Scattered Data Approximation on Submanifolds

(Combined) Ambient Solutions

Lars-B. Maier

Technische Universität Darmstadt

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Part I: Sparse Scattered Data

Part II: Regular well-sampled Scattered Data

Part III: Irregular Scattered Data

PART I: SPARSE SCATTERED DATA

PROBLEM STATEMENT

Setting:

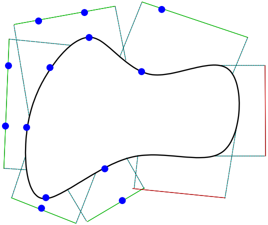
- $\mathbb{M} \subseteq \mathbb{R}^d$ an embedded submanifold with $q = \dim \mathbb{M} < 4$
  - ◇ closed ◇ compact ◇ without boundary
- $\Xi \subseteq \mathbb{M}$ a set of **sparse** sites scattered over $\mathbb{M}$
- Υ function values to Ξ

Task: Determine a »reasonable« function $f : \mathbb{M} \to \mathbb{R}$ such that

$$f(\xi) = y_\xi \text{ for all } \xi \in \Xi \text{ and corresponding } y_\xi \in \Upsilon$$

Common Approaches: Charts and Blending

- Use charts and define function spaces there to solve problem locally.
- Blend local solutions to obtain global solution.



Problems:

- There might be charts without any sites
- Blending tends to produce undesireable »gluing breaks«

Common Approaches: Intrinsic Functions

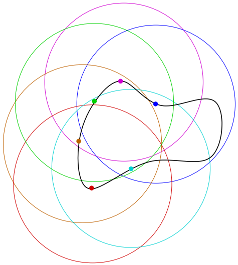
- Determine purely intrinsic function spaces $\mathcal{F}_{\mathbb{M}}$ like spherical harmonics on $\mathbb{S}^q$.
- Use these for a solution

Problems:

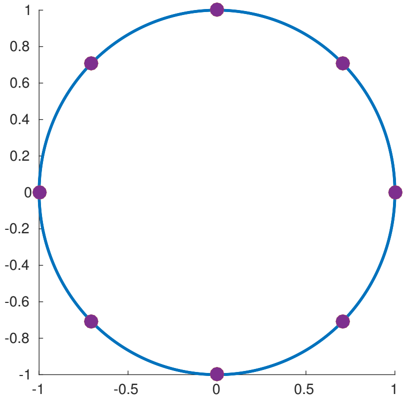
- These spaces are $\mathbb{M}$-specific and for arbitrary $\mathbb{M}$ hard to determine.
- Also, they are often costly to evaluate.

Common Approaches: Extrinsic Direct Interpolation

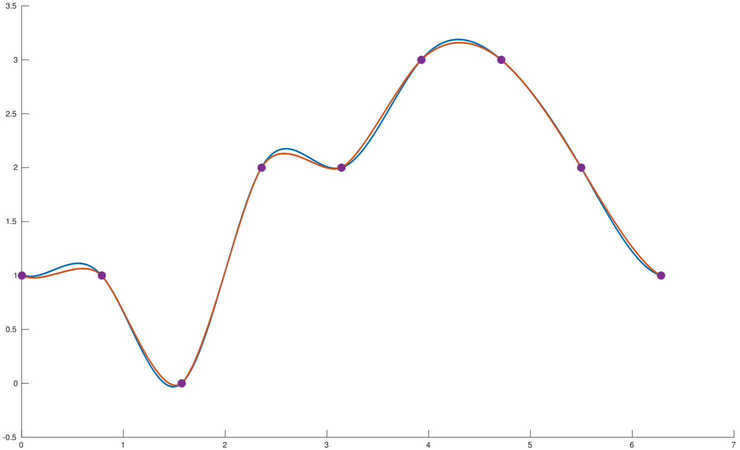
- Use some function space in a suitable ambient neighbourhood of $\mathbb{M}$: RBF, Splines...
- Solve standard interpolation problem in neighbourhood, ignore the geometry of $\mathbb{M}$.
- Restrict the solution to $\mathbb{M}$.



