# Self-Referentiality, Fractels, and Applications

Peter Massopust

Center of Mathematics
Research Unit M15
Technical University of Munich
Germany

**2nd IM-Workshop on Applied Approximation, Signals, and Images,**
**Bernried, Feb 27 - Mar 3, 2017**

Joint work with

- Michael Barnsley (Australian National University)
- Markus Hegland (Australian National University and Institute of Advanced Study, Technical University of Munich, Germany)

# Outline

- Iterated Function Systems (IFSs)
- Fractal Functions
- Fractels
- Local IFSs

# Iterated Function Systems

Standing assumptions:

- $(\mathbb{X}, d_{\mathbb{X}})$ is a complete metric space with metric $d_{\mathbb{X}}$.
- $1 < n \in \mathbb{N}$ and $\mathbb{N}_n := \{1, \ldots, n\}$.

A family $\mathcal{F}$ of continuous mappings $f_i : \mathbb{X} \to \mathbb{X}$, $i \in \mathbb{N}_n$, is called a *iterated function system* (IFS).

If all mappings in $\mathcal{F}$ are contractive on $\mathbb{X}$ then the IFS is called *contractive.*

Assumption: From now on all $f_i$ are contractive.

# Hyperspace of Nonempty Compact Subsets of $\mathbb{X}$

Let $(\mathbb{H}(\mathbb{X}), d_{\mathbb{H}})$ be the hyperspace of nonempty compact subsets of $\mathbb{X}$ endowed with the Hausdorff metric $d_{\mathbb{H}}$:

$$d_{\mathbb{H}}(A, B) := \max\{\max_{a\in A}\min_{b\in B} d_X(a,b), \max_{b\in B}\min_{a\in A} d_X(a,b)\}.$$

Define $\mathcal{F} : \mathbb{H}(\mathbb{X}) \to \mathbb{H}(\mathbb{X})$ by

$$\mathcal{F}(B) := \bigcup_{i=1}^{n} f_i(B).$$

(J. Hutchinson [1981], M. Barnsley & S. Demko [1985])

## Some Results

$(\mathbb{X}, d_{\mathbb{X}})$ complete implies $(\mathbb{H}(\mathbb{X}), d_{\mathbb{H}})$ complete.

If all $f_i : \mathbb{X} \to \mathbb{X}$ are contractive on $\mathbb{X}$ with contractivity constants $s_i \in (-1, 1)$ then $\mathcal{F} : \mathbb{H}(\mathbb{X}) \to \mathbb{H}(\mathbb{X})$ is contractive on $\mathbb{H}(\mathbb{X})$ with contractivity constant $s = \max\{s_i\}$.

The Banach Fixed Point Theorem implies that $\mathcal{F}$ has a unique fixed point in $\mathbb{H}(\mathbb{X})$.

This fixed point $A$ is called the *fractal* generated by the IFS $\mathcal{F}$.

$$A = \mathcal{F}(A) = \bigcup_{i=1}^{n} f_i(A).$$

"Self-referential equation"

# Construction of Fractals Via IFSs

Consequences of the Banach Fixed Point Theorem:

Let $E_0 \in \mathbb{H}(\mathbb{X})$ be *arbitrary*.

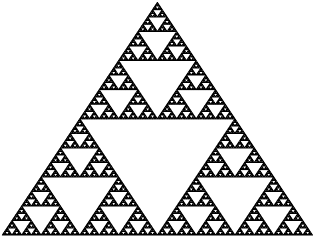
Set $E_k := \mathcal{F}(E_{k-1}) = \bigcup_{i=1}^{n} f_i(E_{k-1})$.

Then $A = \lim_{k\to\infty} E_k = \lim_{k\to\infty} \mathcal{F}^k(E_0)$.

