconstruction error* $e_L = f - f_L$ *satisfies*

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Let $S^*_{\gamma,W,L} \in [0,1]$ denote the *smallest* maximizer of the function $\Phi_{\gamma,W,L}$, defined as

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Assumption

$S^*_{\alpha-\sigma,W,L}$ is uniformly bounded away from 0, i.e., there exists a constant $c_{\alpha-\sigma,W} > 0$ such that

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Under the above assumption follows that

$$\Phi_{\alpha-\sigma,W}(L) \leq c_{\alpha-\sigma,W}^{2(\sigma-\alpha)} \, \|1 - W\|^2_{\infty,[-1,1]} \, L^{2(\sigma-\alpha)} = \mathcal{O}\big(L^{2(\sigma-\alpha)}\big) \quad \text{for} \quad L \longrightarrow \infty.$$

Order of Convergence

Theorem (Rate of convergence)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ *for some* $\alpha > 0$ *and* $W \in \mathcal{C}([-1,1])$ *with* $W(0) = 1$. *Further, let the above assumption be satisfied. Then, for* $0 \leq \sigma \leq \alpha$, *the* H^σ*-norm of the FBP reconstruction error* $e_L = f - f_L$ *is bounded above by*

$$\|e_L\|_\sigma \leq \left(c_{\alpha-\sigma,W}^{\sigma-\alpha} \, \|1 - W\|_{\infty,[-1,1]} + 1 \right) L^{\sigma-\alpha} \, \|f\|_\alpha.$$

In particular,

$$\|e_L\|_\sigma = \mathcal{O}(L^{\sigma-\alpha}) \quad \text{for} \quad L \longrightarrow \infty,$$

i.e., the decay rate is determined by the difference between the smoothness α *of the target function* f *and the order* σ *of the Sobolev norm in which the reconstruction error* e_L *is measured.* □

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Example:

Let the window function $W \in \mathcal{C}([-1,1])$ satisfy

$$W(S) = 1 \quad \forall\, S \in (-\varepsilon, \varepsilon)$$

with some $0 < \varepsilon < 1$. Then, the above assumption is fulfilled with $c_{\alpha-\sigma,W} = \varepsilon$.

Numerical Observations

We investigate the behaviour of $S^*_{\gamma,W,L}$ and $\Phi_{\gamma,W}(L)$ numerically for the following low-pass filters:

- Shepp-Logan filter: $W(S) = \mathrm{sinc}\big(\frac{\pi S}{2}\big) \cdot \chi_{[-1,1]}(S)$,
- Cosine filter: $W(S) = \cos\big(\frac{\pi S}{2}\big) \cdot \chi_{[-1,1]}(S)$,
- Hamming filter (for $\beta \in \big[\frac{1}{2}, 1\big]$): $W(S) = (\beta + (1-\beta)\cos(\pi S)) \cdot \chi_{[-1,1]}(S)$,
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For $\gamma < 2$, we observe that the above assumption

$$\exists\, c_{\gamma,W} > 0 \,\forall\, L > 0\colon\; S^*_{\gamma,W,L} \geq c_{\gamma,W}$$

is fulfilled and

$$\Phi_{\gamma,W}(L) = \mathcal{O}(L^{-2\gamma}) \quad \text{for} \quad L \longrightarrow \infty.$$

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is fulfilled and

$$\Phi_{\gamma,W}(L) = \mathcal{O}(L^{-2\gamma}) \quad \text{for} \quad L \longrightarrow \infty.$$

For $\gamma \geq 2$, we have

$$S^*_{\gamma,W,L} \longrightarrow 0 \quad \text{for} \quad L \longrightarrow \infty$$

Numerical Observations

We investigate the behaviour of $S^*_{\gamma,W,L}$ and $\Phi_{\gamma,W}(L)$ numerically for the following low-pass filters:

- Shepp-Logan filter: $W(S) = \operatorname{sinc}\left(\frac{\pi S}{2}\right) \cdot \chi_{[-1,1]}(S)$,
- Cosine filter: $W(S) = \cos\left(\frac{\pi S}{2}\right) \cdot \chi_{[-1,1]}(S)$,
- Hamming filter (for $\beta \in \left[\frac{1}{2}, 1\right]$): $W(S) = (\beta + (1-\beta)\cos(\pi S)) \cdot \chi_{[-1,1]}(S)$,
- Gaussian filter (for $\beta > 1$): $W(S) = \exp\left(-({}^{\pi S}/_{\beta})^2\right) \cdot \chi_{[-1,1]}(S)$.