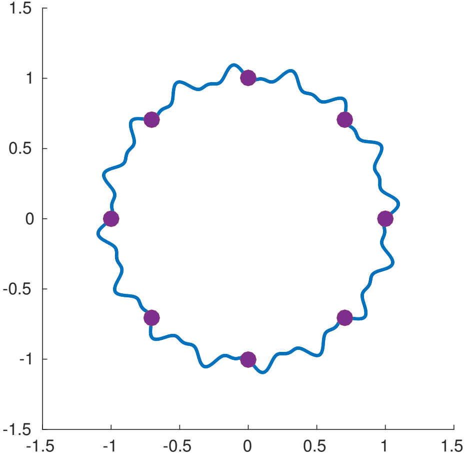
Directly applicable and works always — but how good?



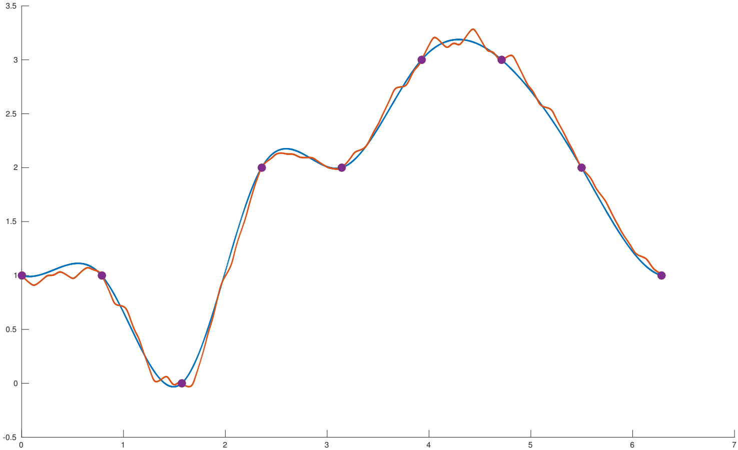
Data sites with function values: 1, 1, 0, 2, 2, 3, 3, 2



Periodic cubic spline | Restricted thin-plate spline | Interpolation sites



Data sites with function values: 1, 1, 0, 2, 2, 3, 3, 2



Periodic cubic spline interpolant

Restricted thin-plate spline interpolant

Common Approaches: Extrinsic Interpolation

- Use some function space in a suitable ambient neighbourhood of $\mathbb{M}$: RBF, Splines...
- Solve standard interpolation problem in neighbourhood, ignore the geometry of $\mathbb{M}$.
- Restrict the solution to $\mathbb{M}$.

Problems:

- Difficulties occur for sparse data.
- Suffers extremely from *intricate* geometries.

NEW APPROACH (M., REIF)

- Use some function space in a suitable ambient neighbourhood of $\mathbb{M}$: RBF, Splines...
- Transfer intrinsic properties into extrinsic (ambient) properties — approximately.
- Solve approximately intrinsic problem with extrinsic methods.

Pro's:

- Extrinsic function spaces are well understood.
- Extrinsic function space are applicable to any submanifold.
- Easily understood and implemented even for non-mathematicians.

SOLUTION IDEA: BACKGROUND

DEFINITION (Tangent Derivative)

Let $f : \mathbb{M} \to \mathbb{R}$ be a sufficiently (weakly) differentiable function and $\tilde{f}$ an extension into $U(\mathbb{M})$. The **Tangent Derivative Operator $\mathbf{d}_{\mathbb{M}}$** is defined as

$$\mathbf{d}_{\mathbb{M}} f := \mathbf{d}\tilde{f} - \pi_N(\mathbf{d}\tilde{f})$$

and independent of the choice of $\tilde{f}$.

SOLUTION IDEA: BACKGROUND

THEOREM (e.g. [Dziuk/Elliot, 2013])

Let $f \in C^2(\mathbb{M})$, and $\bar{f}$ an extension that is **constant in normal directions** of $\mathbb{M}$.

Then the 1^{st} and 2^{nd} **tangent** derivatives of f coincide with the **euclidean** 1^{st} and 2^{nd} derivatives of $\bar{f}$ on the tangent space.

SOLUTION IDEA: BACKGROUND

THEOREM (M. 2015)

Let $f \in C^2(\mathbb{M})$, and $\tilde{f}$ an arbitrary extension. Then the deviations of the 1^{st} and 2^{nd} **tangent** derivatives of f from the **euclidean** 1^{st} and 2^{nd} derivatives of $\tilde{f}$ on the tangent space are **Lipschitz functions** of the first normal derivatives.