# Sierpiński Triangle $\mathfrak{T}$

Define $f_i : [0,1]^2 \to [0,1]^2$ by

$$f_0(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right),$$
$$f_1(x,y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\right),$$
$$f_2(x,y) = \left(\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{\sqrt{3}}{4}\right).$$



$$\mathfrak{T} = f_0(\mathfrak{T}) \cup f_1(\mathfrak{T}) \cup f_2(\mathfrak{T})$$

(W. Sierpiński [1915])

# Semi-Group Property

$$\begin{aligned}\mathcal{F}^2(A) &= (\mathcal{F} \circ \mathcal{F})(A) \\ &= (f_1 \circ f_1)(A) \cup (f_1 \circ f_2)(A) \cup \cdots \cup (f_n \circ f_n)(A) \\ &= \bigcup_{i_1, i_2 = 1}^{n} (f_{i_1} \circ f_{i_2})(A).\end{aligned}$$

In general:

$$\mathcal{F}^k(A) = \bigcup_{i_1, \ldots, i_k = 1}^{n} (f_{i_1} \circ \cdots \circ f_{i_k})(A).$$

Hence

$$A = \lim_{k \to \infty} \mathcal{F}^k(A).$$

$A$ is invariant under the semi-group $S[\mathcal{F}]$ generated by $f \in \mathcal{F}$.

# Collage Theorem (Barnsley 1985)

Let $M \in \mathbb{H}(\mathbb{X})$ and $\varepsilon > 0$ be given. Suppose that $\mathcal{F}$ is a (contractive) IFS such that

$$d_{\mathbb{H}}\left(M, \bigcup_{i=1}^{n} f_i(M)\right) \leq \varepsilon.$$

then

$$d_{\mathbb{H}}(M, A) \leq \frac{\varepsilon}{1-s},$$

where $A$ is the fractal generated by the IFS $\mathcal{F}$ and $s := \max\{s_i\}$.

# The Barnsley Fern

The Barnsley fern is a fractal model of a fern belonging to the black spleenwort variety.

It is generated by the IFS $\{f_i : [0,1]^2 \to [0,1]^2\}$, where

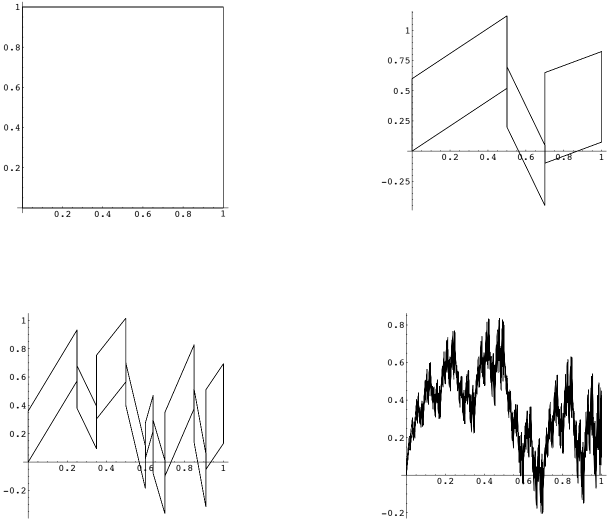
$$f_1 \binom{x}{y} := \frac{4}{5} \begin{pmatrix} \cos\frac{\pi}{60} & \sin\frac{\pi}{60} \\ -\sin\frac{\pi}{60} & \cos\frac{\pi}{60} \end{pmatrix} \binom{x}{y} + \binom{0}{\frac{3}{2}},$$

$$f_2 \binom{x}{y} := \frac{1}{3} \begin{pmatrix} \cos\frac{5\pi}{18} & -\sin\frac{5\pi}{18} \\ \sin\frac{5\pi}{18} & \cos\frac{5\pi}{18} \end{pmatrix} \binom{x}{y} + \binom{0}{\frac{3}{2}},$$

$$f_3 \binom{x}{y} := \begin{pmatrix} \frac{1}{3}\cos\frac{2\pi}{3} & \frac{9}{25}\sin\frac{5\pi}{18} \\ \frac{1}{3}\sin\frac{2\pi}{3} & -\frac{9}{25}\cos\frac{5\pi}{18} \end{pmatrix} \binom{x}{y} + \binom{0}{\frac{2}{5}},$$

$$f_4 \binom{x}{y} := \begin{pmatrix} 0 & 0 \\ 0 & \frac{11}{50} \end{pmatrix} \binom{x}{y}.$$

# Fractal Functions - Geometric Construction



# Fractal Functions - Analytic Construction

Let $\Omega := [0, n) \subset \mathbb{R}$.

Let $u_k : \Omega \to \Omega, \quad u_k(x) := \frac{x}{n} + k, \quad k = 0, 1, \ldots, n-1.$

Let $\lambda_k : \mathbb{R} \to \mathbb{R}$ be bounded functions, and $s_k \in \mathbb{R}$.

