M&M's schemes

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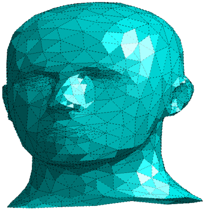
Multivariate subdivision operator

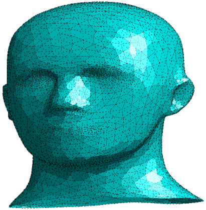
Definition: For $s \in \mathbb{N}$ (dimension), $a \in \ell_0(\mathbb{Z}^s)$ (mask), $M \in \mathbb{Z}^{s \times s}$ with $\rho(M^{-1}) < 1$ (anisoptropic dilation matrix), we define the subdivision operator $S = (a, M) : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$ by

$$Sc(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(\alpha - M\beta) c(\beta), \quad \alpha \in \mathbb{Z}^s. \tag{1}$$

(1) is equivalent to matrix-vector multiplication Sc, where

$$S = (a(\alpha - M\beta))_{\alpha, \beta \in \mathbb{Z}^s}. \tag{2}$$





Multiple subdivision scheme

Definition: Let $\mathcal{S} = \left\{ S_j = (a_j, M_j), j = 1, \ldots, J \right\}$ be a finite set of subdivision operators, and $\{M_j : j = 1, \ldots, J\}$ jointly expanding,

$$\mathrm{JSR}\left(\left\{ M_j^{-1} : j=1,\ldots,J \right\}\right) := \lim_{n\to\infty} \max_{A_k \in \left\{ M_j^{-1} \right\}} \left\| \prod_{k=1}^{n} A_k \right\|^{1/n} < 1. \qquad (3)$$

A sequence $(S_j)_j \in \mathcal{S}^{\mathbb{N}}$ is called a subdivision scheme.

Types of subdivision schemes:

• Stationary schemes are $(S_j)_j \in \mathcal{S}^{\mathbb{N}}$ with $S_j = (a, M)$,
de Rahm (1947), Chaikin (1974).

• Non-stationary schemes are $(S_j)_j \in \mathcal{S}^{\mathbb{N}}$ with $S_j = (a_j, M)$,
N. Dyn, D. Levin (1992), A. Cohen (1996).

• Multiple schemes are $(S_j)_j \in \mathcal{S}^{\mathbb{N}}$ with $S_j = (a_j, M_j)$,
T. Sauer (2012).

Convergence of multiple subdivision schemes: joint spectral radius, restricted spectral radius

Definition: A multiple subdivision scheme $(S_j)_j \in \mathcal{S}^{\mathbb{N}}$ is convergent[a] if $\forall c \in \ell_\infty(\mathbb{Z}^s)\ \exists g \in C^0(\mathbb{R}^s) : g = \lim_{n\to\infty} S_n \cdots S_1 c$

- Stationary schemes (Daubechies, Lagarias, CDM):
  $(S = (a, M))_j \in \mathcal{S}^{\mathbb{N}}$ convergent $\Leftrightarrow$ Joint spectral radius < 1.
  $\Leftrightarrow$ Restricted spectral radius < 1
- Non-stationary schemes (Charina, Conti, Guglielmi, Protasov):
  $(S_j = (a_j, M))_j \in \mathcal{S}^{\mathbb{N}}$ convergent $\Leftarrow$ Joint spectral radius < 1
- Multiple schemes (Sauer):
  All $(S_j = (a_j, M_j))_j \in \mathcal{S}^{\mathbb{N}}$ convergent $\Leftrightarrow$ Restricted spectral radius < 1

Result (Mejstrik 2016): JSR $=$ RSR for multiple schemes

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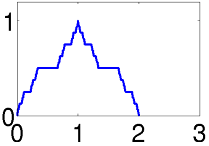
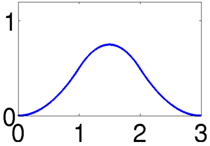
[a] $g = \lim_{n\to\infty} S_n \cdots S_1 c \Leftrightarrow \lim_{n\to\infty} \left|\left| g(M_1^{-1} \cdots M_n^{-1} \cdot) - S_n \cdots S_1 c(\cdot) \right|\right|_{\ell_\infty} = 0.$

Multiple subdivision schemes: univariate example

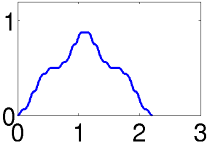
$$\mathcal{S} = \{S_1 = (a_1, M_1 = 3), S_2 = (a_2, M_2 = 2)\},$$
$$a_1 = 1/2 \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \end{pmatrix} \text{ and } a_2 = 1/4 \begin{pmatrix} 1 & 3 & 3 & 1 \end{pmatrix}.$$



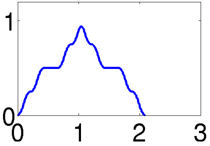
$g_1 = \lim_{n\to\infty} S_1^n c$



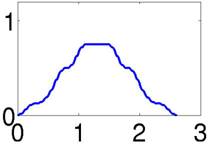
$g_2 = \lim_{n\to\infty} S_2^n c$



$g_3 = \lim_{n\to\infty} (S_2 S_1)^n c$



$g_4 = \lim_{n\to\infty} (S_2^2 S_1^2)^n c$



$g_5 = \lim_{n\to\infty} (S_1 S_2)^n c$

Starting sequence c is the Dirac-delta sequence.

