Analysis of Shearlet Coorbit Spaces in Dimension Three

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Outline

Overview

Shearlet Groups in Dimension 2

The original shearlet group in dimension 2 was introduced by Dahlke, Kutyniok, Steidl, Teschke as

$$H = \left\{ \pm \begin{pmatrix} a & ab \\ 0 & a^{1/2} \end{pmatrix} : \begin{array}{c} a > 0, \\ b \in \mathbb{R} \end{array} \right\}$$
$$= \left\{ \pm \begin{pmatrix} a & 0 \\ 0 & a^{1/2} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : \begin{array}{c} a > 0, \\ b \in \mathbb{R} \end{array} \right\}.$$

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Motivation

The anisotropic scaling inherent in the dilation group gives rise to shearlet systems whose approximation-theoretic properties improve on the classical wavelets.

Shearlet Transform in Dimension 2

- $G = \mathbb{R}^2 \rtimes H$ semidirect product of $\mathbb{R}^2$ and H with group law

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$$\mathcal{S}(\psi) := \{\pi(x,h)\psi : (x,h) \in G\}$$

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A function $0 \neq \psi \in L^2(\mathbb{R}^2)$ is called *admissible shearlet* if

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$$f = \frac{1}{C_\psi} \int_G \mathcal{S}_\psi f(x,h) \, \pi(x,h)\psi \, \mathrm{d}\mu_G(x,h).$$

Coorbit Theory

For measurable, locally bounded, submultiplicative weight $v : H \to (0,\infty)$ and $p, q \in (0,\infty)$ define the weighted mixed L^p-(quasi-)norm

$$\|f\|_{L_v^{p,q}} := \left(\int_H \left(\int_{\mathbb{R}^2} v(h)^p \, |f(x,h)|^p \, \mathrm{d}x \right)^{q/p} \frac{\mathrm{d}h}{|\det(h)|} \right)^{1/q}$$

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Features of Coorbit theory

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Standard Shearlet Group

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Toeplitz Shearlet Group

$$H^{\delta} := \left\{ \pm \begin{pmatrix} a & ab & ac \\ 0 & a^{1-\delta} & a^{1-\delta} b \\ 0 & 0 & a^{1-2\delta} \end{pmatrix} \middle| \begin{array}{l} a > 0, \\ b, c \in \mathbb{R} \end{array} \right\} < \mathrm{GL}(3, \mathbb{R})$$

for $\delta \in \mathbb{R}$.

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Dual Action and Dual Orbit

The dual action is given by $H \times \mathbb{R}^3 \to \mathbb{R}^3, (h,\xi) \mapsto h^{-t}\xi$ and for a shearlet group this action has a unique open dual orbit $H^{-t}\xi_0 = \mathcal{O} = \mathbb{R}^* \times \mathbb{R}^2$.

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If $\mathrm{supp}\ \hat{\psi} \subset U$, then $\mathrm{supp}\ \widehat{S_\psi f}(\cdot, h) \subset h^{-t}U$.

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$$\mathcal{D}(\mathcal{Q}, L^p, \ell^q_u) = \left\{ f \in \mathcal{D}'(\mathcal{O}) : \|f\|_{\mathcal{D}(\mathcal{Q}, L^p, \ell^q_u)} < \infty \right\}$$

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Intuition: The relevant aspect of the group H for the coorbit space is this covering.

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Let $\mathcal{Q}$ be a covering induced by H and define the discrete weight $u = (u_i)_{i \in I}$ by

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Theorem (Führ, Voigtlaender)

The Fourier transform

$$\mathcal{F} : \mathrm{Co}(L_v^{p,q}(G)) \to \mathcal{D}(\mathcal{Q}, L^p, \ell_u^q)$$

is an isomorphism of (quasi) Banach spaces.

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Define the intersection sets of $\mathcal{Q}$ and $\mathcal{P}$ for $i \in I$ and $j \in J$ by

$$I_j := \{ i \in I : Q_i \cap P_j \neq \emptyset \} \text{ and } J_i := \{ j \in J : Q_i \cap P_j \neq \emptyset \} .$$

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Definition (Weak Equivalence)

We call the coverings $\mathcal{Q}$ and $\mathcal{P}$ weakly equivalent if

$$\sup_{j \in J} |I_j| < \infty \text{ and } \sup_{i \in I} |J_i| < \infty.$$

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Let $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ be induced coverings, $p_1, p_2, q_1, q_2 \in (0, \infty)$ and $u = (u_i)_{i \in I}, u' = (u'_j)_{j \in J}$ discrete weights. If

$$\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right) = \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell^{q_2}_{u'}\right),$$

then

i) $p_1 = p_2$ and $q_1 = q_2$,

ii) there is $C > 0$ with $C^{-1} u_i \le u'_j \le C u_i$ for all $i \in I, j \in J$ such that $Q_i \cap P_j \neq \emptyset$,

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iii) in the case $(p_1, q_1) \neq (2, 2)$ the coverings $\mathcal{Q}, \mathcal{P}$ are weakly equivalent.

