

The classification of anisotropic Besov spaces

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Outline

① Anisotropic Besov spaces

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- 1 Anisotropic Besov spaces
- 2 Decomposition spaces: Definition and properties

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$$\forall \xi \in \mathbb{R}^d \setminus \{0\} : \sum_{j \in \mathbb{Z}} |\widehat{\psi}((A^T)^{-j}\xi)|^2 = 1 .$$

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$$\forall \xi \in \mathbb{R}^d : |\varphi(\xi)|^2 + \sum_{j \in \mathbb{N}} |\widehat{\psi}((A^T)^{-j}\xi)|^2 = 1 .$$

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Note: A -wavelets and scaling functions always exist.

Defining homogeneous anisotropic Besov spaces

Definition (M. Bownik, 2005)

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- (a) For $j \in \mathbb{Z}$, define $\psi_j(x) = |\det(A)|^j \psi(A^j x)$.
Given $f \in \mathcal{S}'(\mathbb{R}^d)$, let

$$\|f\|_{\dot{B}_{p,q}^\alpha(A)} = \left\| \left(|\det(A)|^{j\alpha} \|f * \psi_j\|_p \right)_{j \in \mathbb{Z}} \right\|_{\ell^q} \quad (1)$$

denote the **homogeneous Besov quasi-norm**. The **homogeneous Besov space** $\dot{B}_{p,q}^\alpha(A)$ associated to A consists of all f with $\|f\|_{\dot{B}_{p,q}^\alpha(A)} < \infty$.

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Given $f \in \mathcal{S}'(\mathbb{R}^d)$, define

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The **inhomogeneous anisotropic Besov space** $B_{p,q}^\alpha(A)$ associated to A consists of all tempered distributions f with $\|f\|_{B_{p,q}^\alpha(A)} < \infty$.

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Main questions for this talk

Given two expansive matrices A, B , we call them **homogeneously equivalent** ($A \sim_h B$) whenever $\dot{B}_{p,q}^\alpha(A) = \dot{B}_{p,q}^\alpha(B)$ holds for all $0 < p, q \leq \infty, \alpha \in \mathbb{R}$.

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- Can one decide whether $A \sim_h B$ or $A \sim_i B$ holds?
- Do \sim_h and \sim_i differ?

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Anisotropic coverings of frequency space

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Informal description: Cover the frequencies by a family of open relatively compact sets.

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Decomposition spaces (Feichtinger/Gröbner)

Informal description: Cover the frequencies by a family of open relatively compact sets. Decompose functions using a subordinate partition of unity. Introduce a norm by locally taking L^p -norms, and then globally combine using weighted ℓ^q norm.

Admissible coverings

Definition

Let $\mathcal{O} \subset \mathbb{R}^d$ be open, and $\mathcal{Q} = (Q_i)_{i \in I}$ a family of open, subsets $Q_i \subset \mathcal{O}$ with compact closure in \mathcal{O} .

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(b) \mathcal{Q} is called an **almost structured admissible covering (a.s.a.c.)** if it is an admissible covering, and there exist $(Q'_i)_{i \in I}$, nonempty open bounded sets and $T_i \in GL(d, \mathbb{R})$ and $b_i \in \mathbb{R}^d$ with:

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(i) For all $i \in I$: $\overline{T_i Q'_i + b_i} \subset Q_i$.

(ii) The quantity $\sup_{i,j: Q_i \cap Q_j \neq \emptyset} \|T_i^{-1} T_j\|$ is finite.

(iii) The set $\{Q'_i : i \in I\}$ is finite.

(iv) The family $(T_i Q'_i + b_i)_{i \in I}$ is an admissible covering.

The tuple $((T_i)_{i \in I}, (b_i)_{i \in I}, (Q'_i)_{i \in I})$ are called **standardization** of \mathcal{Q} .

Partitions of unity

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- (i) For all $i \in I$: $\varphi_i \in C_c^\infty(\mathcal{O})$.
- (ii) For all $i \in I$: $\varphi_i \equiv 0$ on $\mathbb{R}^d \setminus Q_i$.
- (iii) $\sum_{i \in I} \varphi_i \equiv 1$ on \mathcal{O} .
- (iv) $\sup_{i \in I} |\det(T_i)|^{\frac{1}{t}-1} \|\mathcal{F}^{-1} \varphi_i\|_{L^p} < \infty$. Here $t = \min(p, 1)$.

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Lemma (Feichtinger, Voigtlaender)

BAPU's exist.

