A Tale of Couples and Other Syzygies

Jesús Carnicer & Tomas Sauer*

Lehrstuhl für Mathematik mit Schwerpunkt Digitale Bildverarbeitung
Fakultät für Informatik und Mathematik
&
FORWISS

Universität Passau

UNIVERSITÄT PASSAU
Fakultät für Informatik und Mathematik

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Poynomial Interpolation

Definition

1. *Polynomials*: $\Pi = \mathbb{K}[x] = \mathbb{K}[x_1, \dots, x_s] = \operatorname{span}_{\mathbb{K}} \left\{ (\cdot)^\alpha : \alpha \in \mathbb{N}_0^s \right\}$.
2. *Total degree* $\leq n$: $\Pi_n = \operatorname{span}_{\mathbb{K}} \{ (\cdot)^\alpha : |\alpha| \leq n \}$.

Interpolation

1. Interpolation at $X \subset \mathbb{K}^s$: given $y \in \mathbb{K}^X$ find f such that
$$f(X) = y, \qquad \text{i.e.,} \qquad f(x) = y_x, \quad x \in X.$$
2. X *correct* for subspace $P \subset \Pi$: there exists *unique* $f \in P$ such that
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3. X is n–correct if and only if there exist $\ell_x \in \Pi_n$ such that

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4. n–correct sets: *open and dense*.

Problem

1. Characterize n–correct subsets of $\mathbb{K}^s$.
2. Give *explicit* constructions for n–correct sets.
3. **Example:** triangular grid $X = \{\alpha/n : |\alpha| \le n\}$.

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Inductive construction in 2d: Berzolari [1914], Radon [1948]

1. Let X_{n-1} be $(n-1)$–correct set.
2. Pick line $L = \{x : \ell(x) = 0\}$, $\ell \in \Pi_1$.
3. Choose $x_0, \dots, x_n$ on $L \setminus X_{n-1}$.
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L. Berzolari, *Sulla determinazione di una curva o di una superficie algebrica e su algune questioni di postulazione*, Lomb. Ist. Rend. **47** (1914), 556–564.
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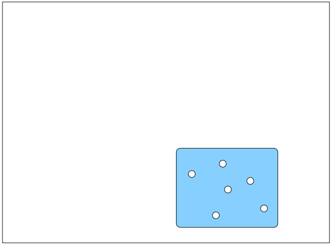

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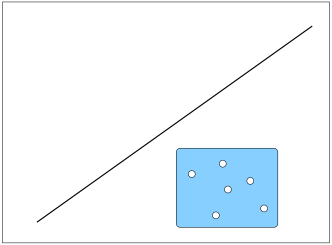

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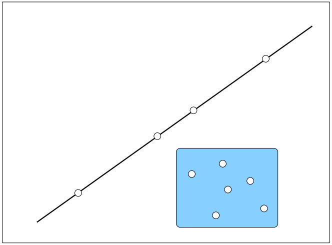

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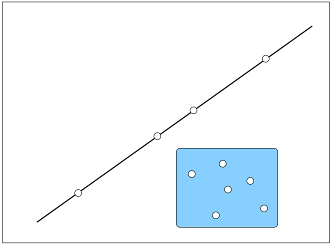

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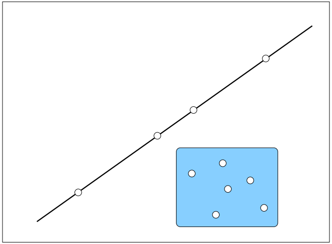

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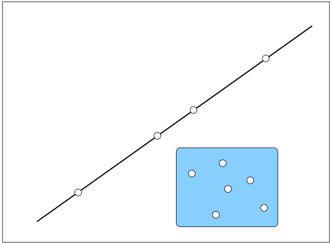

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$$\left. \begin{array}{rcl} x & \notin & L_{x,1} \cup \cdots \cup L_{x,n} \\ X \setminus \{x\} & \subset & L_{x,1} \cup \cdots \cup L_{x,n} \end{array} \right\} \qquad \Rightarrow \qquad \ell_x = \prod_{j=1} \frac{\ell_{x,j}}{\ell_{x,y}(x)}$$

Theorem

GC_n is n–correct.

Couple #2: Chung & Yao

K. C. Chung and T. H. Yao, *On lattices admitting unique Lagrange interpolation*, SIAM J. Num. Anal. **14** (1977), 735–743.