Overview

Comparison of Shearlet Coorbit Spaces in Dimension 3

Question

Do different shearlet groups induce different shearlet coorbit spaces?

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In order to answer this question, we

- find a well-spread family in H^{δ} and H^{λ_1,λ_2},
- compute the induced coverings and
- check whether they are weakly equivalent.

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For definiteness we will focus on the class of groups

$$H^{\lambda_1,\lambda_2} := \left\{ \pm \begin{pmatrix} a & ab & ac \\ 0 & a^{\lambda_1} & 0 \\ 0 & 0 & a^{\lambda_2} \end{pmatrix} \middle| \begin{array}{l} a > 0, \\ b, c \in \mathbb{R} \end{array} \right\}$$

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Well-spread set in H^{λ_1,λ_2}

For $n, m_1, m_2 \in \mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$ define

$$B^{\lambda_1,\lambda_2}_{n,m_1,m_2,\varepsilon} := \varepsilon \begin{pmatrix} 2^n & 2^n m_1 & 2^n m_2 \\ 0 & 2^{n\lambda_1} & 0 \\ 0 & 0 & 2^{n\lambda_2} \end{pmatrix}$$

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Since $\mathcal{Q} = (h_i^{-T} Q)_{i \in I}$, more important is

$$A^{\lambda_1,\lambda_2}_{n,m_1,m_2,\epsilon} := \left(B^{\lambda_1,\lambda_2}_{-n,-m_1,-m_2,\varepsilon} \right)^{-T} = \left(B^{\lambda_1,\lambda_2}_{n,m_1,m_2,\varepsilon} \right)^{T}$$

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Induced Covering

For

$$Q := \left\{ (x, y, z)^T \in \mathbb{R}^3 : x \in (1/2, 2) \text{ and } y/x, z/x \in (-1, 1) \right\}$$

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$$\mathcal{Q}^{\lambda_1,\lambda_2} := \left(Q^{\lambda_1,\lambda_2}_{n,m_1,m_2,\varepsilon}\right)_{(n,m_1,m_2,\epsilon)\in\mathbb{Z}^3\times\{\pm1\}}$$

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$$\begin{aligned}
\mathcal{Q}^{\lambda_1,\lambda_2} &:= \left(Q^{\lambda_1,\lambda_2}_{n,m_1,m_2,\varepsilon} \right)_{(n,m_1,m_2,\epsilon)\in\mathbb{Z}^3\times\{\pm1\}} \\
&:= \left(A^{\lambda_1,\lambda_2}_{n,m_1,m_2,\epsilon} Q \right)_{(n,m_1,m_2,\epsilon)\in\mathbb{Z}^3\times\{\pm1\}} \\
&= \left(\left\{ \varepsilon \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{array}{rl} x & \in 2^n(1/2,2) \\ y/x & \in 2^{n(\lambda_1-1)}\left[m_1 + (-1,1)\right] \\ z/x & \in 2^{n(\lambda_2-1)}\left[m_2 + (-1,1)\right] \end{array} \right\} \right)_{(n,m_1,m_2,\epsilon)}
\end{aligned}$$

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Theorem

Shearlet coorbit spaces with respect to different shearlet groups in dimension three give rise to different coorbit spaces (for $(p, q) \neq (2, 2)$).

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1. We identify the coorbit spaces associated to the groups H^{λ_1,λ_2} and $H^{\lambda_1',\lambda_2'}$ with certain decomposition spaces.

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2. Because of the rigidity theorem it suffices to show that coverings induced by different shearlet groups are not weakly equivalent.
3. In order to achieve this we determine conditions that ensure

$$Q^{\lambda_1,\lambda_2}_{n,m_1,m_2,\varepsilon} \cap Q^{\lambda_1',\lambda_2'}_{n',m_1',m_2',\varepsilon'} \neq \emptyset.$$

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4. These conditions allow the construction of a sequence $(j(k))_{k\in\mathbb{N}} \in \mathbb{Z}^3 \times \{\pm 1\}$ such that

$$\left| \left\{ (n, m_1, m_2, \varepsilon) : Q^{\lambda_1,\lambda_2}_{n,m_1,m_2,\varepsilon} \cap Q^{\lambda_1',\lambda_2'}_{j(k)} \neq \emptyset \right\} \right| \xrightarrow[k\to\infty]{} \infty.$$

Similar reasoning can be used for the comparison of the groups H^{λ_1,λ_2} and H^{δ}.

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Main point

Different shearlet groups lead to an essentially different covering of the dual orbit through the dual action.

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