For $g \in L^\infty(\Omega)$ define a Read-Bajractarević (RB) operator by

$$T(g) := \sum_{k=0}^{n-1} (\lambda_k \circ u_k^{-1}) \chi_{u_k(\Omega)} + \sum_{k=0}^{n-1} s_k (g \circ u_k^{-1}) \chi_{u_k(\Omega)}.$$

If $\max |s_k| < 1$, then $T$ is contractive on $L^\infty(\Omega)$ and its unique fixed point $f : \Omega \to \mathbb{R}$ satisfies

$$f = \sum_{k=0}^{n-1} (\lambda_k \circ u_k^{-1}) \chi_{u_k(\Omega)} + \sum_{k=0}^{n-1} s_k (f \circ u_k^{-1}) \chi_{u_k(\Omega)}$$

$f$ is called a *fractal function.*

Equivalent characterization of a fractal function:

$$f(u_k(x)) = \lambda_k(x) + s_k f(x), \quad \forall x \in [0, n), \ \forall k.$$

The fixed point $f$ can be evaluated via

$$f = \lim_{k \to \infty} T^k f_0,$$

where $f_0 \in L^\infty$.

The rate of convergence is given by

$$\|f - T^k f_0\|_\infty \leq \frac{s^k}{1 - s} \|T f_0 - f_0\|_\infty.$$

Moreover,

$$\|T\|_\infty \leq \frac{1 + s}{1 - s}.$$

# Inhomogeneous Refinability

Extend $f$ to $\mathbb{R}$ by setting it equal to zero off $\Omega$.

Let

$$\Lambda := \begin{cases} \displaystyle\sum_{k=0}^{n-1} (\lambda_k \circ u_k^{-1})\, \chi_{u_k(\Omega)}, & \text{on } \Omega \\ \\ 0, & \text{on } \mathbb{R} \setminus \Omega. \end{cases}$$

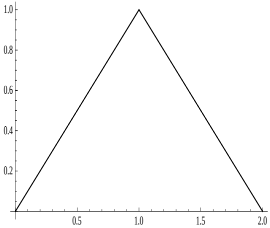
The fixed point equation for $f$ can then be written as

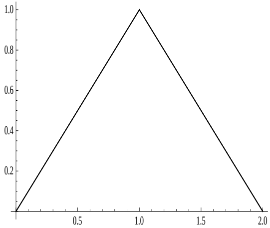
$$f(x) = \Lambda(x) + \sum_{k=0}^{n-1} s_k\, f\,(Nx - k)\,.$$

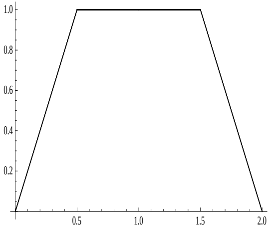
# Examples

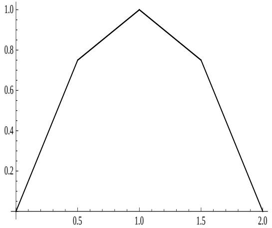
$$(T_t f)(x) = \begin{cases} x + \frac{1}{2}\, f(2x), & x \in [0,1); \\ (2-x) + \frac{1}{2}\, f(2x-1), & x \in [1,2). \end{cases}$$

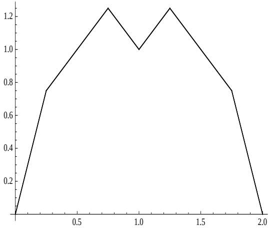
$$(T_p f)(x) = \begin{cases} x + \frac{1}{4}\, f(2x), & x \in [0,1); \\ (2-x) + \frac{1}{4}\, f(2x-1), & x \in [1,2). \end{cases}$$


1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0

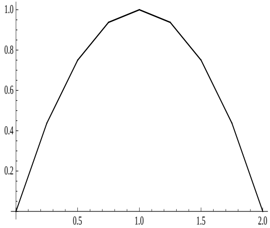

1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0


1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0

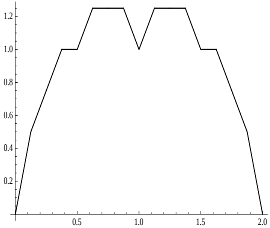

1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0

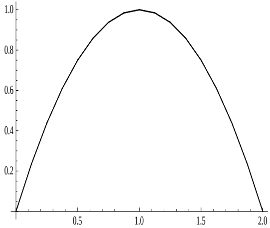


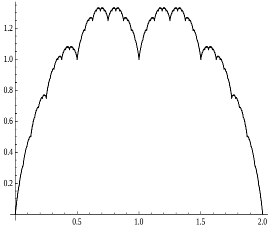
1.2
1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0