For $\gamma < 2$, we observe that the above assumption

$$\exists\, c_{\gamma,W} > 0\ \forall\, L > 0\colon\ S^*_{\gamma,W,L} \geq c_{\gamma,W}$$

is fulfilled and

$$\Phi_{\gamma,W}(L) = \mathcal{O}(L^{-2\gamma}) \quad \text{for} \quad L \longrightarrow \infty.$$

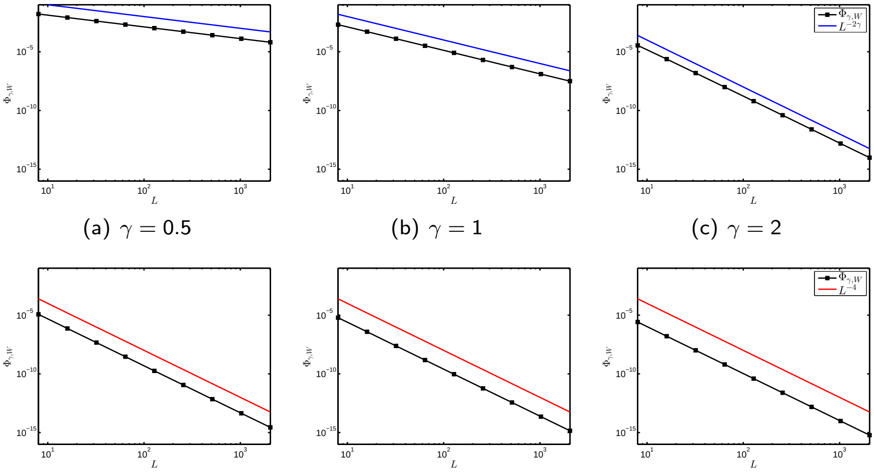
For $\gamma \geq 2$, we have

$$S^*_{\gamma,W,L} \longrightarrow 0 \quad \text{for} \quad L \longrightarrow \infty$$

and the convergence rate of $\Phi_{\gamma,W}$ stagnates at

$$\Phi_{\gamma,W}(L) = \mathcal{O}(L^{-4}) \quad \text{for} \quad L \longrightarrow \infty.$$

Numerical Observations



(a) $\gamma = 0.5$ (b) $\gamma = 1$ (c) $\gamma = 2$

(d) $\gamma = 2.5$ (e) $\gamma = 3$ (f) $\gamma = 4$

Fig.: Decay rate of $\Phi_{\gamma,W}$ for the Shepp-Logan filter

H^σ-Error Analysis for $\mathcal{C}^k$-Windows

Theorem (Convergence rate of $\Phi_{\gamma,W}$ for $\mathcal{C}^k$-windows)

Let the window function W be k-times continuously differentiable on $[-1,1]$, $k \geq 2$, with

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1$$

and let $\gamma \geq 0$ be given. Then, we have

$$\Phi_{\gamma,W}(L) \leq \begin{cases} \frac{1}{(k!)^2} \, \|W^{(k)}\|^2_{\infty,[-1,1]} \, L^{-2\gamma} & \text{for } \gamma \leq k \\ \frac{c^2_{\gamma,k}}{(k!)^2} \, \|W^{(k)}\|^2_{\infty,[-1,1]} \, L^{-2k} & \text{for } \gamma > k \end{cases}$$

with the strictly monotonically decreasing constant

$$c_{\gamma,k} = \Big(\frac{k}{\gamma - k}\Big)^{k/2} \Big(\frac{\gamma - k}{\gamma}\Big)^{\gamma/2} \qquad \text{for } \gamma > k.$$

In particular,

$$\Phi_{\gamma,W}(L) = \mathcal{O}\Big(L^{-2\min\{k,\gamma\}}\Big) \qquad \text{for} \quad L \longrightarrow \infty.$$

□

H^σ-Error Analysis for $\mathcal{C}^k$-Windows

Corollary (H^σ-error estimate for $\mathcal{C}^k$-windows)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ *for some* $\alpha > 0$ *and* $W \in \mathcal{C}^k([-1,1])$, $k \geq 2$, *with*

