The classification of anisotropic Besov spaces

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Outline

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1. Anisotropic Besov spaces

2. Decomposition spaces: Definition and properties

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3. Rigidity theorem for decomposition spaces

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4. Classifying anisotropic coverings

Overview

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A Schwartz function φ is called *A*-scaling function if

$$\forall \xi \in \mathbb{R}^d \ : \ |\varphi(\xi)|^2 + \sum_{j \in \mathbb{N}} |\widehat{\psi}((A^T)^{-j}\xi)|^2 = 1 \ .$$

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Note: *A*-wavelets and scaling functions always exist.

Defining homogeneous anisotropic Besov spaces

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Given $f \in \mathcal{S}'(\mathbb{R}^d)$, let

$$\|f\|_{\dot{B}^{\alpha}_{p,q}(A)} = \left\| \left(|\det(A)|^{j\alpha} \|f * \psi_j\|_p \right)_{j\in\mathbb{Z}} \right\|_{\ell^q} \tag{1}$$

denote the homogeneous Besov quasi-norm. The homogeneous Besov space $\dot{B}^{\alpha}_{p,q}(A)$ associated to A consists of all f with $\|f\|_{\dot{B}^{\alpha}_{p,q}(A)} < \infty$.

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We identify elements of $\dot{B}^{\alpha}_{p,q}(A)$ that only differ by a polynomial.

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Given $f \in \mathcal{S}'(\mathbb{R}^d)$, define

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The inhomogeneous anisotropic Besov space $B^\alpha_{p,q}(A)$ associated to A consists of all tempered distributions f with $\|f\|_{B^\alpha_{p,q}(A)} < \infty$.

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Main questions for this talk

Given two expansive matrices A, B, we call them homogeneously equivalent $(A \sim_h B)$ whenever $\dot{B}^{\alpha}_{p,q}(A) = \dot{B}^{\alpha}_{p,q}(B)$ holds for all $0 < p, q \leq \infty, \alpha \in \mathbb{R}$.

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- Do $\sim_h$ and $\sim_i$ differ?

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Anisotropic coverings of frequency space

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Informal description: Cover the frequencies by a family of open relatively compact sets. Decompose functions using a subordinate partition of unity. Introduce a norm by locally taking L^p-norms, and then globally combine using weighted ℓ^q norm.

Admissible coverings

Definition

Let $\mathcal{O} \subset \mathbb{R}^d$ be open, and $\mathcal{Q} = (Q_i)_{i \in I}$ a family of open, subsets $Q_i \subset \mathcal{O}$ with compact closure in $\mathcal{O}$.

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(b) $\mathcal{Q}$ is called an almost structured admissible covering (a.s.a.c.) if it is an admissible covering, and there exist $(Q_i')_{i \in I}$, nonempty open bounded sets and $T_i \in \mathrm{GL}(d, \mathbb{R})$ and $b_i \in \mathbb{R}^d$ with:

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   (i) For all $i \in I$: $\overline{T_i Q_i' + b_i} \subset Q_i$.
   (ii) The quantity $\sup_{i,j: Q_i \cap Q_j \neq \emptyset} \| T_i^{-1} T_j \|$ is finite.
   (iii) The set $\{Q_i' : i \in I\}$ is finite.
   (iv) The family $(T_i Q_i' + b_i)_{i \in I}$ is an admissible covering.

   The tuple $((T_i)_{i \in I}, (b_i)_{i \in I}, (Q_i')_{i \in I})$ are called standardization of $\mathcal{Q}$.

Partitions of unity

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(i) For all $i \in I : \varphi_i \in C_c^\infty(\mathcal{O})$.

(ii) For all $i \in I : \varphi_i \equiv 0$ on $\mathbb{R}^d \setminus Q_i$.

(iii) $\sum_{i\in I} \varphi \equiv 1$ on $\mathcal{O}$.

(iv) $\sup_{i\in I} |\mathrm{det}(T_i)|^{\frac{1}{t}-1} \|\mathcal{F}^{-1}\varphi_i\|_{\mathrm{L}^p} < \infty$. Here $t = \min(p, 1)$.