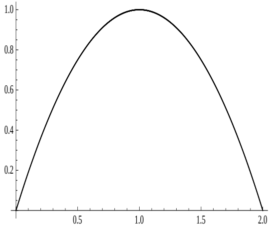
1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0


1.2
1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0


1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0




Takagi function




$f(x) = x(2 - x)$

# Relation Between $\mathcal{F}$ and $T$

$(\mathbb{X}, d_{\mathbb{X}})$ compact metric space and $(\mathbb{Y}, d_{\mathbb{Y}})$ complete metric space

Let $f \in L^\infty(\mathbb{X}, \mathbb{Y})$ be a fractal function generated by the RB operator $T$.

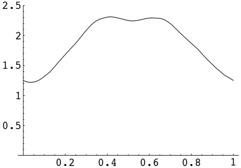
The following diagram commutes:

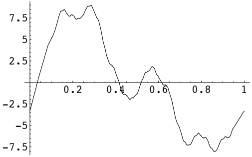
$$\begin{array}{ccc}
\mathbb{X} \times \mathbb{Y} & \xrightarrow{\mathcal{F}} & \mathbb{X} \times \mathbb{Y} \\
\Big\uparrow G & & \Big\uparrow G \\
L^\infty(\mathbb{X}, \mathbb{Y}) & \xrightarrow{T} & L^\infty(\mathbb{X}, \mathbb{Y})
\end{array}$$

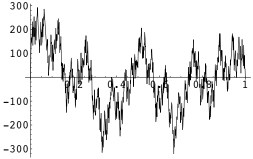
where $G$ is the mapping

$$L^\infty(\mathbb{X}, \mathbb{Y}) \ni g \mapsto G(g) = \{(x, g(x)) \mid x \in \mathbb{X}\} \in \mathbb{X} \times \mathbb{Y}.$$

# Differentiable Fractal Functions



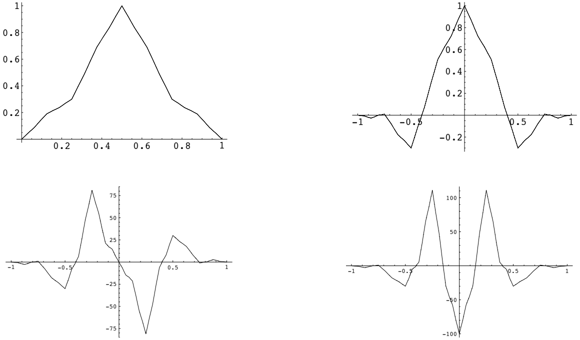



# Applications of Fractal Functions

Fractal functions can be “pieced together” to generate multiresolution analyses and continuous compactly supported refinable functions and wavelets.

In the literature these became known as the GHM scaling vector and the DGHM multiwavelet.



# A Fundamental Result

D. Hardin proved in 2012 that

*every compactly supported refinable function is a piecewise fractal function.*

In particular, the unique compactly supported continuous function determined by the mask of a convergent subdivision scheme is a piecewise fractal function.

# Every Continuous Function is a Fractal Function

Let $f \in C[a,b]$ and $h \in C[a,b]$ any function with $h(a) = f(a)$ and $h(b) = f(b)$.

Then $f$ generates an entire family of fractal functions via the fixed point of

$$Tg := f + \sum_{k=1}^{n} s_k (g - h) \circ u_k^{-1} \chi_{u_k(\Omega)}.$$

This family is parametrized by the scaling factors $s_k$ and its construction is reminiscent of splines.

If $s_k = 0$ for all $k$ then $Tf = f$, i.e., *every continuous function is a fractal function.*

# Approximation with Fractal Functions

By the Collage Theorem one can approximate the graph of any function arbitrarily close by the graph of a fractal function.

But one can do better...

# Fractels

Let $(\mathbb{X}, d_{\mathbb{X}})$ and $(\mathbb{Y}, d_{\mathbb{Y}})$ be complete metric spaces.

Let $u : \mathbb{X} \to \mathbb{X}$ be an injective mapping.