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Then, for $0 \leq \sigma \leq \alpha$*, the* H^σ*-norm of the FBP reconstruction error* $e_L = f - f_L$ *is bounded above by*

$$\|e_L\|_\sigma \leq \begin{cases} \left(\frac{1}{k!}\,\|W^{(k)}\|_{\infty,[-1,1]} + 1\right) L^{\sigma-\alpha}\,\|f\|_\alpha & \text{for } \alpha - \sigma \leq k \\ \left(\frac{c_{\alpha-\sigma,k}}{k!}\,\|W^{(k)}\|_{\infty,[-1,1]}\,L^{-k} + L^{\sigma-\alpha}\right)\|f\|_\alpha & \text{for } \alpha - \sigma > k \end{cases}$$

with the strictly monotonically decreasing constant

$$c_{\alpha-\sigma,k} = \left(\frac{k}{\alpha-\sigma-k}\right)^{k/2}\left(\frac{\alpha-\sigma-k}{\alpha-\sigma}\right)^{(\alpha-\sigma)/2} \qquad \text{for } \alpha - \sigma > k.$$

In particular,

$$\|e_L\|_\sigma \leq \left(c\|W^{(k)}\|_{\infty,[-1,1]}\,L^{-\min\{k,\alpha-\sigma\}} + L^{\sigma-\alpha}\right)\|f\|_\alpha = \mathcal{O}\left(L^{-\min\{k,\alpha-\sigma\}}\right). \qquad \square$$

Numerical Results

We investigate the behaviour of $\Phi_{\gamma,W}$ numerically for the generalized Gaussian filter

$$A_L(S) = |S| \, W(S/L)$$

with the window function

$$W(S) = \exp\left(-\left(\frac{\pi|S|}{\beta}\right)^k\right) \qquad \text{for } S \in [-1, 1]$$

for $k \in \mathbb{N}$ and $\beta > 1$.

Numerical Results

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$$A_L(S) = |S|\, W(S/L)$$

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$$W(S) = \exp\left(-\left(\frac{\pi|S|}{\beta}\right)^k\right) \quad \text{for } S \in [-1,1]$$

for $k \in \mathbb{N}$ and $\beta > 1$.

- If k is even, W satisfies $W \in \mathcal{C}^k([-1,1])$ and

$$W(0) = 1, \quad W^{(j)}(0) = 0 \;\; \forall\, 1 \leq j \leq k-1, \quad W^{(k)}(0) = -k!\left(\frac{\pi}{\beta}\right)^k \neq 0.$$

Numerical Results

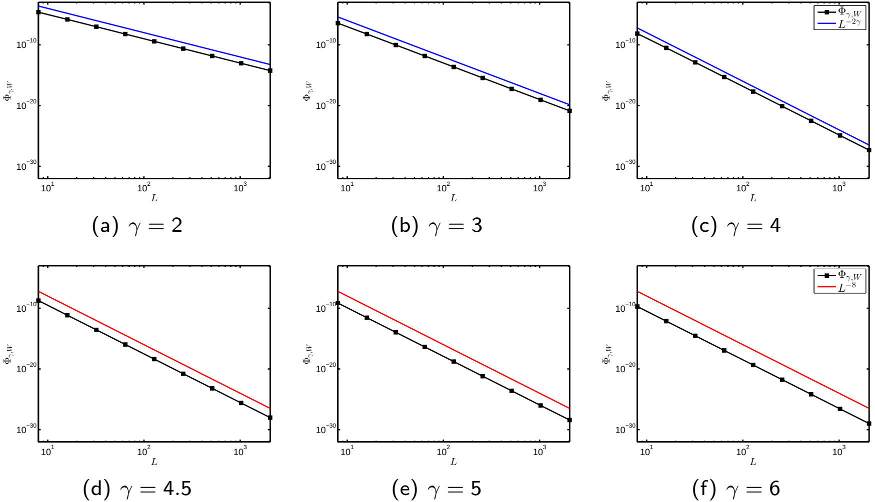


Fig.: Decay rate of $\Phi_{\gamma,W}$ for the generalized Gaussian filter with $k = 4$ and $\beta = 4$

Numerical Results

We investigate the behaviour of $\Phi_{\gamma,W}$ numerically for the generalized Gaussian filter

$$A_L(S) = |S| \, W(S/L)$$

with the window function

$$W(S) = \exp\left(-\left(\frac{\pi|S|}{\beta}\right)^k\right) \quad \text{for } S \in [-1,1]$$

for $k \in \mathbb{N}$ and $\beta > 1$.