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Theorem

GC_n is n–correct. Fundamental polynomials are factorizable.

Chung & Yao II

Remarks

1. Factorizable polynomials are rare.
2. Lattice structures, grids.
3. "Thin sets" in contrast to "open and dense".

Example: natural lattice

1. $H_0, \dots, H_{s+n}$ hyperplanes in *general position*.
2. Any s of the intersect in one point $\Rightarrow \binom{n+s}{s}$ points X.

Further extensions: intersection of lines

1. rincipal lattices.
2. Reversible sets ...
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Couple #3: Gasca & Maeztu

M. Gasca and J. I. Maeztu, *On Lagrange and Hermite interpolation in* $\mathbb{R}^k$, Numer. Math. **39** (1982), 1–14.

Gasca & Maeztu

1. Interpolation on lattices.
2. Newton approach by degree.

Conjeture

1. X a GC_n set $\Rightarrow$ $n+1$ points of X are on a line.
2. Any GC_n set is obtained by the Berzolari–Radon construction.

1. Trivial for $n = 1$.
2. Easy for $n = 2$, combinatoric for $n = 3$.

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The Gasca–Maeztu Conjecture

Conjecture

Any GC_n set is obtained by the Berzolari–Radon construction.

Known?

1. $n = 4$
2. $n = 5$
3. If there is one line, there are three lines.
4. Proofs are *combinatorial*, not really geometric.
5. More work available.
6. Generalization to $s > 2$?

J. R. Busch, *A note on Lagrange interpolation in* $\mathbb{R}^2$, Rev. Union Matem. Argent. **36** (1990), 33–38.
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Couple #3: Hilbert & Burch

Hal Schenck (2015)

1. Geometric approach to Gasca–Maeztu conjecture.
2. Algebraic geometry, syzygies.
3. *Hilbert–Burch* theorem for zero dimensional ideals.
4. Syzygies.

Advantage

Check for maximal lines in a systematic way.

Disadvantage

Sophisticated, projective.

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Advantage

Check for maximal lines in a systematic way.

Disadvantage

Sophisticated, projective. $\Rightarrow$ Give elementary and affine approach.

Ideals & Bases

Definition (Ideal)

1. $\mathscr{I} \subset \Pi$ is called *ideal* if $\mathscr{I} + \mathscr{I} = \mathscr{I}$ and $\mathscr{I} \cdot \Pi = \mathscr{I}$.
2. $\mathscr{I}(X) = \{f \in \Pi : f(X) = 0\}$.

Definition (Bases)

1. $F \subset \Pi$ is called *basis* for $\mathscr{I}$ if

$$\mathscr{I} = \langle F \rangle = \Big\{ \sum_{f \in F} g_f f : g_f \in \Pi \Big\}.$$

2. $H \subset \Pi$ is called *H–basis* for $\mathscr{I}$ if

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$$\mathscr{I} \ni f = \sum_{h \in H} f_h h, \qquad \deg f_h \le \deg f - \deg h, \quad h \in H.$$

Ideals & Bases

Definition (Ideal)

1. $\mathscr{I} \subset \Pi$ is called *ideal* if $\mathscr{I} + \mathscr{I} = \mathscr{I}$ and $\mathscr{I} \cdot \Pi = \mathscr{I}$.
2. $\mathscr{I}(X) = \{f \in \Pi : f(X) = 0\}$.

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H–bases & Interpolation

Lemma

If $\Pi_{n-1} \cap \mathscr{I} = \{0\}$ and $\dim(\Pi_n \cap \mathscr{I}) = \dim \Pi_n^0 = \binom{n+s}{s-1}$ then

$$F \text{ is vector space basis of } \Pi_n \cap \mathscr{I} \quad \Leftrightarrow \quad F \text{ is H–basis of } \mathscr{I}.$$

Theorem (Berzolari–Radon extension, ideal version)

X $(n-1)$–correct, Y on hyperplane H, $\#Y = \binom{n+s}{s-1}$. Then:

$$\{\ell_y : y \in Y\} \subset \Pi_n \text{ is H–basis for } \mathscr{I}(X).$$

Proposition (Schenck, Sauer&Xu)

1. If X is GC_n then $\mathscr{I}(X)$ has a factorizable H–basis.
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Syzygies

Definition

$S \in \Pi^n$ is *syzygy* of $S \in \Pi^n$ if

$$0 = S \cdot P = \sum_{j=1}^{n} s_j p_j.$$

Module

$S, S' \in \Sigma(P) \quad \Rightarrow \quad S + S' \in S(P), S\,\Pi \subset \Sigma(P).$

Dimension theory in one line

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Syzygies & Interpolation in $s = 2$

Observation: Zero dimensional ideals

$\#X < \infty$ implies $\dim \mathscr{I}(X) = 0$.