SOLUTION IDEA: BACKGROUND

What does that mean?

- Euclidean derivatives of $\tilde{f}$ give **approximate** access to tangent derivatives of f
- Standard methods can be used to handle intrinsic problems **approximately**
- Intrinsic functionals are easily **approximated** by standard functionals

NEW APPROACH (M., REIF): SOLUTION IDEA

Let $U(\mathbb{M})$ be a suitable neighbourhood of $\mathbb{M}$, such that any point there has a unique closest point on $\mathbb{M}$.

Consider a suitable function space $\mathcal{F}(U(\mathbb{M}))$ on $U(\mathbb{M})$.

Minimize squared 2^{nd} derivative in tangent directions over $\mathbb{M}$

such that:

Interpolation in Ξ holds

Normal derivatives $\to 0$

Optimization Functional: Tensor-Product B-Splines

Minimize for grid width $h > 0$ and $\tau_1, ...\tau_q$ an ONB of $T_p(\mathbb{M})$, $\nu_1, ..., \nu_{d-q}$ ONB of $N_p(\mathbb{M})$

$$\int\limits_{\mathbb{M}} \sum_{i,j=1}^{q} \left| \frac{\partial^2}{\partial\tau_i \partial\tau_j} s_h \right|^2 + Ch^{-\sigma} \int\limits_{\mathbb{M}} \sum_{i=1}^{d-q} \left| \frac{\partial}{\partial\nu_i} s_h \right|^2 + \int\limits_{U_h(\mathbb{M})} w(h^{-\sigma}, N_x) \sum_{i=1}^{d-q} \left| \frac{\partial}{\partial\nu_i(clp(\cdot))} s_h \right|^2$$

for suitable penalty parameter σ and radial weight function w, under side conditions

$$s_h(\xi) = y_\xi \quad \forall \xi \in \Xi$$

THEORY: SOLVABILITY AND CONVERGENCE

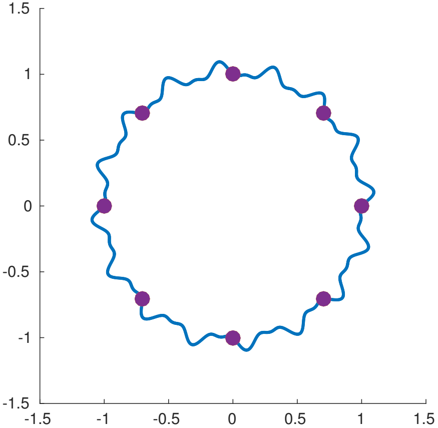
THEOREM (M. 2015):

1. For TP-B-Splines and sufficiently small h, the above problem is uniquely solvable.

2. For $h \to 0$ the squared second derivative energy converges to the unique optimal energy in $\mathcal{H}^2(\mathbb{M})$.

