Contractivity and convergence of refinement schemes in Riemannian geometry

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DOCTORAL PROGRAM
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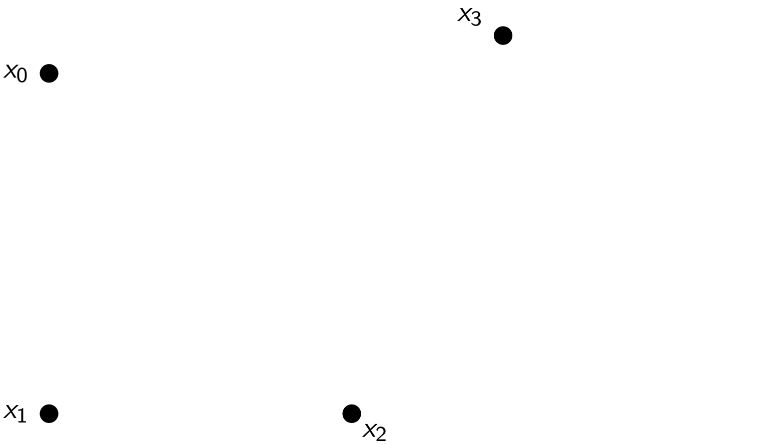
What is a subdivision scheme?



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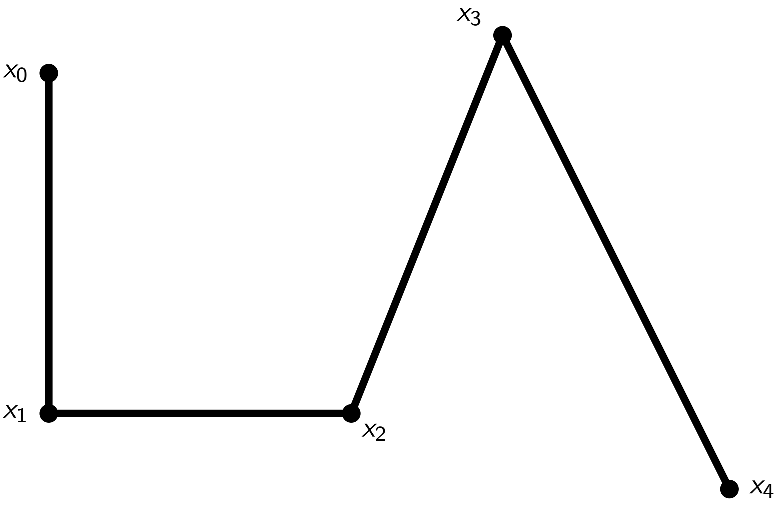

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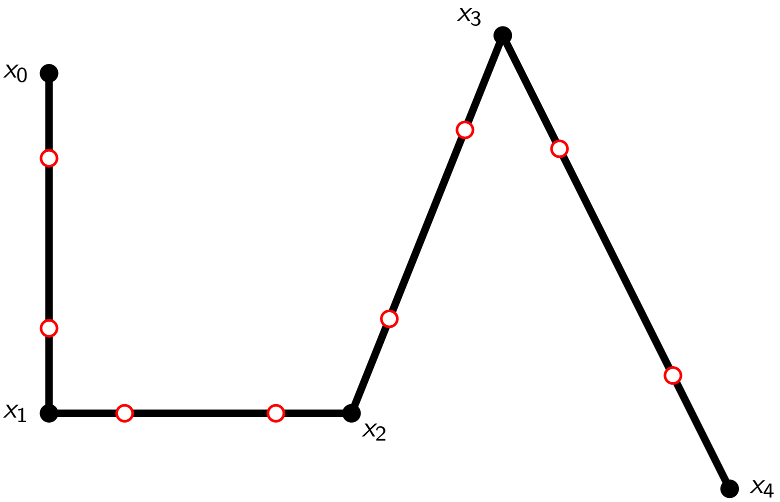

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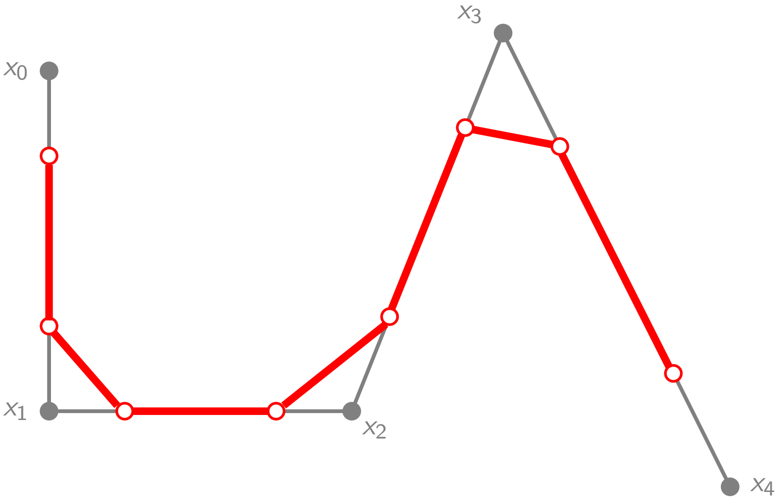