- If k is even, W satisfies $W \in \mathcal{C}^k([-1,1])$ and

$$W(0) = 1, \quad W^{(j)}(0) = 0 \;\; \forall\, 1 \leq j \leq k-1, \quad W^{(k)}(0) = -k!\left(\frac{\pi}{\beta}\right)^k \neq 0.$$

- If k is odd, W satisfies $W \in \mathcal{C}^{k-1}([-1,1])$ with

$$W(0) = 1, \quad W^{(j)}(0) = 0 \;\; \forall\, 1 \leq j \leq k-1,$$

  but $W^{(k-1)}$ is not differentiable at zero.

Numerical Results

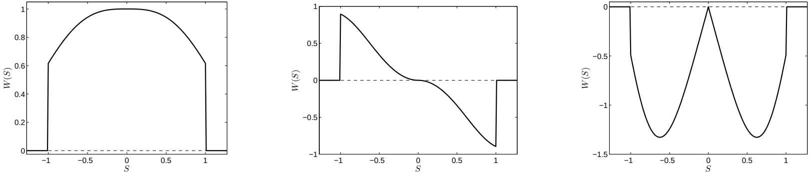
We investigate the behaviour of $\Phi_{\gamma,W}$ numerically for the generalized Gaussian filter

$$A_L(S) = |S|\, W(S/L)$$

with the window function

$$W(S) = \exp\left(-\left(\frac{\pi|S|}{\beta}\right)^k\right) \qquad \text{for } S \in [-1,1]$$

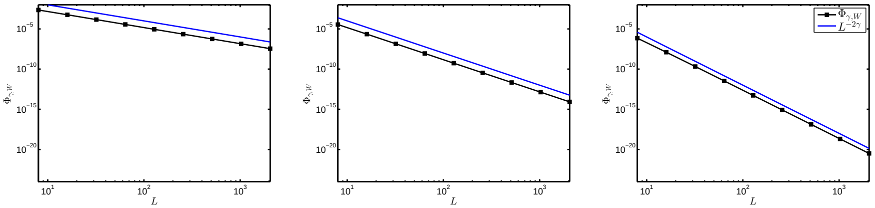
for $k \in \mathbb{N}$ and $\beta > 1$.



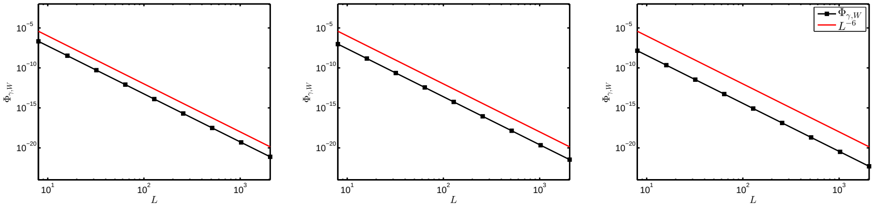
(a) Window (b) 1st Derivative (c) 2nd Derivative

Fig.: Window function of the generalized Gaussian filter with $k = 3$ and $\beta = 4$

Numerical Results



(a) $\gamma = 1$ (b) $\gamma = 2$ (c) $\gamma = 3$



(d) $\gamma = 3.5$ (e) $\gamma = 4$ (f) $\gamma = 6$

Fig.: Decay rate of $\Phi_{\gamma,W}$ for the generalized Gaussian filter with $k = 3$ and $\beta = 4$

H^σ-Error Analysis for Lipschitz-Windows

Theorem (Convergence rate of $\Phi_{\gamma,W}$ for Lipschitz-windows)

Let the window function W satisfy $W^{(j)} \in \mathcal{AC}([-1,1])$ for all $0 \leq j \leq k-1$ and