Let $f : \operatorname{dom} f \subseteq \mathbb{X} \to \mathbb{Y}$.

An invertible mapping $\mathfrak{w} \in C(\mathbb{X} \times \mathbb{Y})$ of the form

$$\mathfrak{w}(x, y) = (u(x), v(x, y)),$$

for some $v : \mathbb{X} \times \mathbb{Y} \to \mathbb{Y}$, and which satisfies

$$\mathfrak{w}(\operatorname{graph} f) \subseteq \operatorname{graph} f,$$

is called a *fractel* for $f$.

# Characterization of Fractels

Fractel is an abbreviation for "**fract**al **el**ement."

The class of fractels is not empty; the identity function $\mathrm{id}_{\mathbb{X}\times\mathbb{Y}}$ on $\mathbb{X}\times\mathbb{Y}$ is a fractel for any function $f$. This fractel is called the *trivial fractel.*

If $\mathfrak{w} = (u, v)$ is a fractal for $f$, then $u(\mathrm{dom}\, f) \subsetneq \mathrm{dom}\, f$.

*The function* $\mathfrak{w} : \mathbb{X}\times\mathbb{Y} \to \mathbb{X}\times\mathbb{Y}$ *is a fractel for* $f$ *if and only if*

$$v(x, f(x)) = f(u(x)),$$

*for all* $x \in \mathrm{dom}\, f \subseteq \mathbb{X}$.

# Example of a Fractel

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be given by $f(x) := ax^p$, where $a, p \in \mathbb{R}$.

Then, a nontrivial fractel for $f$ is $\mathfrak{w}(x, y) = (\frac{x}{2}, \frac{y}{2^p})$, where $v(x, y) = \frac{y}{2^p}$:

$$v(x, ax^p) = \frac{ax^p}{2^p} = f(\tfrac{x}{2}).$$

The fractel $(\frac{x}{2}, \frac{y}{2^p})$ is independent of $a$ and linear in $(x, y)$.

Note that different functions may share the same fractel.

# Semi-Group of Fractels

*Suppose that* $\mathfrak{w}_1 = (u_1, v_1)$ *and* $\mathfrak{w}_2 = (u_2, v_2)$ *are fractels for* $f : \mathbb{X} \to \mathbb{Y}$. *Then* $\mathfrak{w}_1 \circ \mathfrak{w}_2 := (u_1 \circ u_2, v_1(u_2, v_2))$ *is also a fractel for* $f$.

The collection of all nontrivial fractels of a given function $f$ forms a semi-group under composition $\circ$.

The collection of all fractels of a function $f$ forms a monoid under $\circ$ where the identity element is the trivial fractel.

# Self-Referential Functions

A function $f : \mathbb{X} \to \mathbb{Y}$ is called *self-referential* if it has a nontrivial semigroup of fractels.

Example

$f : [0,1] \to \mathbb{R}$, $x \mapsto 4x(1-x)$.

$f(\frac{x}{2}) = x + \frac{1}{4}f(x)$ and $f(\frac{x+1}{2}) = 1 - x + \frac{1}{4}f(x)$.

Two fractels: $\mathfrak{w}_1(x,y) = (\frac{1}{2}x, x + \frac{1}{4}y)$,

$$\mathfrak{w}_2(x,y) = (\tfrac{1}{2}(x+1), 1 - x + \tfrac{1}{4}y).$$

Recall: graph $f$ is invariant under the semi-group generated by $\{\mathfrak{w}_1, \mathfrak{w}_2\}$.

# Linear Fractels

Let $\mathbb{X} := \mathbb{R}^m$ and $\mathbb{Y} := \mathbb{R}^n$.

Let $\mathrm{GL}(n, \mathbb{R})$ be the general linear group on $\mathbb{R}^n$ and $\mathrm{Aff}(m, \mathbb{R}) = \mathbb{R}^m \rtimes \mathrm{GL}(m, \mathbb{R})$ the affine group over $\mathbb{R}^m$.

Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a function.

A fractel $\mathfrak{w} = (u, v)$ of $f$ is called *linear* if

(i) $u \in \mathrm{Aff}(m, \mathbb{R})$ and

(ii) $v(x, \cdot\,) \in \mathrm{GL}(n, \mathbb{R})$, $x \in \mathbb{R}^m$.

A linear fractel $\mathfrak{w}$ has the explicit form

$$\mathfrak{w}(x, y) = (Ax + b, My),$$

where $A$ and $M$ are invertible matrices and $b \in \mathbb{R}^m$.