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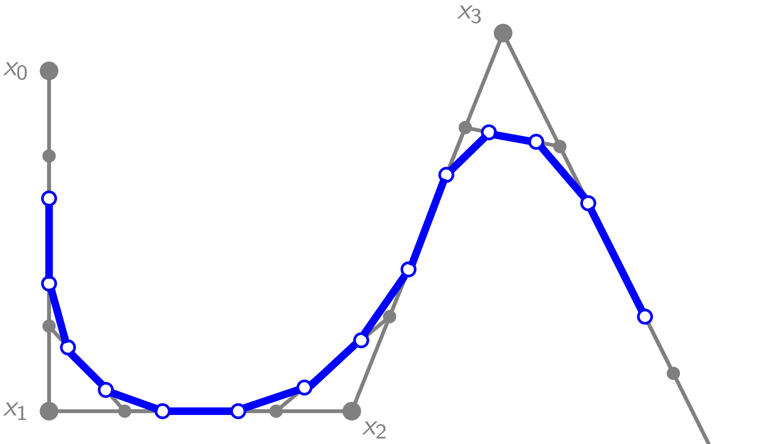
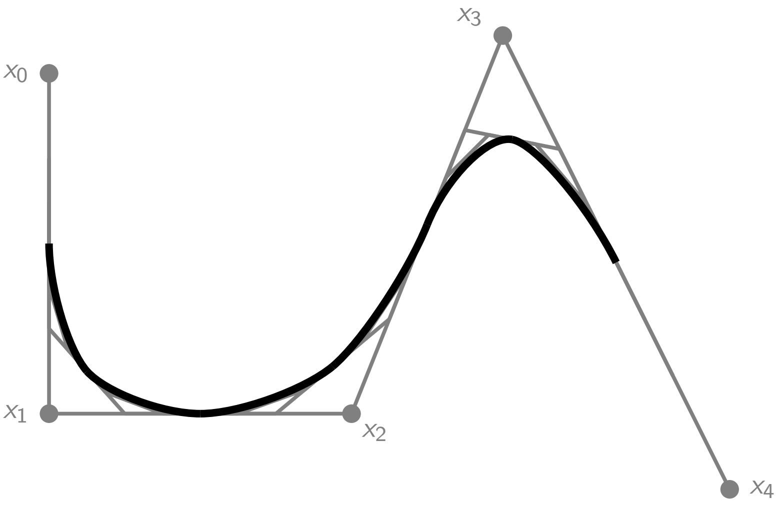
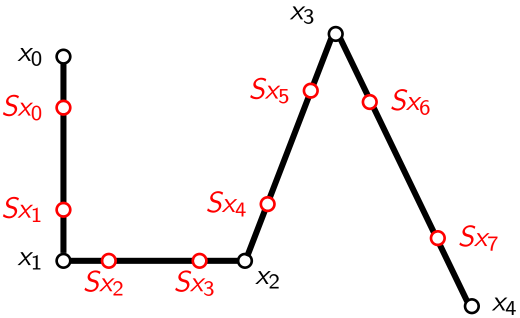



x_0
x_1
x_2
x_3
x_4

Chaikin's algorithm



Subdivision scheme S:

$$(Sx)_i = \sum_{j\in\mathbb{Z}} a_{i-2j} x_j$$

with $\sum_{j\in\mathbb{Z}} a_{i-2j} = 1$ for all i

Here: $a_{-2} = \frac{1}{4} \quad a_{-1} = \frac{3}{4} \quad a_0 = \frac{3}{4} \quad a_1 = \frac{1}{4}$

Subdivision scheme S

Linear case:

$$x = (x_i)_{i\in\mathbb{Z}} \xrightarrow[\text{step}]{\text{refinement}} Sx = (Sx_i)_{i\in\mathbb{Z}}$$

where $Sx_i = \sum_{j\in\mathbb{Z}} a_{i-2j} x_j$

with $\sum_{j\in\mathbb{Z}} a_{i-2j} = 1$ and finitely supported mask $(a_\ell)_{\ell\in\mathbb{Z}}$