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Further, let $W^{(k-1)}$ be Lipschitz-continuous on $[-1,1]$. Then, for $\gamma \geq 0$ we have

$$\Phi_{\gamma,W}(L) \leq \begin{cases} \frac{1}{(k!)^2} \, \|W^{(k)}\|^2_{\infty,[-1,1]} \, L^{-2\gamma} & \text{for } \gamma \leq k \\ \frac{c^2_{\gamma,k}}{(k!)^2} \, \|W^{(k)}\|^2_{\infty,[-1,1]} \, L^{-2k} & \text{for } \gamma > k \end{cases}$$

with the strictly monotonically decreasing constant

$$c_{\gamma,k} = \Big(\frac{k}{\gamma - k}\Big)^{k/2} \Big(\frac{\gamma - k}{\gamma}\Big)^{\gamma/2} \qquad \text{for } \gamma > k.$$

In particular,

$$\Phi_{\gamma,W}(L) = \mathcal{O}\Big(L^{-2\min\{k,\gamma\}}\Big) \qquad \text{for} \quad L \longrightarrow \infty.$$

□

H^σ-Error Analysis for Lipschitz-Windows

Corollary (H^σ-error estimate for Lipschitz-windows)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for $\alpha > 0$ and let $W^{(j)} \in \mathcal{AC}([-1,1])$ for all $0 \leq j \leq k-1$ with

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Further, let $W^{(k-1)}$ be Lipschitz-continuous on $[-1,1]$. Then, for $0 \leq \sigma \leq \alpha$, the H^σ-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_\sigma \leq \begin{cases} \left(\frac{1}{k!}\,\|W^{(k)}\|_{\infty,[-1,1]} + 1\right) L^{\sigma-\alpha}\,\|f\|_\alpha & \text{for } \alpha - \sigma \leq k \\ \left(\frac{c_{\alpha-\sigma,k}}{k!}\,\|W^{(k)}\|_{\infty,[-1,1]}\,L^{-k} + L^{\sigma-\alpha}\right)\|f\|_\alpha & \text{for } \alpha - \sigma > k \end{cases}$$

with the strictly monotonically decreasing constant

$$c_{\alpha-\sigma,k} = \left(\frac{k}{\alpha-\sigma-k}\right)^{k/2}\left(\frac{\alpha-\sigma-k}{\alpha-\sigma}\right)^{(\alpha-\sigma)/2} \qquad \text{for } \alpha - \sigma > k.$$

In particular,

$$\|e_L\|_\sigma \leq \left(c\|W^{(k)}\|_{\infty,[-1,1]}\,L^{-\min\{k,\alpha-\sigma\}} + L^{\sigma-\alpha}\right)\|f\|_\alpha = \mathcal{O}\left(L^{-\min\{k,\alpha-\sigma\}}\right). \qquad \square$$

Asymptotic H^σ-Error Analysis

Theorem (Asymptotic H^σ-error estimate)

Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for $\alpha > 0$ and let $W \in \mathrm{L}^\infty(\mathbb{R})$ be k-times differentiable at the origin, $k \geq 2$, with

$$W(0) = 1, \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Then, for $0 \leq \sigma \leq \alpha$, the H^σ-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_\sigma \leq \begin{cases} \left(\frac{\sqrt{2}}{k!}\,|W^{(k)}(0)| + 1\right) L^{\sigma-\alpha}\,\|f\|_\alpha + o(L^{\sigma-\alpha}) & \text{for } \alpha - \sigma \leq k \\ \left(\frac{\sqrt{2}}{k!}\,c_{\alpha-\sigma,k}\,|W^{(k)}(0)|\,L^{-k} + L^{\sigma-\alpha}\right)\|f\|_\alpha + o(L^{-k}) & \text{for } \alpha - \sigma > k \end{cases}$$

with the strictly monotonically decreasing constant

$$c_{\alpha-\sigma,k} = \left(\frac{k}{\alpha-\sigma-k}\right)^{k/2}\left(\frac{\alpha-\sigma-k}{\alpha-\sigma}\right)^{(\alpha-\sigma)/2} \qquad \text{for } \alpha - \sigma > k.$$

In particular,

$$\|e_L\|_\sigma \leq \left(c\,|W^{(k)}(0)|\,L^{-\min\{k,\alpha-\sigma\}} + L^{\sigma-\alpha}\right)\|f\|_\alpha + o\left(L^{-\min\{k,\alpha-\sigma\}}\right).$$

□

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Thank you for your attention!