3. Restrictions of optimal splines $s_h|_{\mathbb{M}}$ approach unique optimum $f^* \in \mathcal{H}^2(\mathbb{M})$:

$$\|s_h - f^*\|_{\mathcal{H}^2(\mathbb{M})} \to 0 \quad \text{as} \quad h \to 0.$$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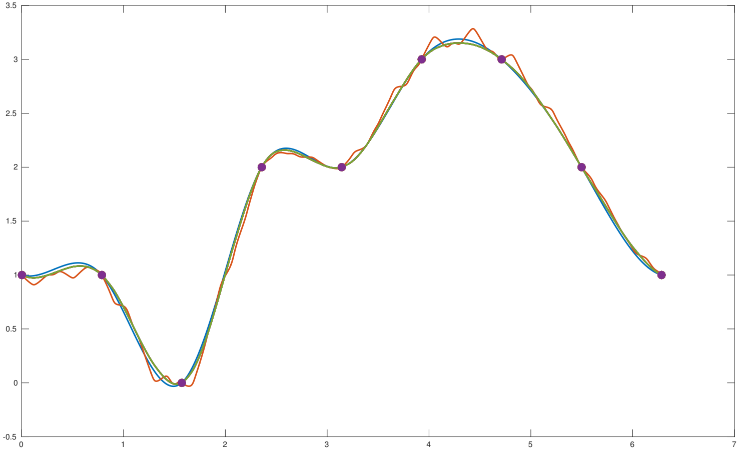
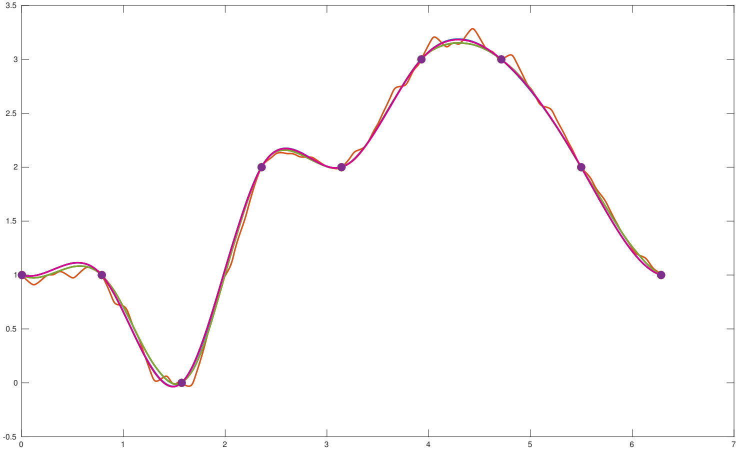


Arc length periodic cubic $\mathbb{R}^1$-spline

TPS in $\mathbb{R}^2$-plane restricted to curve

Opt. for $h = 0.02$ ($err_{RMS} \approx 10^{-2}$)

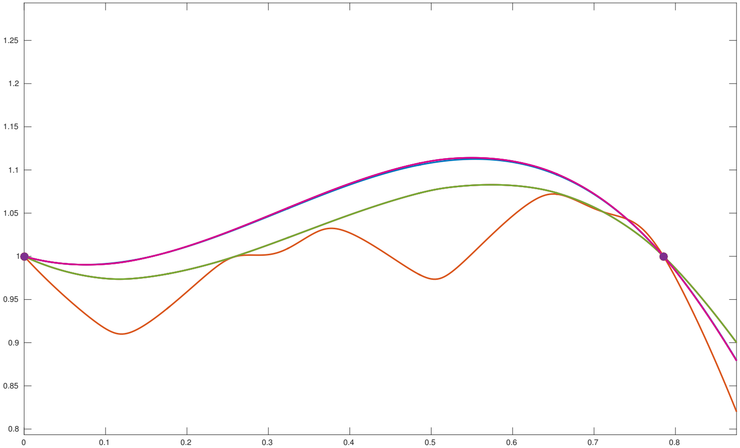


Arc-length periodic cubic $\mathbb{R}^1$-spline

TPS in $\mathbb{R}^2$-plane restricted to curve

Opt. for $h = 0.02$ ($err_{RMS} \approx 10^{-2}$)

Opt. for $h = 0.01$ ($err_{RMS} \approx 10^{-3}$)



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TPS in $\mathbb{R}^2$-plane restricted to curve

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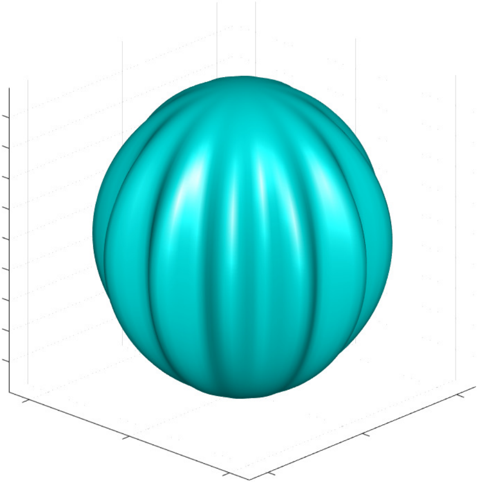
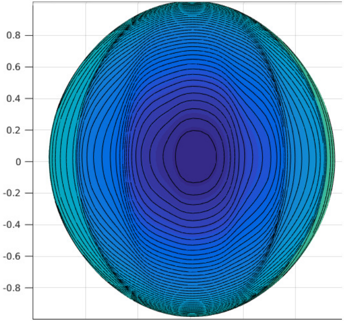
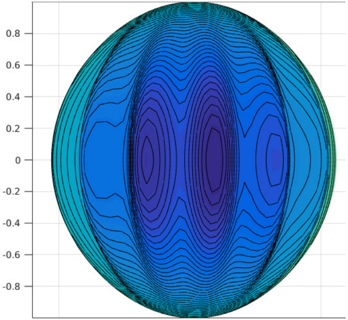