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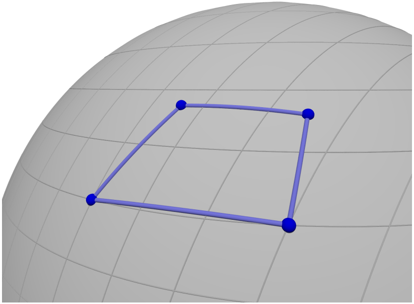
Nonlinear case: $\rightarrow$ geodesics



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where $Sx_i = \sum_{j\in\mathbb{Z}} a_{i-2j} x_j \Leftrightarrow$ Sx_i minimizer of $f(x) = \sum_{j\in\mathbb{Z}} a_{i-2j}\|x_j - x\|^2$

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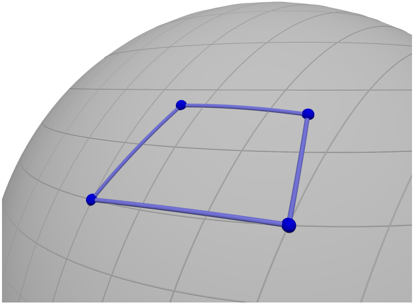

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replacing

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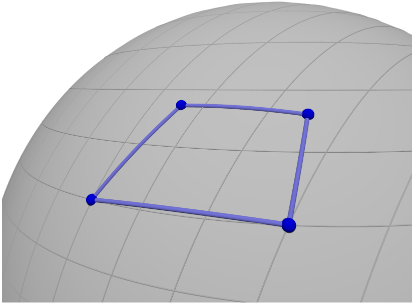

$(\tilde{S}x)_i$ minimizer of

$$f(x) = \sum_{j\in\mathbb{Z}} a_{i-2j}\, \mathrm{dist}(x_j, x)^2$$

where $\mathrm{dist}(\cdot,\cdot)$ denotes the Riemannian distance

$\rightarrow$ called Riemannian center of mass

Binary case

What happens if only two coefficients are nonzero?

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Linear case:

$\rightarrow$ Minimize $f(x) = a_1\|x_1 - x\|^2 + a_2\|x_2 - x\|^2$ with $a_1 + a_2 = 1$

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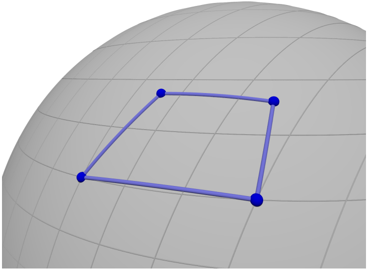

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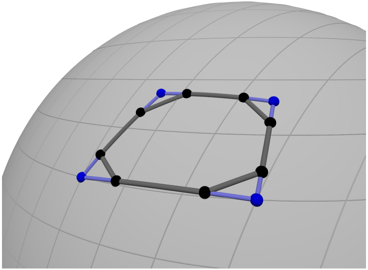

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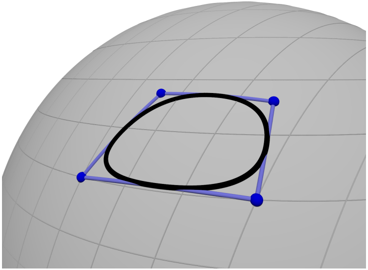

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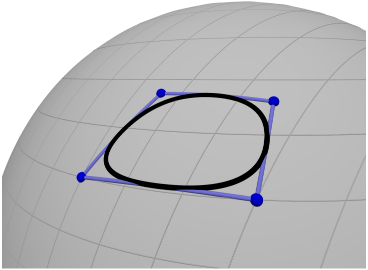
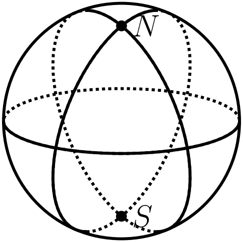


Not well-defined:



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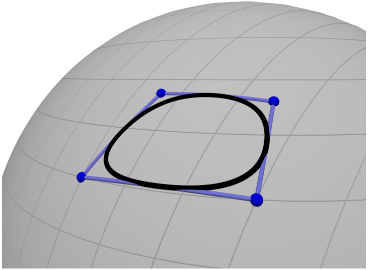

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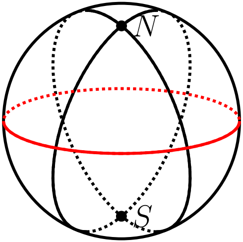

Convergence results on manifolds

Different ways of transferring linear schemes to manifold-valued data.