# A Property of Linear Fractels

*Let $w$ be a linear fractel. Then all the eigenvalues of the matrix $A \in \mathrm{GL}(d, \mathbb{R})$ have modulus $\leq 1$ and at least one eigenvalue has modulus strictly less than 1. If $d = 1$, then $|A| < 1$.*

This proposition together with $A(\mathrm{dom}(f)) \subsetneq \mathrm{dom}(f)$ implies that the collection of all fractels, including the trivial fractel, cannot form a group under function composition.

# Algebra of Fractels

*If* $f : \mathbb{X} \to \mathbb{Y}$ *is bijective, then a fractel for* $f$ *is given by*

$$\mathfrak{w}(x, y) := (u(x), f \circ u \circ f^{-1}(y)).$$

*If* $\mathfrak{w} = (u, v)$, $u \in \mathrm{GL}(d, \mathbb{R})$, *is a fractel for* $f : \mathbb{R}^d \to \mathbb{Y}$, *then*

$$(A^{-1} \circ u \circ A(x) + A^{-1} \circ (u - I)b, v(Ax + b, y))$$

*is a fractel for* $f(Ax + b)$, *where* $I$ *is the unit of* $\mathrm{GL}(d, \mathbb{R})$.

# Fractels for the Ring of Functions $f : \mathbb{X} \to \mathbb{C}$

*Let* $f_1, f_2 : \mathbb{X} \to \mathbb{C}$*. Let* $\mathfrak{w}_i : \mathbb{X} \times \mathbb{Y} \to \mathbb{X} \times \mathbb{Y}$*, where* $\mathfrak{w}_i = (u, v_i)$ *is a fractel for* $f_i$*,* $i = 1, 2$*.*

(i) *A fractel for* $f_1 + f_2$ *is*

$$(u(x), F_1(x, y - f_2(x)) + v_2(x, y - f_1(x))).$$

(ii) *A fractel for* $af_1$*,* $a \in \mathbb{R}\backslash\{0\}$*, is*

$$(u(x), av_1(x, y/a).$$

(iii) *If* $\operatorname{dom}(f_1) \cap \operatorname{dom}(f_2) \neq \emptyset$ *and if* $f_1 \cdot f_2 \neq 0$ *on* $\operatorname{dom}(f_1) \cap \operatorname{dom}(f_2)$*, then a fractel for* $f_1 \cdot f_2$ *is*

$$\left(u(x), v_1\left(x, \frac{y}{f_2(x)}\right) \cdot v_2\left(x, \frac{y}{f_1(x)}\right)\right).$$

# From Fractels to Fractals

Let $f:[0,1]\to\mathbb{R}$, $x\mapsto x^m+g_1(x)$, where $m\in\mathbb{R}\backslash\{0\}$ and $g_1:[0,1]\to\mathbb{R}$. Let $s_1,s_2>0$ with $s_1+s_2\geq 1$.

Let

$$\mathfrak{w}_1:\mathbb{R}^2\to\mathbb{R}^2,\quad (x,y)\mapsto(s_1x, s_1^m y+g_1(s_1x)-s_1^m g_1(x)),$$

and

$$\mathfrak{w}_2:\mathbb{R}^2\to\mathbb{R}^2,\quad (x,y)\mapsto(s_2x-s_2+1, {s_2}^m y+g_2(s_2x)-{s_2}^m g_2(x)),$$

where $g_2(x)=x^m-(x-1)^m+g_1(x)$.

The IFS

$$\{[0,1]\times\mathbb{R};\mathfrak{w}_1(x,y),\mathfrak{w}_2(x,y)\}$$

is contractive and its attractor is graph $f$.

# Local IFSs

Let $\{\mathbb{X}_i : i \in \mathbb{N}_n\}$ be a family of nonempty subsets of $\mathbb{X}$.

Let for each $\mathbb{X}_i$ exist a contractive mapping $f_i : \mathbb{X}_i \to \mathbb{X}$.

Then $\mathcal{F}_{\mathrm{loc}} := \{(\mathbb{X}_i, f_i) : i \in \mathbb{N}_n\}$ is called a *local IFS*.

Note that if $\mathbb{X}_i = \mathbb{X}$ for all $i$ then one obtains an IFS.