Sites with corresponding interpolation values:

$$s(\pm e_1) = \pm 1, \; s(\pm e_2) = 0, \; s(\pm e_3) = 0$$

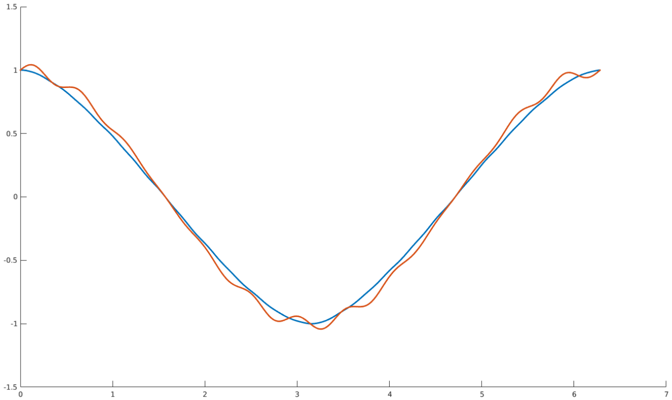




Reprojections onto $\mathbb{S}^2$:

Our optimum for $h = 0.08$

Standard ambient interpolation



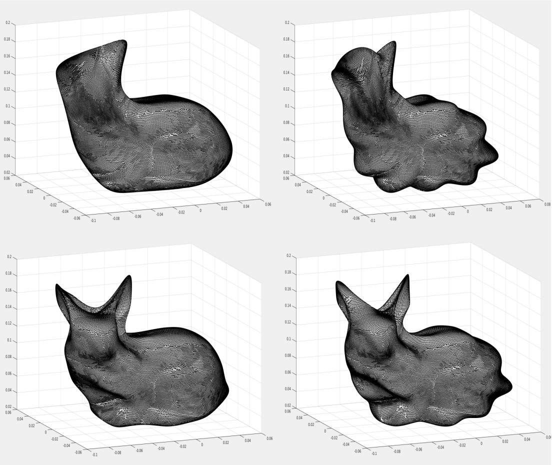
Equator evaluation:

Our optimum for $h = 0.08$

Standard ambient interpolation

Our optimum for $h = 0.0625$

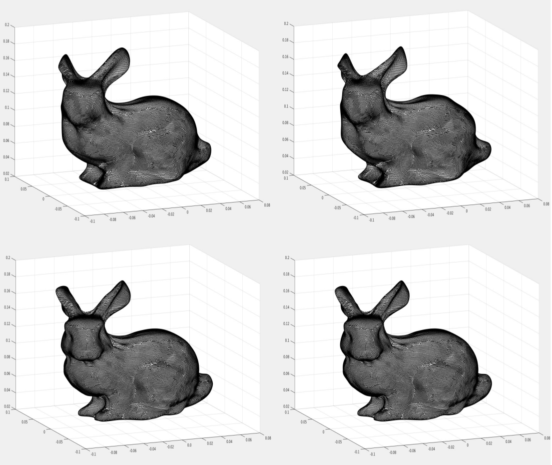
Standard ambient interpolation



Stanford bunny as an image of the pumpkin, using ≈ 80 (upper row) and ≈ 160 sites (lower row)

Our optimum for $h = 0.0625$

Standard ambient interpolation



Stanford bunny as an image of the pumpkin, using ≈ 350 (upper row) and ≈ 700 sites (lower row)

Further remarks

► With fill distance $h_{\Xi,\mathbb{M}} = \max_{p\in\mathbb{M}} \min_{\xi\in\Xi} \|p - \xi\|_2$ decreasing, the convergence is about that of thin-plate splines

But: One will have to increase the number of DOF correspondingly to meet

- More interpolation conditions
- Sufficient decrease of derivatives in normal directions

► One can use RBF or other approaches instead of TPBS — but their locality makes them a very good choice.