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Lemma (Feichtinger, Voigtlaender)

BAPU's exist.

Weights associated to coverings

Definition

Let $\mathcal{Q} = (Q_i)_{i \in I}$ denote an admissible covering, and let $v : I \to \mathbb{R}^+$ denote a weight. The weight is called $\mathcal{Q}$-moderate if

$$\sup_{i,j \in I : Q_i \cap Q_j \neq \emptyset} \frac{v(i)}{v(j)} < \infty .$$

Given $0 < q \leq \infty$, let $\ell^q_v(I) = \{ c = (c_i)_{i \in I} \in \mathbb{C}^I \ : \ (c_i v(i))_{i \in I} \in \ell^q(I) \}$, with the obvious (quasi-)norm.

Definition of decomposition spaces

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$$\|u\|_{\mathcal{D}(\mathcal{Q},\mathrm{L}^p,\ell^q_v)} = \left\| \left(\|\mathcal{F}^{-1}(\varphi_i \cdot u)\|_{\mathrm{L}^p} \right)_{i \in I} \right\|_{\ell^q_v} . \tag{3}$$

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$$\|u\|_{\mathcal{D}(\mathcal{Q}, \mathrm{L}^p, \ell^q_v)} = \left\| \left(\|\mathcal{F}^{-1}(\varphi_i \cdot u)\|_{\mathrm{L}^p} \right)_{i \in I} \right\|_{\ell^q_v} . \tag{3}$$

The decomposition space $\mathcal{D}(\mathcal{Q}, , \mathrm{L}^p, \ell^q_v)$ is the space of all $u \in \mathcal{D}'(\mathcal{O})$ for which this (quasi-)norm is finite.

Properties of decomposition spaces

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Theorem (Feichtinger/Gröbner)

$\mathcal{D}(\mathcal{Q}, , \mathrm{L}^p, \ell^q_v)$ *is a quasi-Banach space, that is* independent *of the choice of BAPU.*

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- Large variety of admissible coverings allows diverse ways of measuring the decay. Describes homogeneous and inhomogeneous isotropic Besov spaces, α-modulation spaces, shearlet and curvelet approximation spaces, shearlet coorbit spaces (see upcoming talk by René Koch), and anisotropic Besov spaces.
- Decomposition spaces provide a unified framework for many embedding results, either between different decomposition spaces, or of decomposition spaces into well-known smoothness spaces such as Sobolev spaces (F. Voigtlaender).

Induced covering

Definition

Let A denote an expansive matrix. Let $Q \Subset \mathbb{R}^d \setminus \{0\}$ be open, and such that

$$\bigcup_{j \in \mathbb{Z}} A^j Q = \mathbb{R}^d \setminus \{0\} \ .$$

Then $\mathcal{Q}_A = (A^j Q)_{j \in \mathbb{Z}}$ is called the homogeneous covering induced by A.

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Intuition

$$Q_j = \widehat{\psi}_j^{-1}(\mathbb{C} \setminus \{0\}) \ , \ Q_j^i = \widehat{\psi_j^i}^{-1}(\mathbb{C} \setminus \{0\})$$

and the wavelet systems are BAPUs.

Anisotropic Besov spaces as decomposition spaces

Theorem

Let A denote an expansive matrix, and $\mathcal{Q}_A$ a homogeneous covering induced by A^T.

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Let A denote an expansive matrix, and $\mathcal{Q}_A$ a homogeneous covering induced by A^T. For $\alpha \in \mathbb{Z}$, define

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Denote by $\rho : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ the restriction map. Then $\rho \circ \mathcal{F}$ is a topological isomorphism

$$\rho \circ \mathcal{F} : \dot{B}^{\alpha}_{p,q}(A) \to \mathcal{D}(\mathcal{Q}_A, \mathrm{L}^p, \ell^q_{v_{\alpha,A}}) \ .$$

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Similarly, if $\mathcal{Q}^i_A$ denote an inhomogeneous covering induced by A^T, then

$$\rho \circ \mathcal{F} : B^{\alpha}_{p,q}(A) \to \mathcal{D}(\mathcal{Q}^i_A, \mathrm{L}^p, \ell^q_{v_{\alpha,A}}) \ .$$

is a topological isomorphism, as well. Here $v_{\alpha,A}$ denotes the restriction of the weight for the homogeneous setting to $\mathbb{N}_0$.