Define a set-valued operator $\mathcal{F}_{\mathrm{loc}} : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ by

$$\mathcal{F}_{\mathrm{loc}}(S) := \bigcup_{i=1}^{n} f_i(S \cap \mathbb{X}_i).$$

$A \in 2^{\mathbb{X}}$ is called a *local fractal* if

$$A = \mathcal{F}_{\mathrm{loc}}(A) = \bigcup_{i=1}^{n} f_i(A \cap \mathbb{X}_i).$$

# Construction of Local Fractals

Assumptions:

- $\mathbb{X}$ is compact.
- All $\mathbb{X}_i$ are closed, i.e., compact in $\mathbb{X}$.
- Local IFS $\{(\mathbb{X}_i, f_i) : i \in \mathbb{N}_n\}$ is contractive.

Let $K_0 := \mathbb{X}$ and $K_k := \mathcal{F}_{\text{loc}}(K_{k-1}) = \bigcup_{i \in \mathbb{N}_n} f_i(K_{k-1} \cap \mathbb{X}_i)$

Assume that $K_k \neq \emptyset$. (E.g.: $f_i(\mathbb{X}_i) \subset \mathbb{X}_i$, $i \in \mathbb{N}_n$.)

Then $K := \bigcap_{n \in \mathbb{N}_0} K_k \neq \emptyset$ and $K = \lim_{k \to \infty} K_k$.

$$K = \lim_{k \to \infty} K_k = \lim_{k \to \infty} \bigcup_{i \in \mathbb{N}_n} f_i(K_{k-1} \cap \mathbb{X}_i) = \bigcup_{i \in \mathbb{N}_n} f_i(K \cap \mathbb{X}_i) = \mathcal{F}_{\text{loc}}(K).$$

# Approximation of Functions with Fractels

Consider fractels which have a domain $\operatorname{dom} \mathfrak{w} = \operatorname{dom} u \times \mathbb{R}$ where the domain of $u$ is an interval $[a, b]$.

$u(x) := \frac{x+\tau}{2}$ for some $\tau \in [a, b]$.

$v(x, y) := \sigma y + (1 - \sigma)G(x)$ for some $0 \leq \sigma < 1$ and $G \in C[a, b]$.

Approximations are obtained by approximating $G(x)$, for instance, constant approximations $G(x) \approx \gamma$.

The functions

$$\mathfrak{w}(x, y) = (\tfrac{1}{2}(x + \tau), \sigma y + (1 - \sigma)\gamma)$$

are fractels for functions $h$ of the form

$$h(x) = \alpha(x - \tau)^{\theta} + \gamma, \quad \alpha \in \mathbb{R},$$

where $2^{-\theta} = \sigma$.

# Approximation Procedure

Generate an approximation of a function $f$ by first constructing its fractels of the form

$$\mathfrak{w}(x,y) = (\tfrac{1}{2}(x+\tau), \sigma y + (1-\sigma)G(x))$$

and then obtain an approximation of $\mathfrak{w}$ by approximating $G$ by a constant $\gamma$:

Let $f : \Omega \to \mathbb{R}$ have fractel $\mathfrak{w}(x,y) = (\frac{1}{2}(x+\tau), \sigma y + \lambda(x))$, where $\lambda$ is given by $(1-\sigma)G$.

Further suppose that $\widetilde{\lambda} = (1-\sigma)\gamma$ is an approximation for $\lambda$.

Denote by $\widetilde{f}$ the function with fractel

$$\widetilde{\mathfrak{w}}(x,y) = (\tfrac{1}{2}(x+\tau), \sigma y + \widetilde{\lambda}(x)) = (\tfrac{1}{2}(x+\tau), \sigma y + (1-\sigma)\gamma)).$$

Then

$$\|f - \widetilde{f}\|_\infty \le \|G - \gamma\|_\infty.$$

# A Proposition

*A function $f$ defined by*

$$f(x) = \alpha(x-\tau)^{\theta} + g(x),$$

*for $x \in [a,b]$ with $\theta > 0$ and $\tau \in [a,b]$, admits a fractel $\mathfrak{w}$ with*

$$\mathfrak{w}(x,y) = (\tfrac{1}{2}(x+\tau), \sigma y + (1-\sigma)G(x)),$$

*where $\sigma = 2^{-\theta}$ and*

$$G(x) = \frac{g\left(\frac{1}{2}(x+\tau)\right) - \sigma g(x)}{1-\sigma}.$$

# An Example

Consider the function $f(x) = \sqrt{x}$, $x \in [0, 1]$.