► Application for $\mathbb{M}$ **with** boundary is possible, but requires reasonable boundary conditions (ongoing research).

► Similar approaches work for **Smoothing** and **Elliptical PDEs** on $\mathbb{M}$.

PART II: WELL SAMPLED REGULAR SCATTERED DATA

PROBLEM STATEMENT

Setting:

- $\mathbb{M} \subseteq \mathbb{R}^d$ an embedded submanifold

  ◇ closed ◇ compact ◇ without boundary

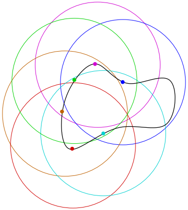
- $\Xi \subseteq \mathbb{M}$ a set of quasi-uniform sites scattered over $\mathbb{M}$, $|\Xi|$ rather large
- Υ function values to Ξ

Task: Determine a function $f : \mathbb{M} \to \mathbb{R}$ such that

- $f(\xi) \approx y_\xi$ for all $\xi \in \Xi$ and corresponding $y_\xi \in \Upsilon$
- y_ξ sampled from function g: $f \approx g$

Common Approaches: Extrinsic Direct Interpolation

Pleasant conv. rates for RBF: Only loss of $O(h_{\Xi}^{\mathrm{codim}(\mathbb{M})/p})$ [Fuselier/Wright '12]

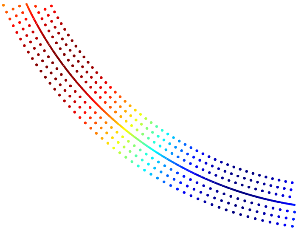


Problems:

- Polynomials/Splines/MLS presumably unstable
- RBF still face a loss of order — although it is moderate

Recent Approach (Reif et al.): Constant extension into Ambient Space

- Use some function space in ambient neighbourhood of $\mathbb{M}$: RBF, MLS, Splines...
- Extend function constantly into ambient space and solve problem there.

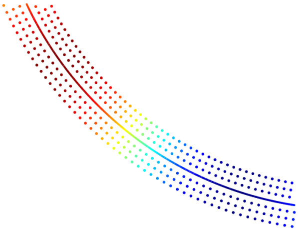


Extension along normals:

- Scattered sites are extended along the normals of $\mathbb{M}$

RECENT APPROACH (REIF ET AL.): CONSTANT EXTENSION INTO AMBIENT SPACE

- Use some function space in ambient neighbourhood of $\mathbb{M}$: RBF, MLS, Splines...
- Extend function constantly into ambient space and solve problem there.



Extension along some suitable flow:

- Scattered sites are extended along e.g. gradient flow of implicit function

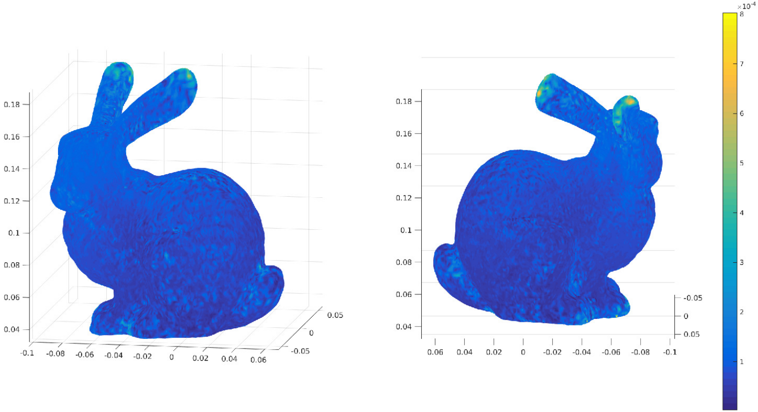
Recent Approach:
Constant extension into Ambient Space

- Use some function space in ambient neighbourhood of $\mathbb{M}$: RBF, MLS, Splines...
- Extend function constantly into ambient space and solve problem there.

THEOREM (M., 2016)

- Let $f \in W_p^m(\mathbb{M})$ extended constantly along normals.
- Use quasi-interpolation with **TPB-Splines of order** m.

$\Longrightarrow$ Optimal convergence order $m - k$ in $W_p^k(\mathbb{M})$ for $k < m - 1$ **regardless** codim($\mathbb{M}$).