Convergence for general schemes but only 'dense enough' input data:

- Proximity conditions [Wallner, Dyn, Xie, Yu, Grohs, Weinmann, 2005ff]

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Convergence for special subdivision schemes and all input data:

- Interpolatory subdivision schemes [Wallner, 2014]
- Schemes with nonnegative mask coefficients on Cartan-Hadamard spaces [Wallner, Nava Yazdani, Weinmann, Ebner 2011ff]

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1) Extension of linear scheme
to CH manifold
↓
Riemannian center of mass:
$(\tilde{S}x)_i$ minimizer of
$\sum_{j\in\mathbb{Z}} a_{i-2j}\,\mathrm{dist}(x_j, x)^2$
↓
well-defined

Convergence analysis on CH manifolds

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2) convergence analysis

↓

contractivity condition

On CH manifold: $(\tilde{S}x)_i$ minimizer of $f(x) = \sum_{j \in \mathbb{Z}} a_{i-2j} \operatorname{dist}(x_j, x)^2$

Riemannian center of mass

On CH manifold: $(\tilde{S}x)_i$ minimizer of $f(x) = \sum_{j\in\mathbb{Z}} a_{i-2j} \operatorname{dist}(x_j, x)^2$

Question: Existence and uniqueness of minimum of

$$f_\alpha(x) := \sum_{j=0}^{m} \alpha_j \operatorname{dist}(x_j, x)^2$$

with $\sum_{j=0}^{m} \alpha_j = 1$ on CH manifolds?

Riemannian center of mass

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  $\rightarrow$ Yes, for nonnegative mask coefficients on CH manifolds
- Uniqueness corresponds to convexity of f_α, i.e.

$$\frac{d^2}{ds^2} f_\alpha(\gamma(s)) \geq 0$$

for any geodesic γ.

Question: What happens if $\sum_{j=0}^{m} \alpha_j = 1$ and $\alpha_j \in \mathbb{R}$?

$$f_\alpha(x) = \sum_{j=0}^{m} \alpha_j \operatorname{dist}(x_j, x)^2$$
$$= -\int_0^{\alpha_-} \operatorname{dist}(x_{\tau(t)}, x)^2 dt + \int_{\alpha_-}^{\alpha_+ + \alpha_-} \operatorname{dist}(x_{\tau(t)}, x)^2 dt$$

with $\alpha_+ := \sum_j \alpha_j$ for all $\alpha_j \geq 0$ and $\alpha_- := \sum_j |\alpha_j|$ for all $\alpha_j < 0$.

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- Existence of minimum of f_α shown by H. Hardering, 2015.
- Uniqueness follows by convexity using $\alpha_+ - \alpha_- = 1$:

$$\frac{1}{2}\frac{d^2}{ds^2} f_\alpha(\gamma(s)) = -\int_0^{\alpha_-} \langle J(1), J'(1)\rangle \, dt + \int_{\alpha_-}^{\alpha_+ + \alpha_-} \langle J(1), J'(1)\rangle \, dt \geq 0.$$

Convergence analysis on CH manifolds

1) Extension of linear scheme to CH manifold

↓

Riemannian center of mass:

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contractivity condition

Contractivity condition:

$$\operatorname{dist}\left((\tilde{S}x)_{2i}, (\tilde{S}x)_{2i+1}\right) \leq \max_k \left(\operatorname{dist}(x_k, x_{k+1})\right) \cdot \left[\int_I |\nu_2(t) - \nu_1(t)|\, dt\right]$$

where ν_1 and ν_2 only depend on the mask coefficients.

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Convergence result:

$$\int_I |\nu_2(t) - \nu_1(t)| dt < 1 \implies \tilde{S} \text{ converges to continuous limit function for all input data.}$$

Example

Four-point scheme given as

$$(Sx)_{2i} = x_i$$
$$(Sx)_{2i+1} = -\omega x_{i-1} + \left(\frac{1}{2} + \omega\right) x_i + \left(\frac{1}{2} + \omega\right) x_{i+1} - \omega x_{i+2}$$

with a parameter ω.

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On CH manifold:

- $\omega \in (-\frac{1}{2}, 0] \Rightarrow \tilde{S}$ converges [Ebner, 2014]

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- We obtain $\int_I |\nu_2(t) - \nu_1(t)|\ dt = \frac{1}{2} + 2|\omega|$ and follow that

$$-\frac{1}{4} < \omega < \frac{1}{4} \implies \tilde{S} \text{ converges.}$$

Thank you for your attention!

References

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