A fractel is $\mathfrak{w}_1(x, y) = (\frac{1}{2}x,\ \frac{1}{\sqrt{2}}y)$ with domain $[0, 1] \times \mathbb{R}$.

$\mathfrak{w}_1$ recovers the function $f$ for $x \in [0, \frac{1}{2}]$.

We now need to recover $f$ for $x \in (\frac{1}{2}, 1]$.

For this we introduce two fractels defined over the domain $[\frac{1}{2}, 1] \times \mathbb{R}$.

First. choose the components $u_i$ as $u_i(x) = \frac{1}{2}(x + \tau_i)$ where $\tau_i = \frac{1}{2}(i - 1)$, $i = 2, 3$.

Set

$$g_i(x) = \sqrt{x} - \alpha_i(x - \tau_i)^{\theta_i},$$

where $\theta_i > 0$ is a free parameter.

This yields

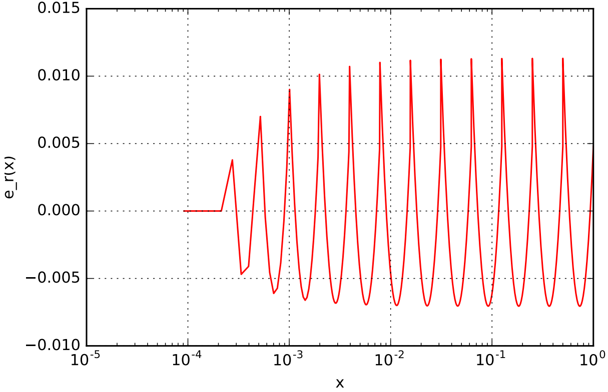
$$G_i(x) = \frac{\sqrt{\frac{1}{2}(x+\tau_i)} - \sigma_i\sqrt{x}}{1-\sigma_i}.$$

and therefore the fractels $\mathfrak{w}_2$ and $\mathfrak{w}_3$ (and a local IFS):

$$\begin{aligned}
\mathfrak{w}_1(x,y) &= (\tfrac{1}{2}x, \tfrac{y}{\sqrt{2}}), && x \in [0,1],\\
\mathfrak{w}_2(x,y) &= (\tfrac{1}{4}(2x+1), \sigma_2 y + \tfrac{1}{2}\sqrt{2x+1} - \sigma_2\sqrt{x}), && x \in [\tfrac{1}{2},1],\\
\mathfrak{w}_3(x,y) &= (\tfrac{1}{2}(x+1), \sigma_3 y + \tfrac{1}{\sqrt{2}}\sqrt{x+1} - \sigma_3\sqrt{x}), && x \in [\tfrac{1}{2},1].
\end{aligned}$$

Using approximation by the midpoint rule and setting $\sigma_i = \frac{1}{2}$:

$$\begin{aligned}
\mathfrak{w}_1(x,y) &= (\tfrac{1}{2}x, \tfrac{y}{\sqrt{2}}), && x \in [0,1],\\
\mathfrak{w}_2(x,y) &= (\tfrac{1}{4}(2x+1), (y + \sqrt{\tfrac{5}{2}} - \tfrac{1}{2}\sqrt{\tfrac{3}{2}}), && x \in [\tfrac{1}{2},1],\\
\mathfrak{w}_3(x,y) &= (\tfrac{1}{2}(x+1), (y + \sqrt{5} - \tfrac{1}{2}\sqrt{\tfrac{3}{2}}), && x \in [\tfrac{1}{2},1].
\end{aligned}$$




The relative error of the fractal approximation of $f(x) = \sqrt{x}$ based on the midpoint formula.

# References


- M. F. Barnsley, M. Hegland, P. Massopust: *Numerics and Fractals*, Bull. Inst. Math., Academica Sinica (N.S.), Vol. 9 (3)(2014), (389-430)
- M. F. Barnsley, M. Hegland, P. Massopust: *Self-Referential Functions*, arXiv:1610.01369.
- P. Massopust: Interpolation and Approximation with Splines and Fractals, Oxford University Press, January 2010.
- P. Massopust: Fractal Functions, Fractal Surfaces, and Wavelets, Academic Press, 2nd ed., 2016.


THANK YOU!