Stanford Bunny Approximation

Locally weighted Least Squares and TPBS cells of edge length 0.025,

applied to about 30.000 data sites

Further remarks

- Comparable convergence results hold for other approximation methods and/or extension flows at least if $k < m - \mathrm{codim}(\mathbb{M})$.
- Convergence requires the data to increase as the degrees of freedom
- Application for $\mathbb{M}$ **with** boundary is possible, but requires reasonable boundary handling, e.g. coupling of splines.

PART III: IRREGULAR SCATTERED DATA

PROBLEM STATEMENT

Setting:

- $\mathbb{M} \subseteq \mathbb{R}^d$ an embedded submanifold with $q = \dim \mathbb{M} < 4$

  ◇ closed ◇ compact ◇ without boundary

- $\Xi \subseteq \mathbb{M}$ a set of sites scattered over $\mathbb{M}$
- Ξ contains clusters or considerable holes / gaps
- Υ function values to Ξ

Task: Determine a function $f : \mathbb{M} \to \mathbb{R}$ such that

- $f(\xi) \approx y_\xi$ for all $\xi \in \Xi$ and corresponding $y_\xi \in \Upsilon$
- y_ξ sampled from function g: $f \approx g$

Approach (M., Reif): Multi-Level/Hierachy Approximation

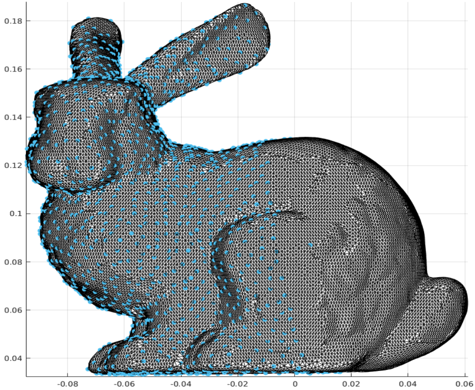
- Thin data sites until suitably sparse
- Apply »Sparse Data« method on thinned sites
- Approximate error in well-sampled regions using **all** sites

- Hierarchical approach might be used in second level

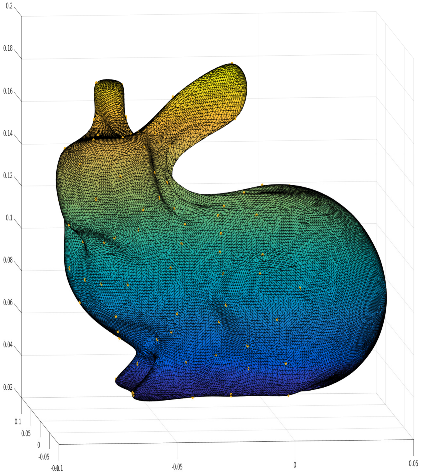
Advantage:

- Use of the same function space in first and following levels:

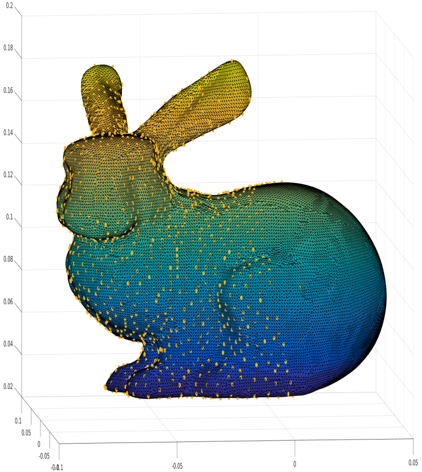
  $\Rightarrow$ single explicit expression for solution



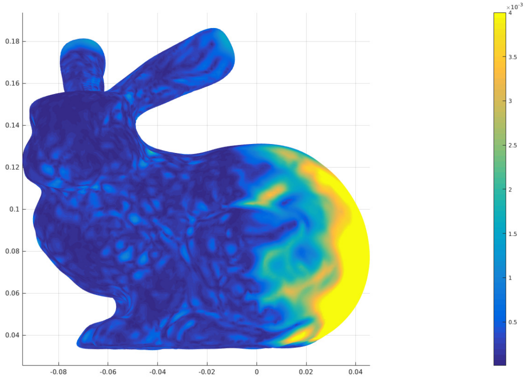
Hole in the data sites ($\approx$ 2.500): Rear side is lost