Theorem (poor men's Hilbert–Burch)

If $X \subset \mathbb{R}^2$ is n–poised then there exist

1. H–basis $H \subset \Pi_{n+1}$ of $\mathscr{I}(X)$,
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such that

$$SH = 0 \qquad \text{and} \qquad h_j = (-1)^j \, w \det S_j, \quad j = 0, \dots, n+1.$$

S_j: jth minor of S, $w \neq 0$.

Hilbert–Burch II

Proof of "Hilbert–Burch"

1. Start with n–poised X.
2. Use Berzolari–Radon extension $Y \subset L$, $L \cap X = \emptyset$.
3. $X \cap Y$ is $(n+1)$–correct.
4. Explicit H–basis for almost any $m \in \Pi_1$

$$\ell_y = \frac{p_y}{p_y(y)} - , \qquad p_y = \prod_{y' \neq y} \left(m - m(y') \right), \qquad y \in Y.$$

5. Explicit syzygies for "S-polynomials", $y \in Y$,

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6. Linear Algebra …

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6. Linear Algebra ...

The Syzygy Matrix

Theorem

Suppose X is n–correct and H, H' two H–bases of $\mathscr{I}(X)$ with syzygy matrices S, S'. Then

$$S' = ASB, \qquad A \in \mathbb{R}^{n+1 \times n+1}, \quad B \in \mathbb{R}^{n+2 \times n+2},$$

with nonsingular A, B.

Corollary

The syzygy matrix $S = S_X$ is unique up to *scalar* similarities.

Corollary

The rows of S_X are linearly independent.

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Syzygy Matrix & Maximal Lines

Theorem

For an n–correct X are equivalent:

1. X contains a *maximal line*, i.e., $X \cap L = n+1$ for some L.
2. The exists a syzygy matrix S such that

$$S = \begin{bmatrix} \ell & * \\ 0 & * \end{bmatrix}.$$

Remark

1. Need only that some column contains ℓe_j for some j.
2. Read maximal line from syzygy matrix.
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$$S = \begin{bmatrix} \ell & * \\ 0 & * \end{bmatrix}.$$

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1. Need only that some column contains $\ell \, e_j$ for some j.
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Syzygy Matrix & Maximal Lines

Theorem

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Example I: Natural Lattices

Setup

1. Lines $L_j = \{x : \ell_j(x) = 0\}$, $j = 0, \dots, n-1$, in general position.
2. $X = \{L_j \cap L_k : j \neq k\}$, $\#X = \binom{n+2}{2}$.

Bases

1. Fundamental polynomials: $\ell_x = \prod_{r \neq j,k} \frac{\ell_r}{\ell_r(x)}$, $x = L_j \cap L_k \in X$.
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Consequence

1. All lines are maximal.
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Example II: Generalized Principal Lattices

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Projective setup

1. Sets of lines $L_{j,k}$, $j = 0, \ldots, n$, $k = 0, 1, 2$.
2. Points

$$x_\beta = \bigcap_{k=0,1,2} L_{\beta_j,j}, \qquad \beta \in \mathbb{N}_0^3,\ |\beta| = n.$$

Bases

1. Fundamental: $\ell_\beta = \displaystyle\prod_{j=0,1,2} \prod_{\gamma_j < \beta_j} \frac{\ell_{\gamma_j,j}}{\ell_{\gamma_j,j}(x_\beta)}$.
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Consequence

1. Find the two maximal lines $\ell_{0,1}$, $\ell_{0,2}$.
2. For $\ell_{0,0}$ we need another H–basis that contains $\ell_{j,0}$.
3. Direct operations on S? Combination of independent lines.

Summary

What's the point?

1. Does not prove Gasca–Maeztu conjecture.
2. Advantage: geometry instead of combinatorics.
3. No case distinctions and classifications.
4. Algebraic condition to be checked.
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