Weights associated to coverings

Definition

Let $\mathcal{Q} = (Q_i)_{i \in I}$ denote an admissible covering, and let $v : I \rightarrow \mathbb{R}^+$ denote a weight. The weight is called **\mathcal{Q} -moderate** if

$$\sup_{i, j \in I : Q_i \cap Q_j \neq \emptyset} \frac{v(i)}{v(j)} < \infty .$$

Given $0 < q \leq \infty$, let $\ell_v^q(I) = \{c = (c_i)_{i \in I} \in \mathbb{C}^I : (c_i v(i))_{i \in I} \in \ell^q(I)\}$, with the obvious (quasi-)norm.

Definition of decomposition spaces

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$$\|u\|_{\mathcal{D}(\mathcal{Q}, L^p, \ell_v^q)} = \left\| \left(\|\mathcal{F}^{-1}(\varphi_i \cdot u)\|_{L^p} \right)_{i \in I} \right\|_{\ell_v^q}. \quad (3)$$

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The **decomposition space** $\mathcal{D}(\mathcal{Q}, L^p, \ell_v^q)$ is the space of all $u \in \mathcal{D}'(\mathcal{O})$ for which this (quasi-)norm is finite.

Properties of decomposition spaces

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Theorem (Feichtinger/Gröbner)

$\mathcal{D}(\mathcal{Q}, \cdot, L^p, \ell_v^q)$ is a quasi-Banach space, that is *independent* of the choice of BAPU.

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- Large variety of admissible coverings allows diverse ways of measuring the decay. Describes homogeneous and inhomogeneous isotropic Besov spaces, α -modulation spaces, shearlet and curvelet approximation spaces, shearlet coorbit spaces (see upcoming talk by René Koch), and **anisotropic Besov spaces**.
- Decomposition spaces provide a unified framework for many **embedding results**, either between different decomposition spaces, or of decomposition spaces into well-known smoothness spaces such as Sobolev spaces (F. Voigtlaender).

Induced covering

Definition

Let A denote an expansive matrix. Let $Q \in \mathbb{R}^d \setminus \{0\}$ be open, and such that

$$\bigcup_{j \in \mathbb{Z}} A^j Q = \mathbb{R}^d \setminus \{0\} .$$

Then $\mathcal{Q}_A = (A^j Q)_{j \in \mathbb{Z}}$ is called the **homogeneous covering induced by A** .

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Intuition

$$Q_j = \widehat{\psi}_j^{-1}(\mathbb{C} \setminus \{0\}), \quad Q_j^i = \widehat{\psi}_j^{i-1}(\mathbb{C} \setminus \{0\})$$

and the wavelet systems are BAPUs.

Anisotropic Besov spaces as decomposition spaces

Theorem

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$$v_{\alpha,A} : \mathbb{Z} \rightarrow \mathbb{R}^+ , v_{\alpha,A}(j) = |\det(A)|^{j\alpha} .$$

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Denote by $\rho : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{D}'(\mathbb{R}^d)$ the restriction map. Then $\rho \circ \mathcal{F}$ is a topological isomorphism

$$\rho \circ \mathcal{F} : \dot{B}_{p,q}^\alpha(A) \rightarrow \mathcal{D}(\mathcal{Q}_A, L^p, \ell_{v_{\alpha,A}}^q) .$$

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Similarly, if \mathcal{Q}_A^i denote an inhomogeneous covering induced by A^T , then

$$\rho \circ \mathcal{F} : B_{p,q}^\alpha(A) \rightarrow \mathcal{D}(\mathcal{Q}_A^i, L^p, \ell_{v_{\alpha,A}}^q) .$$

is a topological isomorphism, as well. Here $v_{\alpha,A}$ denotes the restriction of the weight for the homogeneous setting to \mathbb{N}_0 .

Overview

- 1 Anisotropic Besov spaces
- 2 Decomposition spaces: Definition and properties
- 3 Rigidity theorem for decomposition spaces
- 4 Classifying anisotropic coverings

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Weights v on I and w on J are **equivalent** if

$$\sup_{i, j, Q_i \cap P_j \neq \emptyset} \frac{v(i)}{w(j)} + \frac{w(j)}{v(i)} < \infty .$$

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- (c) For all $(p, q) \in [1, \infty]^2$ and all pairs v_1, v_2 of equivalent weights,

$$\mathcal{D}(\mathcal{P}, L^p, \ell_{v_1}^q) = \mathcal{D}(\mathcal{Q}, L^p, \ell_{v_2}^q) .$$