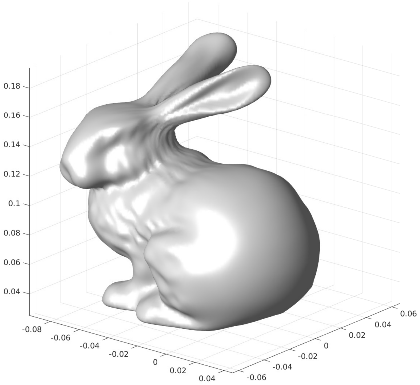
Result of first level with data sites (≈ 160) over »pumpkin«



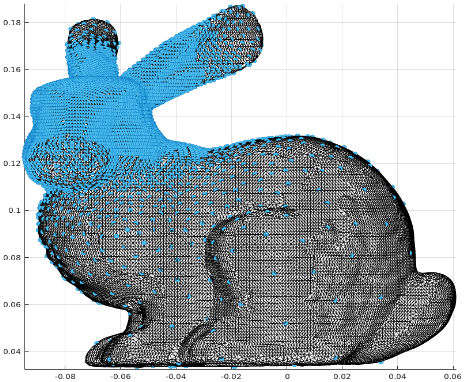
Result of first and second level with data sites



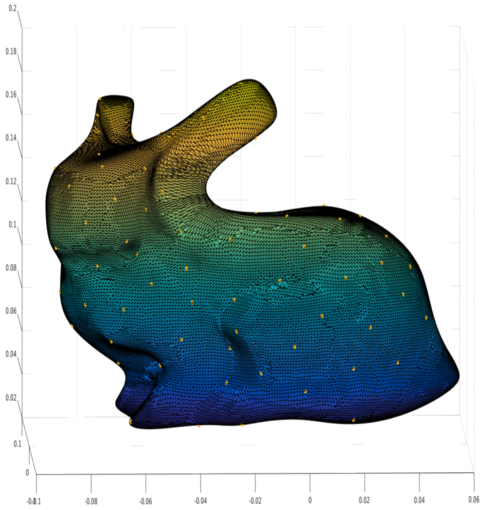
Error plot on resulting model, error cut at $4 \cdot 10^{-3}$ for comparability reasons



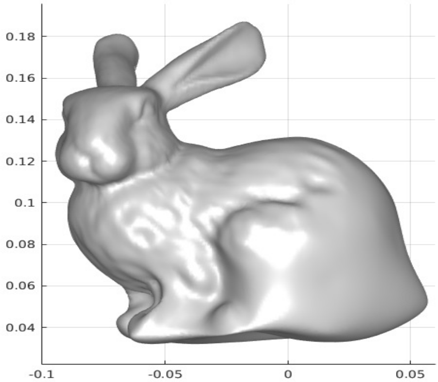
Resulting model after both levels



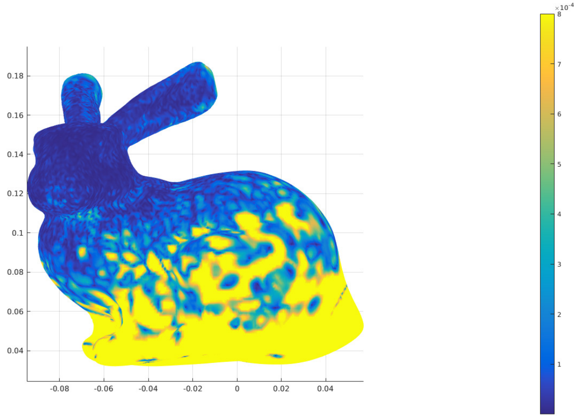
Data sites ($\approx$ 7.000) clustering in one area around the head



Result and sites (≈ 130) of first step ($h = 0.1$)



Resulting model after 3 hierarchy levels $(0.1, 0.05, 0.025)$ in second step



Error plot on resulting model, error cut at $8 \cdot 10^{-4}$ for comparability reasons

SUMMARY

New approaches to all kinds of scattered data approximation and/or interpolation on certain submanifolds:

- Overcome common difficulties
- Apply well-known concepts in novel setting
- Easy to implement
- Produce pleasant results

THANK YOU!