Joint spectral radius approach

$$\mathcal{S} = \{S_1 = (a_1, M_1 = 3), S_2 = (a_2, M_2 = 2)\},$$
$$a_1 = 1/2 \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \end{pmatrix} \text{ and } a_2 = 1/4 \begin{pmatrix} 1 & 3 & 3 & 1 \end{pmatrix}.$$
$$S_j = (a_j(M_j\alpha - \beta))_{\alpha,\beta\in\mathbb{Z}^s},\ j = 1, 2.$$

- Transition matrices:

$$T_{1,0} = 1/2 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ T_{1,1} = 1/2 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},\ T_{1,2} = 1/2 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$
$$T_{2,0} = 1/4 \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix},\ T_{2,1} = 1/4 \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$V = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^{\perp}$$

- Convergence analysis:

$$\mathrm{JSR}(\{T_{1,0}|_V, T_{1,1}|_V, T_{1,2}|_V, T_{2,0}|_V, T_{2,1}|_V\}) = \frac{1}{2} < 1 \Rightarrow \text{Convergence}$$

- Hölder regularity:

$g_3 = \lim_{n\to\infty} (S_2 S_1)^n \delta$	$g_4 = \lim_{n\to\infty} (S_2^2 S_1^2)^n \delta$
0.630	0.712

Joint spectral radius approach

Definition: Let $\mathcal{S} = \{S_j = (a_j, M_j) : j = 1, \dots, J\}$ be a finite set of subdivision operators.

- Set of transition matrices $\mathcal{T} := \left\{ T_{j,d} : d \in D_j, j = 1, \dots, J \right\}$,

$$T_{j,d} = \left(a_j(M_j\beta - \alpha + d)\right)_{\alpha,\beta\in\Omega} \in \mathbb{R}^{|\Omega|\times|\Omega|} \qquad (4)$$

with $\Omega \subset \mathbb{Z}^s$ finite, $d \in D_j \simeq \mathbb{Z}^s / M_j\mathbb{Z}^s$, $j = 1, \dots, J$.

- Difference subspace: $V := (1, \cdots, 1)^\perp \subseteq \mathbb{R}^{|\Omega|}$,
- Restriction of $\mathcal{T}$ to V: $\mathcal{T}|_V := \left\{ T_{j,d}|_V : d \in D_j, j = 1, \dots, J \right\}$
- Joint spectral radius corresponding to $\mathcal{S}$ is defined by

$$\mathrm{JSR}(\mathcal{T}|_V) := \lim_{n\to\infty} \max_{A_i \in \mathcal{T}|_V} \left\| \prod_{i=1}^{n} A_i \right\|^{1/n}. \qquad (5)$$

Goal: JSR and multiple subdivision schemes

$\mathcal{S} = \{S_j = (a_j, M_j) : j = 1, \ldots, J\}$ finite set of subdivision operators,
$\mathcal{T} = \{T_{j,d} : d \in D_j, j = 1, \ldots, J\}$ finite set of transition matrices.

Theorem (Mejstrik, 2016):
All $(S_j)_j \in \mathcal{S}^{\mathbb{N}}$ are convergent $\Leftrightarrow$ $\mathrm{JSR}(\mathcal{T}|_V) < 1$.

Definition: For $k = 1, \ldots, s$ we define $\nabla_k : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$ by

$$\nabla_k c := c(\cdot) - c(\cdot - \varepsilon_k), c \in \ell(\mathbb{Z}^s). \tag{6}$$

Outline of the proof

$\mathrm{JSR}(\mathcal{T}|_V) = \lim_{n\to\infty} \max_{T_{d,j}|_V \in \mathcal{T}|_V} \left|\left| \prod_{i=1}^n T_{d,j}|_V \right|\right|^{1/n}$, where $V = (1,\ldots,1)^{\perp}$.

- $V = \mathrm{span}\{v = (\nabla_k \delta)(\cdot - \beta) : k = 1,\ldots,s,\ \beta \in \mathbb{Z}^s,\ \mathrm{supp}\, v \subseteq \Omega\}$,
- There exists $A, B > 0$, depending on Ω and s, such that

$$A \left|\left| T_{j,d}|_V \right|\right|_\infty \leqslant \max_{k=1,\ldots,s} \left|\left| \nabla_k S_j \delta \right|\right|_\infty \leqslant B \left|\left| T_{j,d}|_V \right|\right|_\infty, \quad \forall d, j.$$

$\mathrm{JSR}(\mathcal{T}|_V) = \lim_{n\to\infty} \max_{(S_j)_j \in \mathcal{S}^{\mathbb{N}}} \max_{k=1,\ldots,s} ||\nabla_k S_n \cdots S_1 \delta||_\infty^{1/n}$

- There exists $\overline{A}, \overline{B} > 0$, depending on $\mathcal{S}$ and s, such that

$$\overline{A} \left|\left| \nabla_k S_j \delta \right|\right|_\infty \leqslant \sup_{\substack{c \in \ell_\infty(\mathbb{Z}^s) \\ ||\nabla_k c||_\infty = 1}} \left|\left| \nabla_k S_j c \right|\right|_\infty \leqslant \overline{B} \left|\left| \nabla_k S_j \delta \right|\right|_\infty, \quad \forall k, j.$$

$\mathrm{JSR}(\mathcal{T}|_V) = \lim_{n\to\infty} \max_{(S_j)_j \in \mathcal{S}^{\mathbb{N}}} \max_{k=1,\ldots,s} \sup_{\substack{c \in \ell_\infty(\mathbb{Z}^s) \\ ||\nabla_k c||_\infty = 1}} ||\nabla_k S_n \cdots S_1 c||_\infty^{1/n} = \mathrm{RSR}(\nabla \mathcal{S})$