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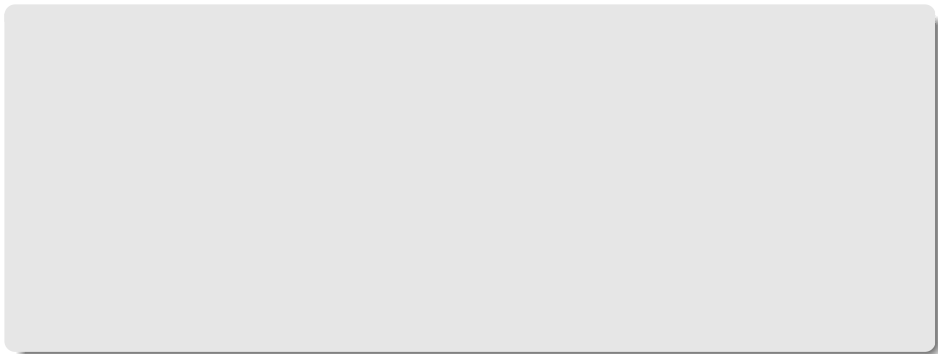
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Note: If v denotes the constant weight, then $\mathcal{D}(\mathcal{P}, L^2, \ell_v^2) = L^2(\mathbb{R}^d)$, for every a.s.a.c. covering \mathcal{P} .

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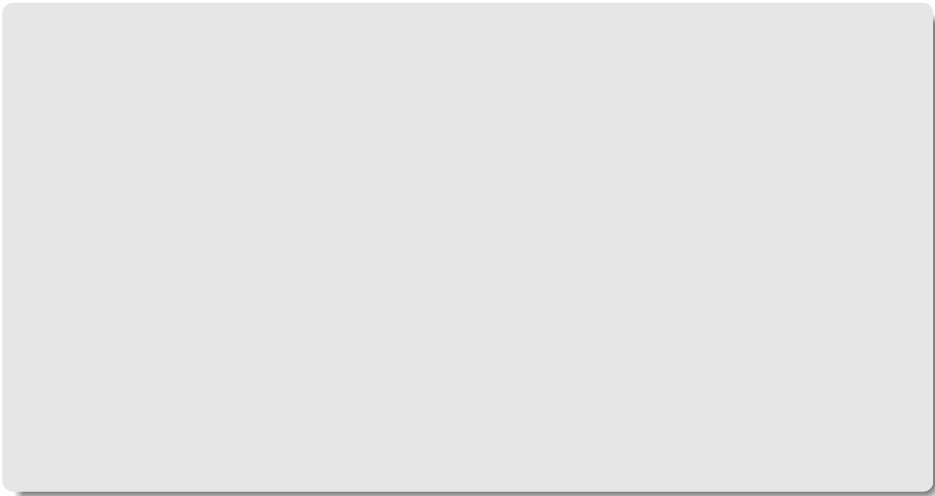
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 - ▶ $A \sim_B B$ iff the induced inhomogeneous coverings $\mathcal{Q}_{A^T}^i$ and $\mathcal{Q}_{B^T}^i$ are weakly equivalent.
- $A \sim_h B$ is equivalent to the property that A, B induce the same scale of **anisotropic Hardy spaces** (\rightsquigarrow M. Bownik).

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- Simple exercise: Checking $A \sim_h B$ or $A \sim_i B$ for A, B diagonal.

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Generalized eigenspaces

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Given a matrix $A \in \mathbb{C}^{d \times d}$, we define for $r > 0$ and $m \in \mathbb{N}_0$

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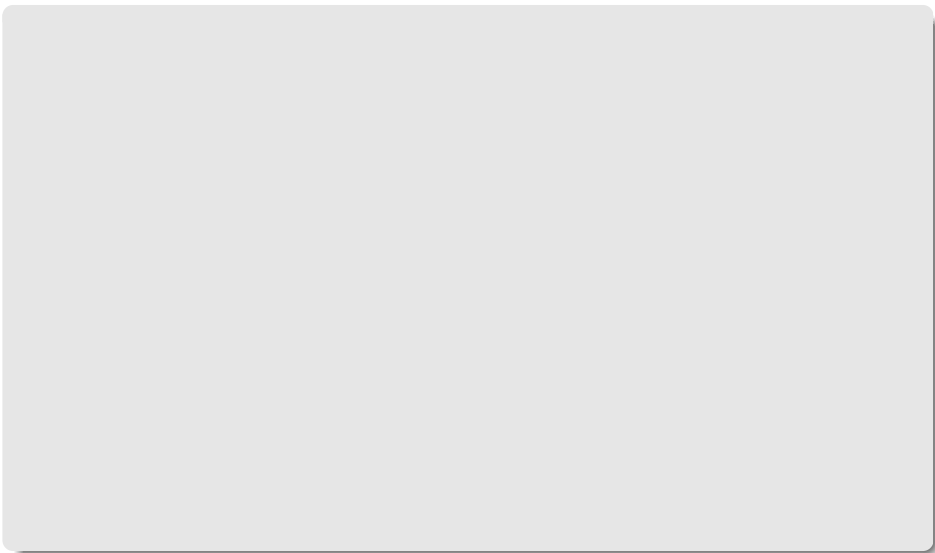
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Given A, B , compute expansive normal forms A', B' , and compare.