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Comparing coverings and weights

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Definition

Admissible coverings $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ are weakly equivalent

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Admissible coverings $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ are weakly equivalent if

$$\sup_{i \in I} |\{j \in J : Q_i \cap P_j \neq \emptyset\}| + \sup_{j \in J} |\{i \in I : Q_i \cap P_j \neq \emptyset\}| < \infty \; .$$

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Weights v on I and w on J are equivalent if

$$\sup_{i,j,Q_i \cap P_j \neq \emptyset} \frac{v(i)}{w(j)} + \frac{w(j)}{v(i)} < \infty .$$

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Note: If v denotes the constant weight, then $\mathcal{D}(\mathcal{P}, L^2, \ell^2_v) = \mathrm{L}^2(\mathbb{R}^d)$, for every a.s.a.c. covering $\mathcal{P}$.

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- $A \sim_h B$ is equivalent to the property that A, B induce the same scale of anisotropic Hardy spaces ($\rightsquigarrow$ M. Bownik).

Overview

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- Simple exercise: Checking $A \sim_h B$ or $A \sim_i B$ for A, B diagonal.

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$A \sim_h B$ *implies* $A \sim_i B$.

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Corollary

$A \sim_h B$ implies $A \sim_i B$. The converse holds if A, B are diagonal.

Generalized eigenspaces

Definition

Given a matrix $A \in \mathbb{C}^{d \times d}$, we define for $r > 0$ and $m \in \mathbb{N}_0$

$$E(A, r, m) = \mathrm{span}\left(\bigcup_{|\lambda|=r} \mathrm{Ker}(A - \lambda I_d)^m \cup \bigcup_{|\lambda|<r} \mathrm{Ker}(A - \lambda I_d)^d \right) .$$

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Let $B \in \mathrm{GL}(d, \mathbb{R})$ be expansive. We call B in expansive normal form if B has only positive eigenvalues, and $\det(B) = 2$.

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Once the spectra of A, B are known, this can be carried out using standard linear algebra methods.

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with $z_1, z_2 \ldots \in \{0, 1\} \subset \mathbb{C}$.

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- Define $A' = \exp(s \log(A_0))$, where $s = \ln(2)/d \ln(r)$.

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- Then $\epsilon(A, B) = 1$, and A, B have the same eigenvalues and associated generalized eigenspaces. (See Lemma above.)
- Hence, w.l.o.g., A, B have the same, single eigenvalue $\lambda > 0$.
- Write $B = \lambda(I_d + N_B), A^{-1} = \lambda^{-1}(I_d + N_A)$, with N_A, N_B nilpotent matrices.
- It follows that

$$A^{-k} = \lambda^{-k} P_k \ , B^k = \lambda^k Q_k \ , A^{-k} B^k = P_k Q_k \ ,$$

  where P_k, Q_k are matrices whose entries are polynomials in k.
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- $A^{-1} B^1 = A^{-2} B^2$ implies $A = B$.

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Theorem (HF & J. Cheshmavar)

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$$A = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix} ,$$

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Then $A^T \sim_i B^T$ if and only if

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fulfill $A \sim_i B$, but $A \not\sim_h B$.
- If A, B are expansive diagonal matrices with positive entries, then $A \sim_h B$ iff $A = B^\epsilon$, for some $\epsilon > 0$.

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- Main results of the talk: Decomposition space description of anisotropic Besov spaces, and resulting classification of dilation matrices.
- Further consequences of the decomposition space description: Sharp embedding theorems into Sobolev spaces, or other Besov spaces (anisotropic and isotropic). (Not yet fully explored.)
- The present talk presents a case study of the power of decomposition space methods for the analysis of function spaces defined in terms of families of convolution products.

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