Once the spectra of A, B are known, this can be carried out using standard linear algebra methods.

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$$CAC^{-1} = \begin{pmatrix} M_\lambda & M_{z_1} & & & & \\ & M_\lambda & M_{z_2} & & & \\ & & \ddots & \ddots & & \\ & & & M_\lambda & M_{z_{d/2-1}} & \\ & & & & M_\lambda & \end{pmatrix},$$

with $z_1, z_2, \dots \in \{0, 1\} \subset \mathbb{C}$.

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- Define $A' = \exp(s \log(A_0))$, where $s = \ln(2)/d \ln(r)$.

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- Hence, w.l.o.g., A, B have the same, single eigenvalue $\lambda > 0$.
- Write $B = \lambda(I_d + N_B), A^{-1} = \lambda^{-1}(I_d + N_A)$, with N_A, N_B nilpotent matrices.

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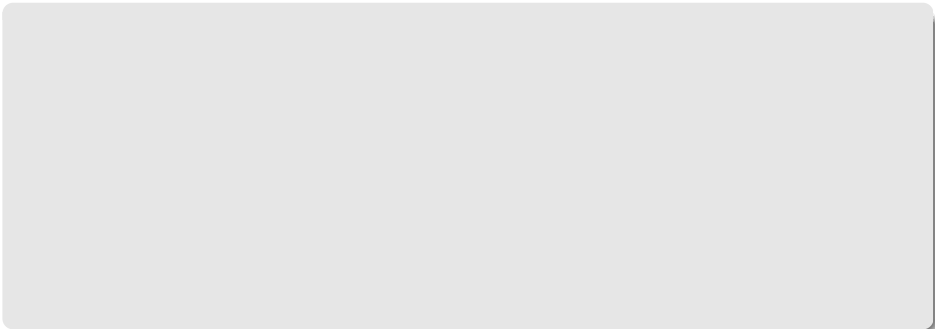
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Then $A^T \sim_i B^T$ if and only if

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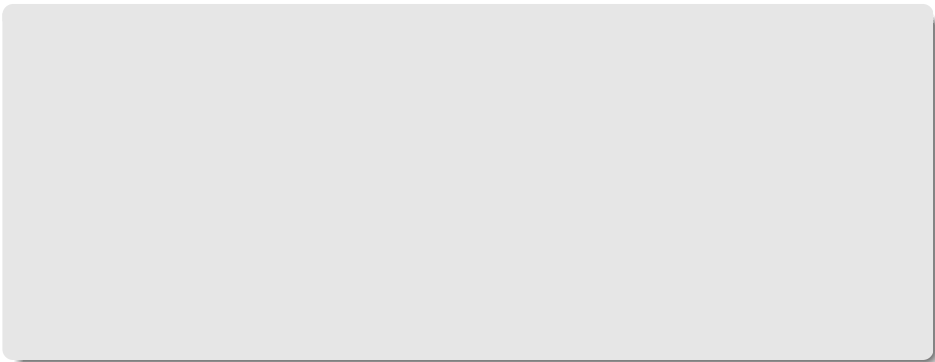
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- If A, B are expansive diagonal matrices with positive entries, then $A \sim_h B$ iff $A = B^\epsilon$, for some $\epsilon > 0$.

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- Further consequences of the decomposition space description: Sharp embedding theorems into Sobolev spaces, or other Besov spaces (anisotropic and isotropic). (Not yet fully explored.)
- The present talk presents a case study of the power of decomposition space methods for the analysis of function spaces defined in terms of families of convolution products.

References

- M. Bownik, *Anisotropic Hardy spaces and wavelets*, Mem. Amer. Math. Soc. **164** (2003).
- M. Bownik, *Atomic and molecular decompositions of anisotropic Besov spaces*, Math. Z. **250**, 539–571 (2005)
- H. Feichtinger, P. Gröbner, *Banach spaces of distributions defined by decomposition methods. I*. Math. Nachr. **123**, 97-120 (1985)
- HF, J. Cheshmavar, *A classification of anisotropic Besov spaces*. Preprint, 2016, available under <https://arxiv.org/abs/1609.06083>
- F. Voigtlaender: *Embedding Theorems for Decomposition Spaces with Applications to Wavelet Coorbit Spaces*. Ph.D. Thesis, RWTH Aachen, 2015
- F. Voigtlaender, *Embeddings of decomposition spaces*. Preprint, available under <http://arxiv.org/abs/1605.09705>
- F. Voigtlaender, *Embeddings of decomposition spaces into Sobolev and BV spaces*. Preprint, available under <http://arxiv.org/abs/1601.02201>
- H.G. Feichtinger, F. Voigtlaender *From Frazier-Jawerth characterizations of Besov spaces to wavelets and decomposition spaces*. Preprint, available under <http://arxiv.org/abs/1606.04924>